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Preface to TC Calculus 1

Calculus 1 text adapted from "Calculus Volume 1" for Texarkana College using the THECB ACGM.

Welcome to *TC Calculus 1*. This textbook has been created from *Calculus Volume 1*, an OpenStax resource, with several goals in mind: accessibility, cost effectiveness, and student engagement. This textbook offers a strong foundation in calculus in an accessible format.

About OpenStax

OpenStax is a non-profit organization committed to improving student access to quality learning materials. Our free textbooks go through a rigorous editorial publishing process. Our texts are developed and peer-reviewed by educators to ensure they are readable, accurate, and meet the scope and sequence requirements of today's college courses. Unlike traditional textbooks, OpenStax resources live online and are owned by the community of educators using them. Through our partnerships with companies and foundations committed to reducing costs for students, OpenStax is working to improve access to higher education for all. OpenStax is an initiative of Rice University and is made possible through the generous support of several philanthropic foundations. Since our launch in 2012 our texts have been used by millions of learners online and thousands of institutions worldwide.

About *TC Calculus 1*

TC Calculus 1 is the first of three textbooks designed for the three-semester calculus course at Texarkana College. For many students, this course provides the foundation to a career in mathematics, science, or engineering. As such, this textbook provides an important opportunity for students to learn the core concepts of calculus and understand how those concepts apply to their lives and the world around them. The text has been developed

to meet the scope and sequence as set forth by the Texas Higher Education Coordinating Board in the Academic Course Guide Manual.

Coverage and Scope

TC Calculus 1 makes calculus interesting and accessible to students while maintaining the mathematical rigor inherent in the subject. The content of the three volumes of *TC Calculus* have been arranged to provide a logical progression from fundamental to more advanced concepts, building upon what students have already learned and emphasizing connections between topics and between theory and applications. The goal of each section is to enable students not just to recognize concepts, but work with them in ways that will be useful in later courses and future careers.

TC Calculus 1

- Chapter 1: Functions and Graphs
- Chapter 2: Limits
- Chapter 3: Derivative Fundamentals
- Chapter 4: More Derivatives
- Chapter 5: Parametric Equations and Conics
- Chapter 6: Fundamental Applications of Derivatives
- Chapter 7: More Applications of Derivatives
- Chapter 8: Integration

TC Calculus 2

- Chapter 1: Integration
- Chapter 2: Fundamental Applications of Integration
- Chapter 3: Further Applications of Integration
- Chapter 4: Polar Coordinates
- Chapter 5: Techniques of Integration
- Chapter 6: Introduction to Differential Equations
- Chapter 7: Sequences and Series
- Chapter 8: Power Series

TC Calculus 3

Chapter 1: Parametric Equations and Polar Coordinates
Chapter 2: Vectors in Space
Chapter 3: Vector-Valued Functions
Chapter 4: Differentiation of Functions of Several Variables
Chapter 5: Multiple Integration
Chapter 6: Vector Calculus

Pedagogical Foundation

Throughout *TC Calculus 1* you will find examples and exercises that present classical ideas and techniques as well as modern applications and methods. Derivations and explanations are based on years of classroom experience on the part of long-time calculus professors, striving for a balance of clarity and rigor that has proven successful with their students. Motivational applications cover important topics in probability, biology, ecology, business, and economics, as well as areas of physics, chemistry, engineering, and computer science. **Student Projects** in each chapter give students opportunities to explore interesting sidelights in pure and applied mathematics, from showing that the number e is irrational, to calculating the center of mass of the Grand Canyon Skywalk or the terminal speed of a skydiver. **Chapter Opening Applications** pose problems that are solved later in the chapter, using the ideas covered in that chapter. Problems include the hydraulic force against the Hoover Dam, and the comparison of the relative intensity of two earthquakes. **Definitions, Rules, and Theorems** are highlighted throughout the text, including over 60 **Proofs** of theorems.

Assessments That Reinforce Key Concepts

In-chapter **Examples** walk students through problems by posing a question, stepping out a solution, and then asking students to practice the skill with a “Checkpoint” question. The book also includes assessments at the end of each chapter so students can apply what they’ve learned through practice problems. Many exercises are marked with a **[T]** to indicate they are suitable for solution by technology, including calculators or Computer

Algebra Systems (CAS). Answers for selected exercises are available in the **Answer Key** at the back of the book.

About the Team that created OpenStax "Calculus"

Senior Contributing Authors

Gilbert Strang, PhD

Dr. Strang received his PhD from UCLA in 1959 and has been teaching mathematics at MIT ever since. His Calculus online textbook is one of eleven that he has published and is the basis from which our final product has been derived and updated for today's student. Strang is a decorated mathematician and past Rhodes Scholar at Oxford University.

Edwin "Jed" Herman, PhD

Dr. Herman earned a BS in Mathematics from Harvey Mudd College in 1985, an MA in Mathematics from UCLA in 1987, and a PhD in Mathematics from the University of Oregon in 1997. He is currently a Professor at the University of Wisconsin-Stevens Point. He has more than 20 years of experience teaching college mathematics, is a student research mentor, is experienced in course development/design, and is also an avid board game designer and player.

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Introduction

class="introduction"

A portion of
the San
Andreas Fault
in California.
Major faults
like this are
the sites of
most of the
strongest
earthquakes
ever
recorded.
(credit:
modification
of work by
Robb
Hannawacker
, NPS)



In the past few years, major earthquakes have occurred in several countries around the world. In January 2010, an earthquake of magnitude 7.3 hit Haiti. A magnitude 9 earthquake shook northeastern Japan in March 2011. In April 2014, an 8.2-magnitude earthquake struck off the coast of northern

Chile. What do these numbers mean? In particular, how does a magnitude 9 earthquake compare with an earthquake of magnitude 8.2? Or 7.3? Later in this chapter, we show how logarithmic functions are used to compare the relative intensity of two earthquakes based on the magnitude of each earthquake (see [\[link\]](#)).

Calculus is the mathematics that describes changes in functions. In this chapter, we review all the functions necessary to study calculus. We define polynomial, rational, trigonometric, exponential, and logarithmic functions. We review how to evaluate these functions, and we show the properties of their graphs. We provide examples of equations with terms involving these functions and illustrate the algebraic techniques necessary to solve them. In short, this chapter provides the foundation for the material to come. It is essential to be familiar and comfortable with these ideas before proceeding to the formal introduction of calculus in the next chapter.

Review of Functions

- Use functional notation to evaluate a function.
- Determine the domain and range of a function.
- Draw the graph of a function.
- Find the zeros of a function.
- Recognize a function from a table of values.
- Make new functions from two or more given functions.
- Describe the symmetry properties of a function.

In this section, we provide a formal definition of a function and examine several ways in which functions are represented—namely, through tables, formulas, and graphs. We study formal notation and terms related to functions. We also define composition of functions and symmetry properties. Most of this material will be a review for you, but it serves as a handy reference to remind you of some of the algebraic techniques useful for working with functions.

Functions

Given two sets A and B , a set with elements that are ordered pairs (x, y) , where x is an element of A and y is an element of B , is a relation from A to B . A relation from A to B defines a relationship between those two sets. A function is a special type of relation in which each element of the first set is related to exactly one element of the second set. The element of the first set is called the *input*; the element of the second set is called the *output*. Functions are used all the time in mathematics to describe relationships between two sets. For any function, when we know the input, the output is determined, so we say that the output is a function of the input. For example, the area of a square is determined by its side length, so we say that the area (the output) is a function of its side length (the input). The velocity of a ball thrown in the air can be described as a function of the amount of time the ball is in the air. The cost of mailing a package is a function of the weight of the package. Since functions have so many uses, it is important to have precise definitions and terminology to study them.

Note:

Definition

A **function** f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the **domain** of the function. The set of outputs is called the **range** of the function.

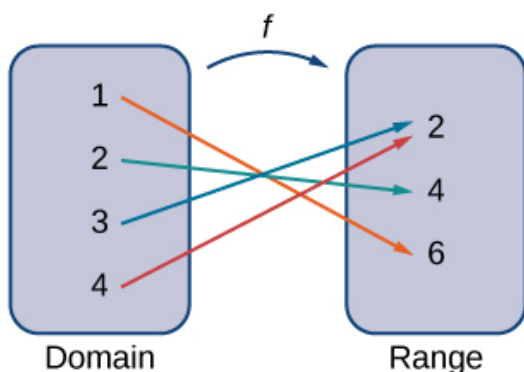
For example, consider the function f , where the domain is the set of all real numbers and the rule is to square the input. Then, the input $x = 3$ is assigned to the output $3^2 = 9$. Since every nonnegative real number has a real-value square root, every nonnegative number is an element of the range of this function. Since there is no real number with a square that is negative, the negative real numbers are not elements of the range. We conclude that the range is the set of nonnegative real numbers.

For a general function f with domain D , we often use x to denote the input and y to denote the output associated with x . When doing so, we refer to x as the **independent variable** and y as the **dependent variable**, because it depends on x . Using function notation, we write $y = f(x)$, and we read this equation as “ y equals f of x .” For the squaring function described earlier, we write $f(x) = x^2$.

The concept of a function can be visualized using [\[link\]](#), [\[link\]](#), and [\[link\]](#).

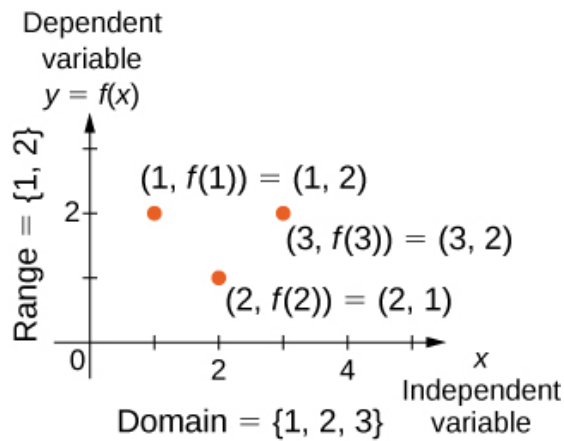


A function can be visualized as an input/output device.



A function maps every element in the domain to exactly one element in the range. Although each input can be sent to only one output, two

different inputs can be sent to the same output.

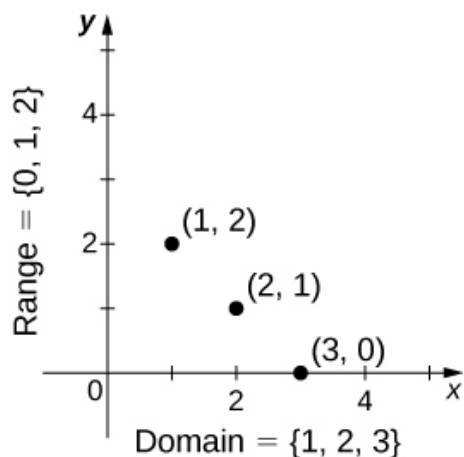


In this case, a graph of a function f has a domain of $\{1, 2, 3\}$ and a range of $\{1, 2\}$. The independent variable is x and the dependent variable is y .

Note:

Visit this [applet link](#) to see more about graphs of functions.

We can also visualize a function by plotting points (x, y) in the coordinate plane where $y = f(x)$. The **graph of a function** is the set of all these points. For example, consider the function f , where the domain is the set $D = \{1, 2, 3\}$ and the rule is $f(x) = 3 - x$. In [\[link\]](#), we plot a graph of this function.



Here we see a graph of the function f with domain $\{1, 2, 3\}$ and rule $f(x) = 3 - x$. The graph consists of the points $(x, f(x))$ for all x in the domain.

Every function has a domain. However, sometimes a function is described by an equation, as in $f(x) = x^2$, with no specific domain given. In this case, the domain is taken to be the set of all real numbers x for which $f(x)$ is a real number. For example, since any real number can be squared, if no other domain is specified, we consider the domain of $f(x) = x^2$ to be the set of all real numbers. On the other hand, the square root function $f(x) = \sqrt{x}$ only gives a real output if x is nonnegative. Therefore, the domain of the function $f(x) = \sqrt{x}$ is the set of nonnegative real numbers, sometimes called the *natural domain*.

For the functions $f(x) = x^2$ and $f(x) = \sqrt{x}$, the domains are sets with an infinite number of elements. Clearly we cannot list all these elements. When describing a set with an infinite number of elements, it is often helpful to use set-builder or interval notation. When using set-builder notation to describe a subset of all real numbers, denoted \mathbb{R} , we write

Equation:

$$\{x \mid x \text{ has some property}\}.$$

We read this as the set of real numbers x such that x has some property. For example, if we were interested in the set of real numbers that are greater than one but less than five,

we could denote this set using set-builder notation by writing

Equation:

$$\{x|1 < x < 5\}.$$

A set such as this, which contains all numbers greater than a and less than b , can also be denoted using the interval notation (a, b) . Therefore,

Equation:

$$(1, 5) = \{x|1 < x < 5\}.$$

The numbers 1 and 5 are called the *endpoints* of this set. If we want to consider the set that includes the endpoints, we would denote this set by writing

Equation:

$$[1, 5] = \{x|1 \leq x \leq 5\}.$$

We can use similar notation if we want to include one of the endpoints, but not the other. To denote the set of nonnegative real numbers, we would use the set-builder notation

Equation:

$$\{x|0 \leq x\}.$$

The smallest number in this set is zero, but this set does not have a largest number. Using interval notation, we would use the symbol ∞ , which refers to positive infinity, and we would write the set as

Equation:

$$[0, \infty) = \{x|0 \leq x\}.$$

It is important to note that ∞ is not a real number. It is used symbolically here to indicate that this set includes all real numbers greater than or equal to zero. Similarly, if we wanted to describe the set of all nonpositive numbers, we could write

Equation:

$$(-\infty, 0] = \{x|x \leq 0\}.$$

Here, the notation $-\infty$ refers to negative infinity, and it indicates that we are including all numbers less than or equal to zero, no matter how small. The set

Equation:

$$(-\infty, \infty) = \{x | x \text{ is any real number}\}$$

refers to the set of all real numbers.

Some functions are defined using different equations for different parts of their domain. These types of functions are known as *piecewise-defined functions*. For example, suppose we want to define a function f with a domain that is the set of all real numbers such that $f(x) = 3x + 1$ for $x \geq 2$ and $f(x) = x^2$ for $x < 2$. We denote this function by writing

Equation:

$$f(x) = \begin{cases} 3x + 1 & x \geq 2 \\ x^2 & x < 2 \end{cases}.$$

When evaluating this function for an input x , the equation to use depends on whether $x \geq 2$ or $x < 2$. For example, since $5 > 2$, we use the fact that $f(x) = 3x + 1$ for $x \geq 2$ and see that $f(5) = 3(5) + 1 = 16$. On the other hand, for $x = -1$, we use the fact that $f(x) = x^2$ for $x < 2$ and see that $f(-1) = 1$.

Example:

Exercise:

Problem:

Evaluating Functions

For the function $f(x) = 3x^2 + 2x - 1$, evaluate

- a. $f(-2)$
- b. $f(\sqrt{2})$
- c. $f(a + h)$

Solution:

Substitute the given value for x in the formula for $f(x)$.

- a. $f(-2) = 3(-2)^2 + 2(-2) - 1 = 12 - 4 - 1 = 7$
- b. $f(\sqrt{2}) = 3(\sqrt{2})^2 + 2\sqrt{2} - 1 = 6 + 2\sqrt{2} - 1 = 5 + 2\sqrt{2}$

c.

$$\begin{aligned}f(a+h) &= 3(a+h)^2 + 2(a+h) - 1 = 3(a^2 + 2ah + h^2) + 2a + 2h - 1 \\&= 3a^2 + 6ah + 3h^2 + 2a + 2h - 1\end{aligned}$$

Note:

Exercise:

Problem: For $f(x) = x^2 - 3x + 5$, evaluate $f(1)$ and $f(a+h)$.

Solution:

$$f(1) = 3 \text{ and } f(a+h) = a^2 + 2ah + h^2 - 3a - 3h + 5$$

Hint

Substitute 1 and $a+h$ for x in the formula for $f(x)$.

Example:

Exercise:

Problem:

Finding Domain and Range

For each of the following functions, determine the i. domain and ii. range.

a. $f(x) = (x-4)^2 + 5$

b. $f(x) = \sqrt{3x+2} - 1$

c. $f(x) = \frac{3}{x-2}$

Solution:

a. Consider $f(x) = (x-4)^2 + 5$.

i. Since $f(x) = (x-4)^2 + 5$ is a real number for any real number x , the domain of f is the interval $(-\infty, \infty)$.

ii. Since $(x-4)^2 \geq 0$, we know $f(x) = (x-4)^2 + 5 \geq 5$. Therefore, the range must be a subset of $\{y|y \geq 5\}$. To show that every element in

this set is in the range, we need to show that for a given y in that set, there is a real number x such that $f(x) = (x - 4)^2 + 5 = y$. Solving this equation for x , we see that we need x such that

Equation:

$$(x - 4)^2 = y - 5.$$

This equation is satisfied as long as there exists a real number x such that

Equation:

$$x - 4 = \pm \sqrt{y - 5}.$$

Since $y \geq 5$, the square root is well-defined. We conclude that for $x = 4 \pm \sqrt{y - 5}$, $f(x) = y$, and therefore the range is $\{y | y \geq 5\}$.

b. Consider $f(x) = \sqrt{3x + 2} - 1$.

- i. To find the domain of f , we need the expression $3x + 2 \geq 0$. Solving this inequality, we conclude that the domain is $\{x | x \geq -2/3\}$.
- ii. To find the range of f , we note that since $\sqrt{3x + 2} \geq 0$, $f(x) = \sqrt{3x + 2} - 1 \geq -1$. Therefore, the range of f must be a subset of the set $\{y | y \geq -1\}$. To show that every element in this set is in the range of f , we need to show that for all y in this set, there exists a real number x in the domain such that $f(x) = y$. Let $y \geq -1$. Then, $f(x) = y$ if and only if

Equation:

$$\sqrt{3x + 2} - 1 = y.$$

Solving this equation for x , we see that x must solve the equation

Equation:

$$\sqrt{3x + 2} = y + 1.$$

Since $y \geq -1$, such an x could exist. Squaring both sides of this equation, we have $3x + 2 = (y + 1)^2$.

Therefore, we need

Equation:

$$3x = (y + 1)^2 - 2,$$

which implies

Equation:

$$x = \frac{1}{3}(y + 1)^2 - \frac{2}{3}.$$

We just need to verify that x is in the domain of f . Since the domain of f consists of all real numbers greater than or equal to $-2/3$, and

Equation:

$$\frac{1}{3}(y + 1)^2 - \frac{2}{3} \geq -\frac{2}{3},$$

there does exist an x in the domain of f . We conclude that the range of f is $\{y|y \geq -1\}$.

c. Consider $f(x) = 3/(x - 2)$.

- i. Since $3/(x - 2)$ is defined when the denominator is nonzero, the domain is $\{x|x \neq 2\}$.
- ii. To find the range of f , we need to find the values of y such that there exists a real number x in the domain with the property that

Equation:

$$\frac{3}{x - 2} = y.$$

Solving this equation for x , we find that

Equation:

$$x = \frac{3}{y} + 2.$$

Therefore, as long as $y \neq 0$, there exists a real number x in the domain such that $f(x) = y$. Thus, the range is $\{y|y \neq 0\}$.

Note:

Exercise:

Problem: Find the domain and range for $f(x) = \sqrt{4 - 2x} + 5$.

Solution:

Domain = $\{x | x \leq 2\}$, range = $\{y | y \geq 5\}$

Hint

Use $4 - 2x \geq 0$.

Representing Functions

Typically, a function is represented using one or more of the following tools:

- A table
- A graph
- A formula

We can identify a function in each form, but we can also use them together. For instance, we can plot on a graph the values from a table or create a table from a formula.

Tables

Functions described using a **table of values** arise frequently in real-world applications. Consider the following simple example. We can describe temperature on a given day as a function of time of day. Suppose we record the temperature every hour for a 24-hour period starting at midnight. We let our input variable x be the time after midnight, measured in hours, and the output variable y be the temperature x hours after midnight, measured in degrees Fahrenheit. We record our data in [\[link\]](#).

Hours after Midnight	Temperature (°F)	Hours after Midnight	Temperature (°F)
-------------------------	---------------------	-------------------------	---------------------

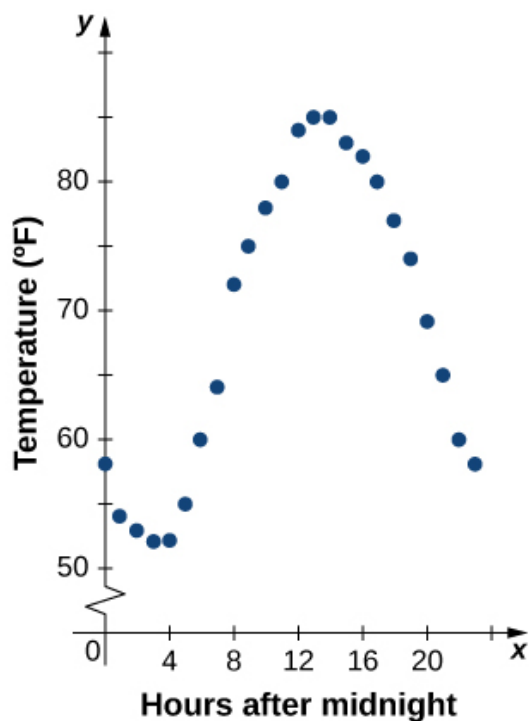
Hours after Midnight	Temperature (° F)	Hours after Midnight	Temperature (° F)
0	58	12	84
1	54	13	85
2	53	14	85
3	52	15	83
4	52	16	82
5	55	17	80
6	60	18	77
7	64	19	74
8	72	20	69
9	75	21	65
10	78	22	60
11	80	23	58

Temperature as a Function of Time of Day

We can see from the table that temperature is a function of time, and the temperature decreases, then increases, and then decreases again. However, we cannot get a clear picture of the behavior of the function without graphing it.

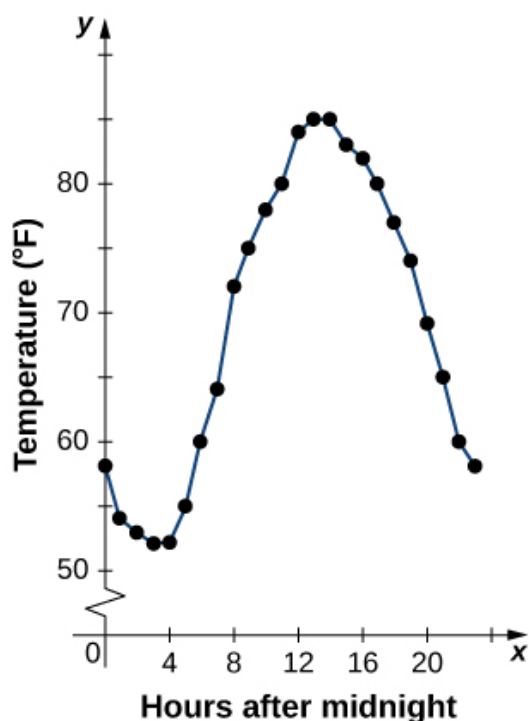
Graphs

Given a function f described by a table, we can provide a visual picture of the function in the form of a graph. Graphing the temperatures listed in [\[link\]](#) can give us a better idea of their fluctuation throughout the day. [\[link\]](#) shows the plot of the temperature function.



The graph of the data from [\[link\]](#) shows temperature as a function of time.

From the points plotted on the graph in [\[link\]](#), we can visualize the general shape of the graph. It is often useful to connect the dots in the graph, which represent the data from the table. In this example, although we cannot make any definitive conclusion regarding what the temperature was at any time for which the temperature was not recorded, given the number of data points collected and the pattern in these points, it is reasonable to suspect that the temperatures at other times followed a similar pattern, as we can see in [\[link\]](#).



Connecting the dots in [\[link\]](#) shows the general pattern of the data.

Algebraic Formulas

Sometimes we are not given the values of a function in table form, rather we are given the values in an explicit formula. Formulas arise in many applications. For example, the area of a circle of radius r is given by the formula $A(r) = \pi r^2$. When an object is thrown upward from the ground with an initial velocity v_0 ft/s, its height above the ground from the time it is thrown until it hits the ground is given by the formula $s(t) = -16t^2 + v_0t$. When P dollars are invested in an account at an annual interest rate r compounded continuously, the amount of money after t years is given by the formula $A(t) = Pe^{rt}$. Algebraic formulas are important tools to calculate function values. Often we also represent these functions visually in graph form.

Given an algebraic formula for a function f , the graph of f is the set of points $(x, f(x))$, where x is in the domain of f and $f(x)$ is in the range. To graph a function given by a formula, it is helpful to begin by using the formula to create a table of inputs and outputs. If the domain of f consists of an infinite number of values, we cannot list

all of them, but because listing some of the inputs and outputs can be very useful, it is often a good way to begin.

When creating a table of inputs and outputs, we typically check to determine whether zero is an output. Those values of x where $f(x) = 0$ are called the **zeros of a function**. For example, the zeros of $f(x) = x^2 - 4$ are $x = \pm 2$. The zeros determine where the graph of f intersects the x -axis, which gives us more information about the shape of the graph of the function. The graph of a function may never intersect the x -axis, or it may intersect multiple (or even infinitely many) times.

Another point of interest is the y -intercept, if it exists. The y -intercept is given by $(0, f(0))$.

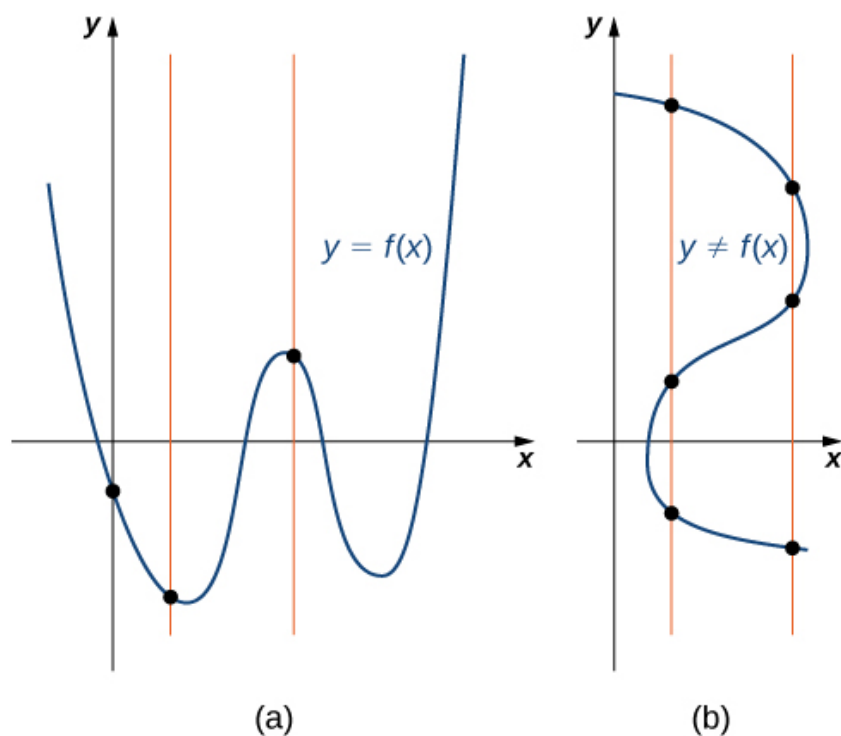
Since a function has exactly one output for each input, the graph of a function can have, at most, one y -intercept. If $x = 0$ is in the domain of a function f , then f has exactly one y -intercept. If $x = 0$ is not in the domain of f , then f has no y -intercept. Similarly, for any real number c , if c is in the domain of f , there is exactly one output $f(c)$, and the line $x = c$ intersects the graph of f exactly once. On the other hand, if c is not in the domain of f , $f(c)$ is not defined and the line $x = c$ does not intersect the graph of f . This property is summarized in the **vertical line test**.

Note:

Rule: Vertical Line Test

Given a function f , every vertical line that may be drawn intersects the graph of f no more than once. If any vertical line intersects a set of points more than once, the set of points does not represent a function.

We can use this test to determine whether a set of plotted points represents the graph of a function ([link](#)).



(a) The set of plotted points represents the graph of a function because every vertical line intersects the set of points, at most, once. (b) The set of plotted points does not represent the graph of a function because some vertical lines intersect the set of points more than once.

Example:

Exercise:

Problem:

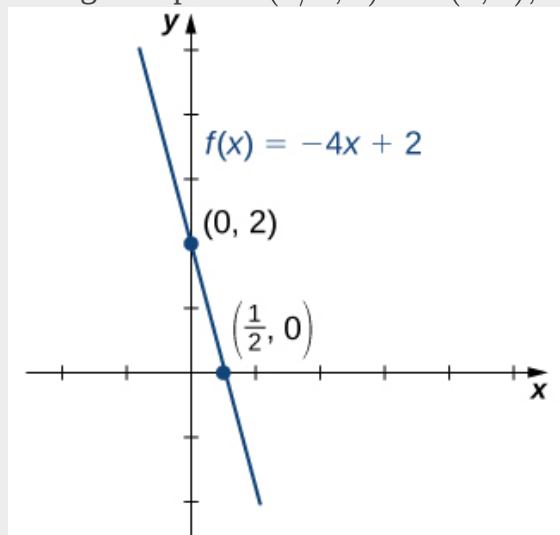
Finding Zeros and y -Intercepts of a Function

Consider the function $f(x) = -4x + 2$.

- Find all zeros of f .
- Find the y -intercept (if any).
- Sketch a graph of f .

Solution:

- a. To find the zeros, solve $f(x) = -4x + 2 = 0$. We discover that f has one zero at $x = 1/2$.
- b. The y -intercept is given by $(0, f(0)) = (0, 2)$.
- c. Given that f is a linear function of the form $f(x) = mx + b$ that passes through the points $(1/2, 0)$ and $(0, 2)$, we can sketch the graph of f ([link](#)).



The function $f(x) = -4x + 2$ is a line with x -intercept $(1/2, 0)$ and y -intercept $(0, 2)$.

Example:

Exercise:

Problem:

Using Zeros and y -Intercepts to Sketch a Graph

Consider the function $f(x) = \sqrt{x+3} + 1$.

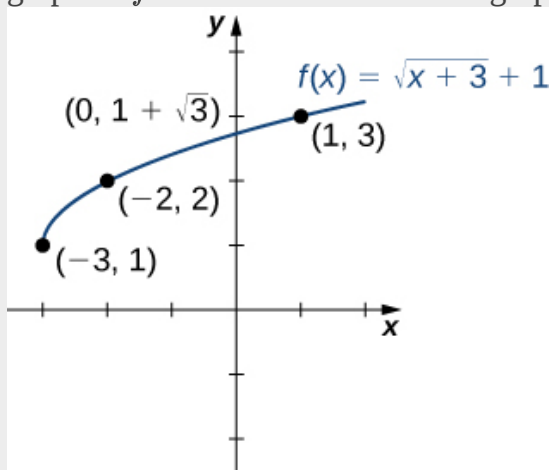
- a. Find all zeros of f .
- b. Find the y -intercept (if any).
- c. Sketch a graph of f .

Solution:

- To find the zeros, solve $\sqrt{x+3} + 1 = 0$. This equation implies $\sqrt{x+3} = -1$. Since $\sqrt{x+3} \geq 0$ for all x , this equation has no solutions, and therefore f has no zeros.
- The y -intercept is given by $(0, f(0)) = (0, \sqrt{3} + 1)$.
- To graph this function, we make a table of values. Since we need $x + 3 \geq 0$, we need to choose values of $x \geq -3$. We choose values that make the square-root function easy to evaluate.

x	-3	-2	1
$f(x)$	1	2	3

Making use of the table and knowing that, since the function is a square root, the graph of f should be similar to the graph of $y = \sqrt{x}$, we sketch the graph ([link](#)).



The graph of $f(x) = \sqrt{x+3} + 1$ has a y -intercept but no x -intercepts.

Note:

Exercise:

Problem: Find the zeros of $f(x) = x^3 - 5x^2 + 6x$.

Solution:

$$x = 0, 2, 3$$

Hint

Factor the polynomial.

Example:

Exercise:

Problem:

Finding the Height of a Free-Falling Object

If a ball is dropped from a height of 100 ft, its height s at time t is given by the function $s(t) = -16t^2 + 100$, where s is measured in feet and t is measured in seconds. The domain is restricted to the interval $[0, c]$, where $t = 0$ is the time when the ball is dropped and $t = c$ is the time when the ball hits the ground.

- Create a table showing the height $s(t)$ when $t = 0, 0.5, 1, 1.5, 2$, and 2.5 .
Using the data from the table, determine the domain for this function. That is, find the time c when the ball hits the ground.
- Sketch a graph of s .

Solution:

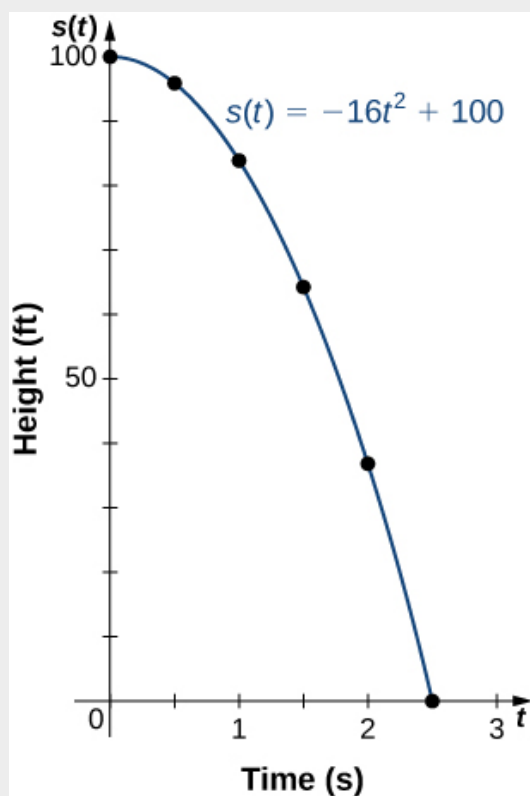
a.

t	0	0.5	1	1.5	2	2.5
$s(t)$	100	96	84	64	36	0

Height s as a Function of Time t

Since the ball hits the ground when $t = 2.5$, the domain of this function is the interval $[0, 2.5]$.

b.



Note that for this function and the function $f(x) = -4x + 2$ graphed in [\[link\]](#), the values of $f(x)$ are getting smaller as x is getting larger. A function with this property is said to be decreasing. On the other hand, for the function $f(x) = \sqrt{x+3} + 1$ graphed in [\[link\]](#), the values of $f(x)$ are getting larger as the values of x are getting larger. A function with this property is said to be increasing. It is important to note, however, that a function can be increasing on some interval or intervals and decreasing over a different interval or intervals. For example, using our temperature function in [\[link\]](#), we can see that the function is decreasing on the interval $(0, 4)$, increasing on the interval $(4, 14)$, and then decreasing on the interval $(14, 23)$. We make the idea of a function increasing or decreasing over a particular interval more precise in the next definition.

Note:
Definition

We say that a function f is **increasing on the interval** I if for all $x_1, x_2 \in I$,

Equation:

$$f(x_1) \leq f(x_2) \text{ when } x_1 < x_2.$$

We say f is strictly increasing on the interval I if for all $x_1, x_2 \in I$,

Equation:

$$f(x_1) < f(x_2) \text{ when } x_1 < x_2.$$

We say that a function f is **decreasing on the interval** I if for all $x_1, x_2 \in I$,

Equation:

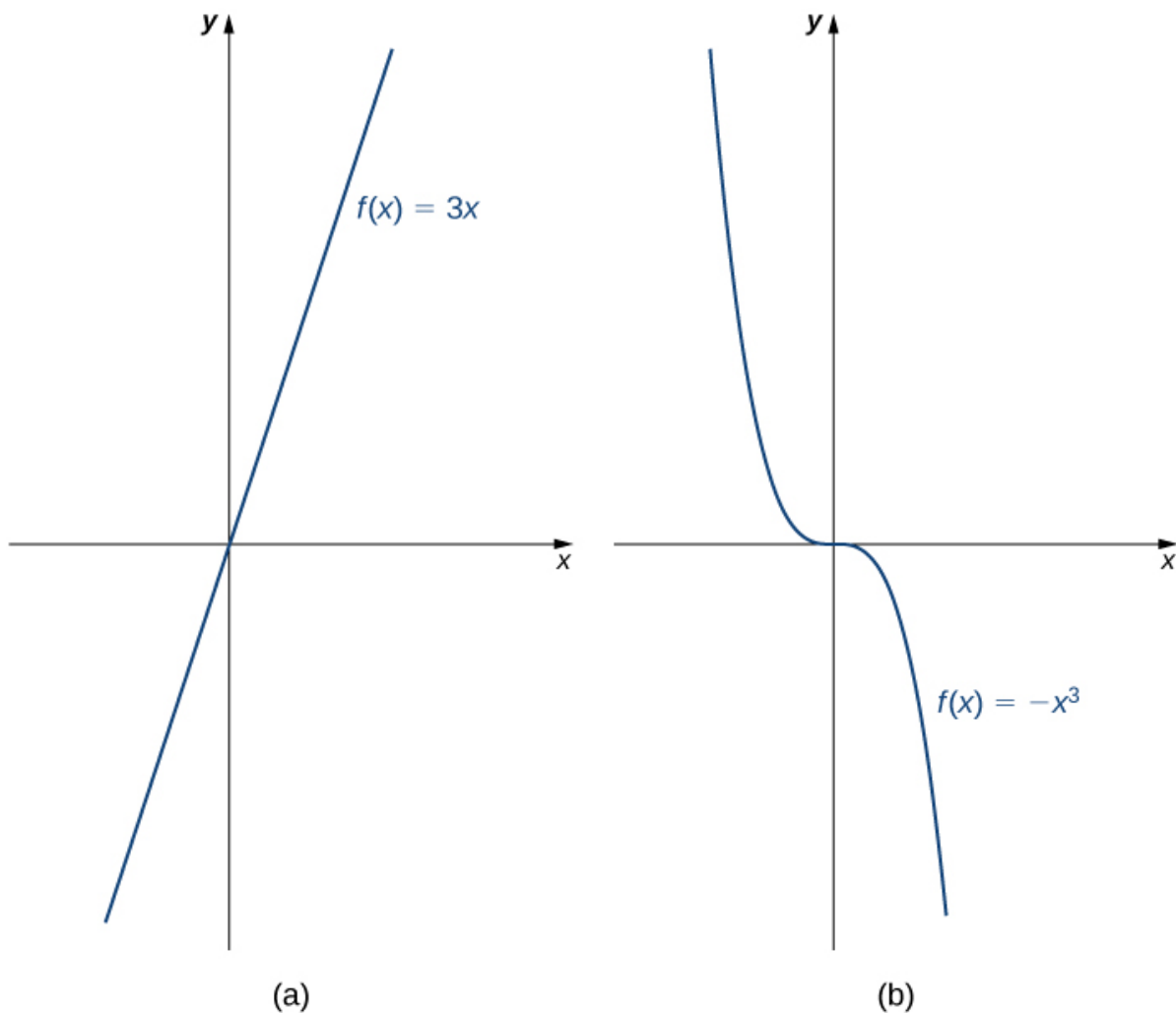
$$f(x_1) \geq f(x_2) \text{ if } x_1 < x_2.$$

We say that a function f is strictly decreasing on the interval I if for all $x_1, x_2 \in I$,

Equation:

$$f(x_1) > f(x_2) \text{ if } x_1 < x_2.$$

For example, the function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$ because $3x_1 < 3x_2$ whenever $x_1 < x_2$. On the other hand, the function $f(x) = -x^3$ is decreasing on the interval $(-\infty, \infty)$ because $-x_1^3 > -x_2^3$ whenever $x_1 < x_2$ ([link](#)).



(a) The function $f(x) = 3x$ is increasing on the interval $(-\infty, \infty)$. (b) The function $f(x) = -x^3$ is decreasing on the interval $(-\infty, \infty)$.

Combining Functions

Now that we have reviewed the basic characteristics of functions, we can see what happens to these properties when we combine functions in different ways, using basic mathematical operations to create new functions. For example, if the cost for a company to manufacture x items is described by the function $C(x)$ and the revenue created by the sale of x items is described by the function $R(x)$, then the profit on the manufacture and sale of x items is defined as $P(x) = R(x) - C(x)$. Using the difference between two functions, we created a new function.

Alternatively, we can create a new function by composing two functions. For example, given the functions $f(x) = x^2$ and $g(x) = 3x + 1$, the composite function $f \circ g$ is defined such that

Equation:

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (3x + 1)^2.$$

The composite function $g \circ f$ is defined such that

Equation:

$$(g \circ f)(x) = g(f(x)) = 3f(x) + 1 = 3x^2 + 1.$$

Note that these two new functions are different from each other.

Combining Functions with Mathematical Operators

To combine functions using mathematical operators, we simply write the functions with the operator and simplify. Given two functions f and g , we can define four new functions:

Equation:

$(f + g)(x) = f(x) + g(x)$	<i>Sum</i>
$(f - g)(x) = f(x) - g(x)$	<i>Difference</i>
$(f \cdot g)(x) = f(x)g(x)$	<i>Product</i>
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ for } g(x) \neq 0$	<i>Quotient</i>

Example:

Exercise:

Problem:

Combining Functions Using Mathematical Operations

Given the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, find each of the following functions and state its domain.

- $(f + g)(x)$
- $(f - g)(x)$

- c. $(f \cdot g)(x)$
 d. $\left(\frac{f}{g}\right)(x)$

Solution:

- a. $(f + g)(x) = (2x - 3) + (x^2 - 1) = x^2 + 2x - 4$. The domain of this function is the interval $(-\infty, \infty)$.
 b. $(f - g)(x) = (2x - 3) - (x^2 - 1) = -x^2 + 2x - 2$. The domain of this function is the interval $(-\infty, \infty)$.
 c. $(f \cdot g)(x) = (2x - 3)(x^2 - 1) = 2x^3 - 3x^2 - 2x + 3$. The domain of this function is the interval $(-\infty, \infty)$.
 d. $\left(\frac{f}{g}\right)(x) = \frac{2x-3}{x^2-1}$. The domain of this function is $\{x|x \neq \pm 1\}$.

Note:

Exercise:

Problem:

For $f(x) = x^2 + 3$ and $g(x) = 2x - 5$, find $(f/g)(x)$ and state its domain.

Solution:

$$\left(\frac{f}{g}\right)(x) = \frac{x^2+3}{2x-5}. \text{ The domain is } \left\{x|x \neq \frac{5}{2}\right\}.$$

Hint

The new function $(f/g)(x)$ is a quotient of two functions. For what values of x is the denominator zero?

Function Composition

When we compose functions, we take a function of a function. For example, suppose the temperature T on a given day is described as a function of time t (measured in hours after midnight) as in [\[link\]](#). Suppose the cost C , to heat or cool a building for 1 hour, can be described as a function of the temperature T . Combining these two functions, we can describe the cost of heating or cooling a building as a function of time by evaluating $C(T(t))$. We have defined a new function, denoted $C \circ T$, which is defined such that

$(C \circ T)(t) = C(T(t))$ for all t in the domain of T . This new function is called a composite function. We note that since cost is a function of temperature and temperature is a function of time, it makes sense to define this new function $(C \circ T)(t)$. It does not make sense to consider $(T \circ C)(t)$, because temperature is not a function of cost.

Note:

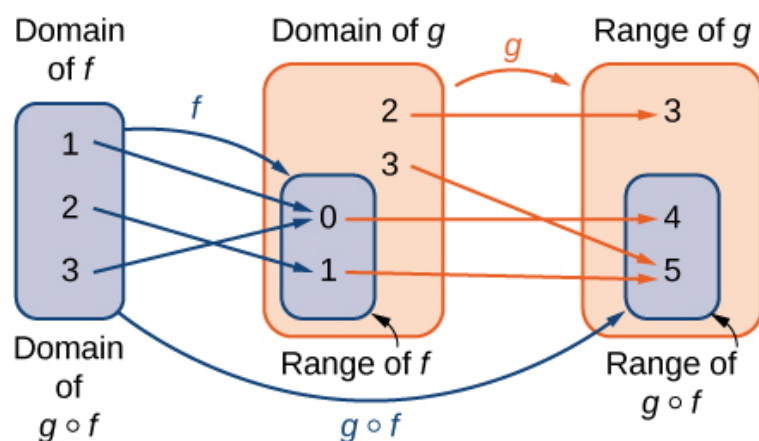
Definition

Consider the function f with domain A and range B , and the function g with domain D and range E . If B is a subset of D , then the **composite function** $(g \circ f)(x)$ is the function with domain A such that

Equation:

$$(g \circ f)(x) = g(f(x)).$$

A composite function $g \circ f$ can be viewed in two steps. First, the function f maps each input x in the domain of f to its output $f(x)$ in the range of f . Second, since the range of f is a subset of the domain of g , the output $f(x)$ is an element in the domain of g , and therefore it is mapped to an output $g(f(x))$ in the range of g . In [\[link\]](#), we see a visual image of a composite function.



For the composite function $g \circ f$, we have
 $(g \circ f)(1) = 4$, $(g \circ f)(2) = 5$, and
 $(g \circ f)(3) = 4$.

Example:

Exercise:

Problem:

Compositions of Functions Defined by Formulas

Consider the functions $f(x) = x^2 + 1$ and $g(x) = 1/x$.

- a. Find $(g \circ f)(x)$ and state its domain and range.
- b. Evaluate $(g \circ f)(4)$, $(g \circ f)(-1/2)$.
- c. Find $(f \circ g)(x)$ and state its domain and range.
- d. Evaluate $(f \circ g)(4)$, $(f \circ g)(-1/2)$.

Solution:

- a. We can find the formula for $(g \circ f)(x)$ in two different ways. We could write

Equation:

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = \frac{1}{x^2 + 1}.$$

Alternatively, we could write

Equation:

$$(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{x^2 + 1}.$$

Since $x^2 + 1 \neq 0$ for all real numbers x , the domain of $(g \circ f)(x)$ is the set of all real numbers. Since $0 < 1/(x^2 + 1) \leq 1$, the range is, at most, the interval $(0, 1]$. To show that the range is this entire interval, we let $y = 1/(x^2 + 1)$ and solve this equation for x to show that for all y in the interval $(0, 1]$, there exists a real number x such that $y = 1/(x^2 + 1)$.

Solving this equation for x , we see that $x^2 + 1 = 1/y$, which implies that

Equation:

$$x = \pm \sqrt{\frac{1}{y} - 1}.$$

If y is in the interval $(0, 1]$, the expression under the radical is nonnegative,

and therefore there exists a real number x such that $1/(x^2 + 1) = y$. We conclude that the range of $g \circ f$ is the interval $(0, 1]$.

b. $(g \circ f)(4) = g(f(4)) = g(4^2 + 1) = g(17) = \frac{1}{17}$

$$(g \circ f)\left(-\frac{1}{2}\right) = g\left(f\left(-\frac{1}{2}\right)\right) = g\left(\left(-\frac{1}{2}\right)^2 + 1\right) = g\left(\frac{5}{4}\right) = \frac{4}{5}$$

c. We can find a formula for $(f \circ g)(x)$ in two ways. First, we could write

Equation:

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^2 + 1.$$

Alternatively, we could write

Equation:

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 + 1 = \left(\frac{1}{x}\right)^2 + 1.$$

The domain of $f \circ g$ is the set of all real numbers x such that $x \neq 0$. To find the range of f , we need to find all values y for which there exists a real number $x \neq 0$ such that

Equation:

$$\left(\frac{1}{x}\right)^2 + 1 = y.$$

Solving this equation for x , we see that we need x to satisfy

Equation:

$$\left(\frac{1}{x}\right)^2 = y - 1,$$

which simplifies to

Equation:

$$\frac{1}{x} = \pm \sqrt{y - 1}.$$

Finally, we obtain

Equation:

$$x = \pm \frac{1}{\sqrt{y-1}}.$$

Since $1/\sqrt{y-1}$ is a real number if and only if $y > 1$, the range of f is the set $\{y|y \geq 1\}$.

d. $(f \circ g)(4) = f(g(4)) = f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^2 + 1 = \frac{17}{16}$
 $(f \circ g)\left(-\frac{1}{2}\right) = f\left(g\left(-\frac{1}{2}\right)\right) = f(-2) = (-2)^2 + 1 = 5$

In [\[link\]](#), we can see that $(f \circ g)(x) \neq (g \circ f)(x)$. This tells us, in general terms, that the order in which we compose functions matters.

Note:

Exercise:

Problem: Let $f(x) = 2 - 5x$. Let $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$.

Solution:

$$(f \circ g)(x) = 2 - 5\sqrt{x}.$$

Example:

Exercise:

Problem:

Composition of Functions Defined by Tables

Consider the functions f and g described by [\[link\]](#) and [\[link\]](#).

--	--	--	--	--	--	--	--	--

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0	4	2	4	-2	0	-2	4

x	-4	-2	0	2	4
$g(x)$	1	0	3	0	5

- Evaluate $(g \circ f)(3)$, $(g \circ f)(0)$.
- State the domain and range of $(g \circ f)(x)$.
- Evaluate $(f \circ f)(3)$, $(f \circ f)(1)$.
- State the domain and range of $(f \circ f)(x)$.

Solution:

- $(g \circ f)(3) = g(f(3)) = g(-2) = 0$
 $(g \circ f)(0) = g(4) = 5$
- The domain of $g \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $g \circ f$ is the set $\{0, 3, 5\}$.
- $(f \circ f)(3) = f(f(3)) = f(-2) = 4$
 $(f \circ f)(1) = f(f(1)) = f(-2) = 4$
- The domain of $f \circ f$ is the set $\{-3, -2, -1, 0, 1, 2, 3, 4\}$. Since the range of f is the set $\{-2, 0, 2, 4\}$, the range of $f \circ f$ is the set $\{0, 4\}$.

Example:

Exercise:

Problem:

Application Involving a Composite Function

A store is advertising a sale of 20 % off all merchandise. Caroline has a coupon that entitles her to an additional 15 % off any item, including sale merchandise. If Caroline decides to purchase an item with an original price of x dollars, how much will she end up paying if she applies her coupon to the sale price? Solve this problem by using a composite function.

Solution:

Since the sale price is 20 % off the original price, if an item is x dollars, its sale price is given by $f(x) = 0.80x$. Since the coupon entitles an individual to 15 % off the price of any item, if an item is y dollars, the price, after applying the coupon, is given by $g(y) = 0.85y$. Therefore, if the price is originally x dollars, its sale price will be $f(x) = 0.80x$ and then its final price after the coupon will be $g(f(x)) = 0.85(0.80x) = 0.68x$.

Note:

Exercise:

Problem:

If items are on sale for 10 % off their original price, and a customer has a coupon for an additional 30 % off, what will be the final price for an item that is originally x dollars, after applying the coupon to the sale price?

Solution:

$$(g \circ f)(x) = 0.63x$$

Hint

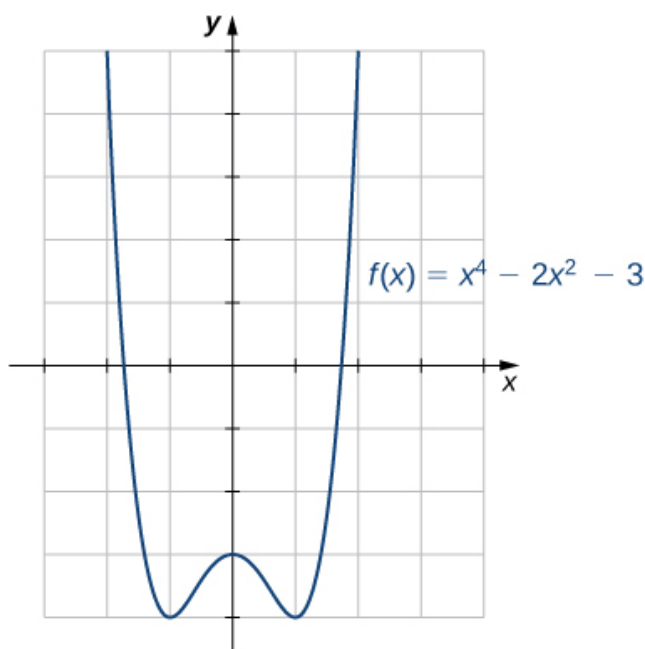
The sale price of an item with an original price of x dollars is $f(x) = 0.90x$. The coupon price for an item that is y dollars is $g(y) = 0.70y$.

Symmetry of Functions

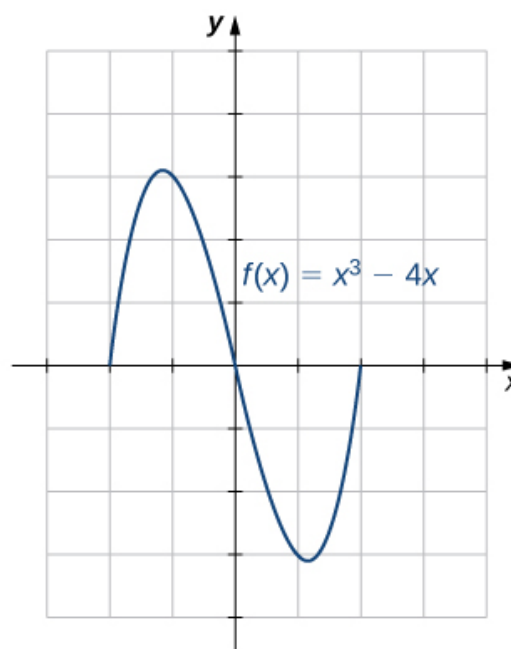
The graphs of certain functions have symmetry properties that help us understand the function and the shape of its graph. For example, consider the function

$f(x) = x^4 - 2x^2 - 3$ shown in [\[link\]](#)(a). If we take the part of the curve that lies to the right of the y -axis and flip it over the y -axis, it lays exactly on top of the curve to the left

of the y -axis. In this case, we say the function has **symmetry about the y -axis**. On the other hand, consider the function $f(x) = x^3 - 4x$ shown in [\[link\]](#)(b). If we take the graph and rotate it 180° about the origin, the new graph will look exactly the same. In this case, we say the function has **symmetry about the origin**.



(a) Symmetry about the y -axis



(b) Symmetry about the origin

(a) A graph that is symmetric about the y -axis. (b) A graph that is symmetric about the origin.

If we are given the graph of a function, it is easy to see whether the graph has one of these symmetry properties. But without a graph, how can we determine algebraically whether a function f has symmetry? Looking at [\[link\]](#) again, we see that since f is symmetric about the y -axis, if the point (x, y) is on the graph, the point $(-x, y)$ is on the graph. In other words, $f(-x) = f(x)$. If a function f has this property, we say f is an even function, which has symmetry about the y -axis. For example, $f(x) = x^2$ is even because

Equation:

$$f(-x) = (-x)^2 = x^2 = f(x).$$

In contrast, looking at [\[link\]](#) again, if a function f is symmetric about the origin, then whenever the point (x, y) is on the graph, the point $(-x, -y)$ is also on the graph. In other words, $f(-x) = -f(x)$. If f has this property, we say f is an **odd function**, which has symmetry about the origin. For example, $f(x) = x^3$ is odd because

Equation:

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Note:

Definition

If $f(x) = f(-x)$ for all x in the domain of f , then f is an **even function**. An even function is symmetric about the y-axis.

If $f(-x) = -f(x)$ for all x in the domain of f , then f is an **odd function**. An odd function is symmetric about the origin.

Example:

Exercise:

Problem:

Even and Odd Functions

Determine whether each of the following functions is even, odd, or neither.

- a. $f(x) = -5x^4 + 7x^2 - 2$
- b. $f(x) = 2x^5 - 4x + 5$
- c. $f(x) = \frac{3x}{x^2+1}$

Solution:

To determine whether a function is even or odd, we evaluate $f(-x)$ and compare it to $f(x)$ and $-f(x)$.

- a. $f(-x) = -5(-x)^4 + 7(-x)^2 - 2 = -5x^4 + 7x^2 - 2 = f(x)$. Therefore, f is even.
- b. $f(-x) = 2(-x)^5 - 4(-x) + 5 = -2x^5 + 4x + 5$. Now, $f(-x) \neq f(x)$. Furthermore, noting that $-f(x) = -2x^5 + 4x - 5$, we see that $f(-x) \neq -f(x)$. Therefore, f is neither even nor odd.

c.

$f(-x) = 3(-x)/((-x)^2 + 1) = -3x/(x^2 + 1) = -[3x/(x^2 + 1)] = -f(x)$.
Therefore, f is odd.

Note:

Exercise:

Problem: Determine whether $f(x) = 4x^3 - 5x$ is even, odd, or neither.

Solution:

$f(x)$ is odd.

Hint

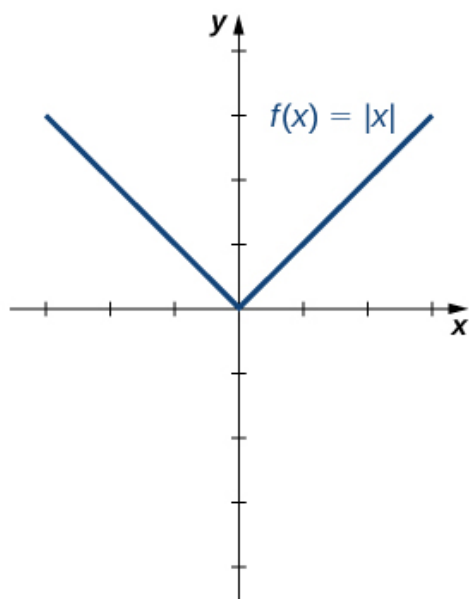
Compare $f(-x)$ with $f(x)$ and $-f(x)$.

One symmetric function that arises frequently is the **absolute value function**, written as $|x|$. The absolute value function is defined as

Equation:

$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}.$$

Some students describe this function by stating that it “makes everything positive.” By the definition of the absolute value function, we see that if $x < 0$, then $|x| = -x > 0$, and if $x > 0$, then $|x| = x > 0$. However, for $x = 0$, $|x| = 0$. Therefore, it is more accurate to say that for all nonzero inputs, the output is positive, but if $x = 0$, the output $|x| = 0$. We conclude that the range of the absolute value function is $\{y|y \geq 0\}$. In [\[link\]](#), we see that the absolute value function is symmetric about the y -axis and is therefore an even function.



The graph of $f(x) = |x|$ is symmetric about the y -axis.

Example:

Exercise:

Problem:

Working with the Absolute Value Function

Find the domain and range of the function $f(x) = 2|x - 3| + 4$.

Solution:

Since the absolute value function is defined for all real numbers, the domain of this function is $(-\infty, \infty)$. Since $|x - 3| \geq 0$ for all x , the function $f(x) = 2|x - 3| + 4 \geq 4$. Therefore, the range is, at most, the set $\{y | y \geq 4\}$. To see that the range is, in fact, this whole set, we need to show that for $y \geq 4$ there exists a real number x such that

Equation:

$$2|x - 3| + 4 = y.$$

A real number x satisfies this equation as long as

Equation:

$$|x - 3| = \frac{1}{2}(y - 4).$$

Since $y \geq 4$, we know $y - 4 \geq 0$, and thus the right-hand side of the equation is nonnegative, so it is possible that there is a solution. Furthermore,

Equation:

$$|x - 3| = \begin{cases} -(x - 3) & \text{if } x < 3 \\ x - 3 & \text{if } x \geq 3 \end{cases}.$$

Therefore, we see there are two solutions:

Equation:

$$x = \pm \frac{1}{2}(y - 4) + 3.$$

The range of this function is $\{y | y \geq 4\}$.

Note:

Exercise:

Problem: For the function $f(x) = |x + 2| - 4$, find the domain and range.

Solution:

Domain = $(-\infty, \infty)$, range = $\{y | y \geq -4\}$.

Hint

$|x + 2| \geq 0$ for all real numbers x .

Key Concepts

- A function is a mapping from a set of inputs to a set of outputs with exactly one output for each input.

- If no domain is stated for a function $y = f(x)$, the domain is considered to be the set of all real numbers x for which the function is defined.
- When sketching the graph of a function f , each vertical line may intersect the graph, at most, once.
- A function may have any number of zeros, but it has, at most, one y -intercept.
- To define the composition $g \circ f$, the range of f must be contained in the domain of g .
- Even functions are symmetric about the y -axis whereas odd functions are symmetric about the origin.

Key Equations

- **Composition of two functions**

$$(g \circ f)(x) = g(f(x))$$

- **Absolute value function**

$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

For the following exercises, (a) determine the domain and the range of each relation, and (b) state whether the relation is a function.

Exercise:

Problem:

x	y	x	y
-3	9	1	1
-2	4	2	4
-1	1	3	9
0	0		

Solution:

a. Domain = $\{-3, -2, -1, 0, 1, 2, 3\}$, range = $\{0, 1, 4, 9\}$ b. Yes, a function

Exercise:

Problem:

x	y	x	y
-3	-2	1	1
-2	-8	2	8
-1	-1	3	-2
0	0		

Exercise:

Problem:

x	y	x	y
1	-3	1	1
2	-2	2	2
3	-1	3	3
0	0		

Solution:

a. Domain = $\{0, 1, 2, 3\}$, range = $\{-3, -2, -1, 0, 1, 2, 3\}$ b. No, not a function

Exercise:

Problem:

x	y	x	y
1	1	5	1
2	1	6	1
3	1	7	1
4	1		

Exercise:

Problem:

x	y	x	y
3	3	15	1
5	2	21	2
8	1	33	3
10	0		

Solution:

a. Domain = $\{3, 5, 8, 10, 15, 21, 33\}$, range = $\{0, 1, 2, 3\}$ b. Yes, a function

Exercise:

Problem:

x	y	x	y
-7	11	1	-2
-2	5	3	4
-2	1	6	11
0	-1		

For the following exercises, find the values for each function, if they exist, then simplify.

a. $f(0)$ b. $f(1)$ c. $f(3)$ d. $f(-x)$ e. $f(a)$ f. $f(a + h)$

Exercise:

Problem: $f(x) = 5x - 2$

Solution:

a. -2 b. 3 c. 13 d. $-5x - 2$ e. $5a - 2$ f. $5a + 5h - 2$

Exercise:

Problem: $f(x) = 4x^2 - 3x + 1$

Exercise:

Problem: $f(x) = \frac{2}{x}$

Solution:

a. Undefined b. 2 c. $\frac{2}{3}$ d. $-\frac{2}{x}$ e. $\frac{2}{a}$ f. $\frac{2}{a+h}$

Exercise:

Problem: $f(x) = |x - 7| + 8$

Exercise:

Problem: $f(x) = \sqrt{6x + 5}$

Solution:

a. $\sqrt{5}$ b. $\sqrt{11}$ c. $\sqrt{23}$ d. $\sqrt{-6x+5}$ e. $\sqrt{6a+5}$ f. $\sqrt{6a+6h+5}$

Exercise:

Problem: $f(x) = \frac{x-2}{3x+7}$

Exercise:

Problem: $f(x) = 9$

Solution:

a. 9 b. 9 c. 9 d. 9 e. 9 f. 9

For the following exercises, find the domain, range, and all zeros/intercepts, if any, of the functions.

Exercise:

Problem: $f(x) = \frac{x}{x^2-16}$

Exercise:

Problem: $g(x) = \sqrt{8x-1}$

Solution:

$x \geq \frac{1}{8}; y \geq 0; x = \frac{1}{8};$ no y -intercept

Exercise:

Problem: $h(x) = \frac{3}{x^2+4}$

Exercise:

Problem: $f(x) = -1 + \sqrt{x+2}$

Solution:

$x \geq -2; y \geq -1; x = -1; y = -1 + \sqrt{2}$

Exercise:

Problem: $f(x) = \frac{1}{\sqrt{x-9}}$

Exercise:

Problem: $g(x) = \frac{3}{x-4}$

Solution:

$x \neq 4; y \neq 0$; no x -intercept; $y = -\frac{3}{4}$

Exercise:

Problem: $f(x) = 4|x+5|$

Exercise:

Problem: $g(x) = \sqrt{\frac{7}{x-5}}$

Solution:

$x > 5; y > 0$; no intercepts

For the following exercises, set up a table to sketch the graph of each function using the following values: $x = -3, -2, -1, 0, 1, 2, 3$.

Exercise:

Problem: $f(x) = x^2 + 1$

x	y	x	y
-3	10	1	2
-2	5	2	5
-1	2	3	10

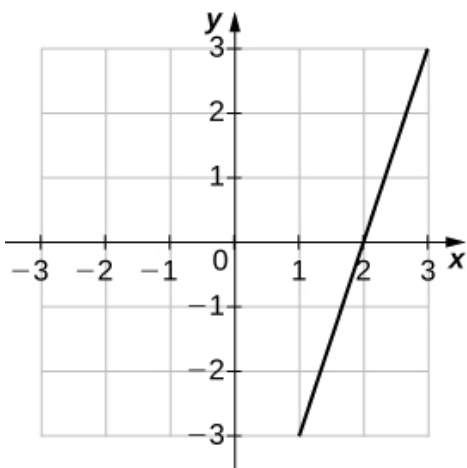
x	y	x	y
0	1		

Exercise:

Problem: $f(x) = 3x - 6$

x	y	x	y
-3	-15	1	-3
-2	-12	2	0
-1	-9	3	3
0	-6		

Solution:



Exercise:

Problem: $f(x) = \frac{1}{2}x + 1$

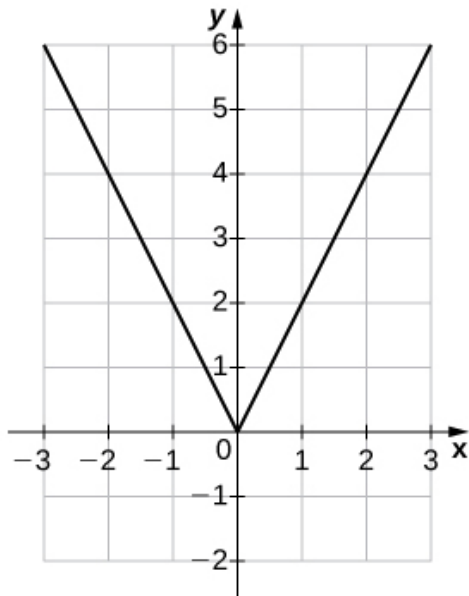
x	y	x	y
-3	$-\frac{1}{2}$	1	$\frac{3}{2}$
-2	0	2	2
-1	$\frac{1}{2}$	3	$\frac{5}{2}$
0	1		

Exercise:

Problem: $f(x) = 2|x|$

x	y	x	y
-3	6	1	2
-2	4	2	4
-1	2	3	6
0	0		

Solution:



Exercise:

Problem: $f(x) = -x^2$

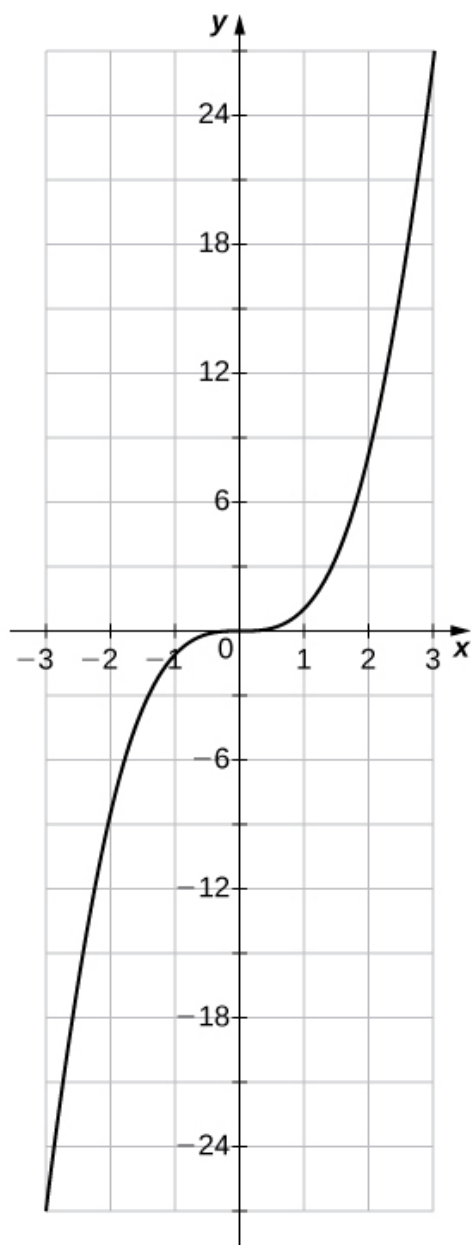
x	y	x	y
-3	-9	1	-1
-2	-4	2	-4
-1	-1	3	-9
0	0		

Exercise:

Problem: $f(x) = x^3$

x	y	x	y
-3	-27	1	1
-2	-8	2	8
-1	-1	3	27
0	0		

Solution:



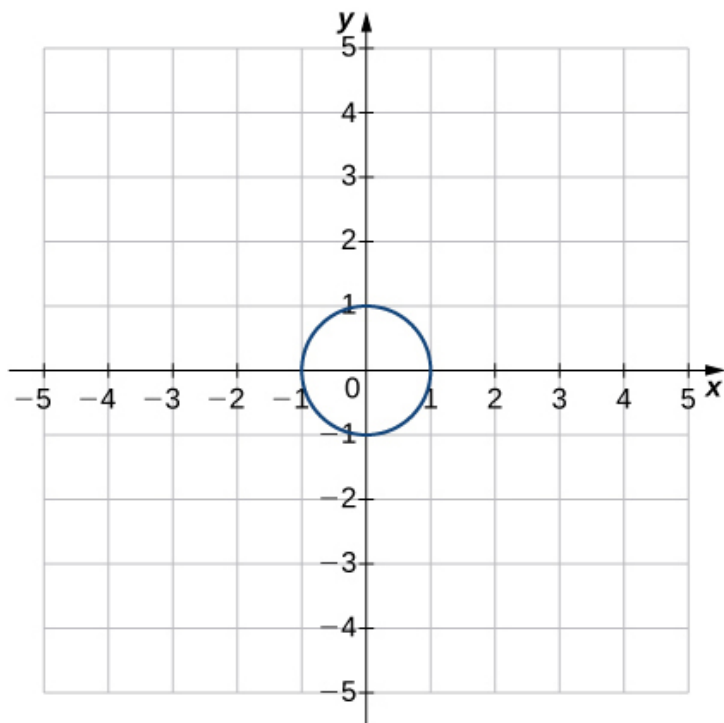
For the following exercises, use the vertical line test to determine whether each of the given graphs represents a function. **Assume that a graph continues at both ends if it extends beyond the given grid.** If the graph represents a function, then determine the following for each graph:

- Domain and range
- x -intercept, if any (estimate where necessary)
- y -Intercept, if any (estimate where necessary)
- The intervals for which the function is increasing
- The intervals for which the function is decreasing

- f. The intervals for which the function is constant
- g. Symmetry about any axis and/or the origin
- h. Whether the function is even, odd, or neither

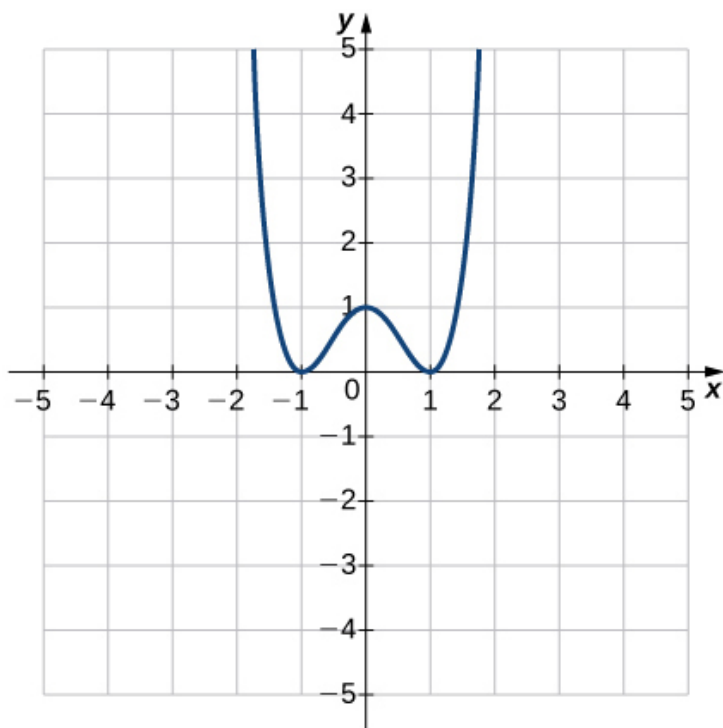
Exercise:

Problem:



Exercise:

Problem:

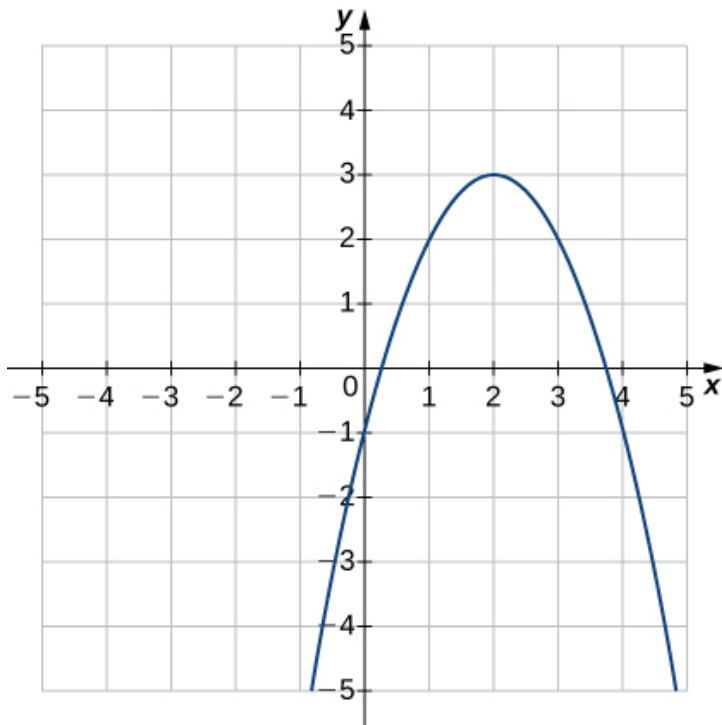


Solution:

Function; a. Domain: all real numbers, range: $y \geq 0$ b. $x = \pm 1$ c. $y = 1$ d. $-1 < x < 0$ and $1 < x < \infty$ e. $-\infty < x < -1$ and $0 < x < 1$ f. Not constant g. y-axis h. Even

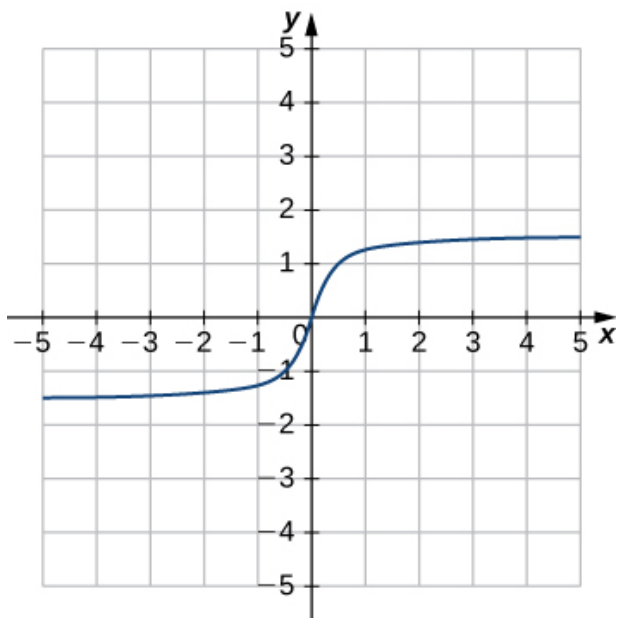
Exercise:

Problem:



Exercise:

Problem:

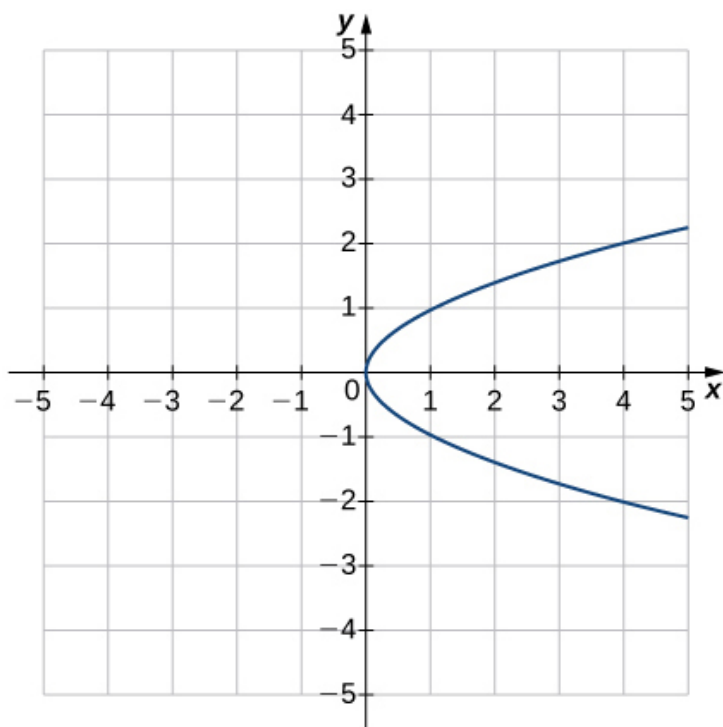


Solution:

Function; a. Domain: all real numbers, range: $-1.5 \leq y \leq 1.5$ b. $x = 0$ c. $y = 0$ d. all real numbers e. None f. Not constant g. Origin h. Odd

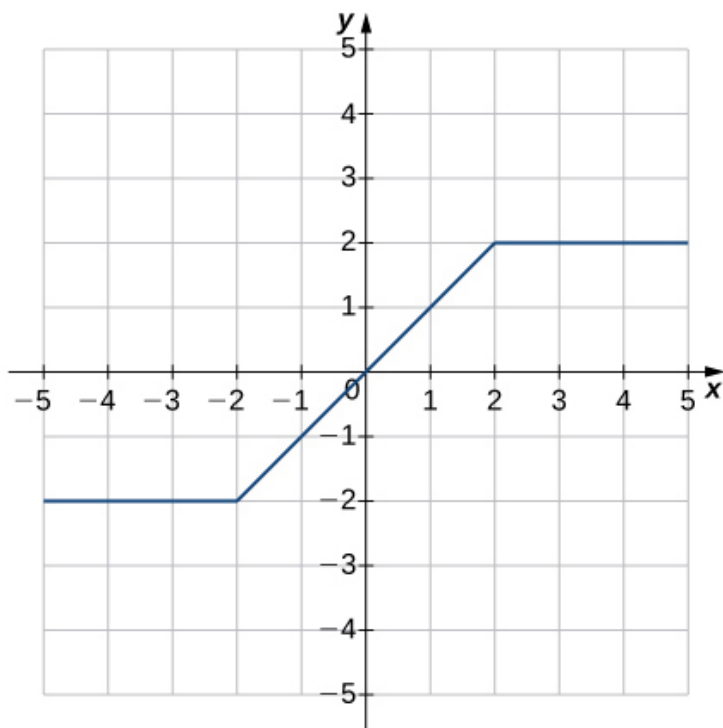
Exercise:

Problem:



Exercise:

Problem:

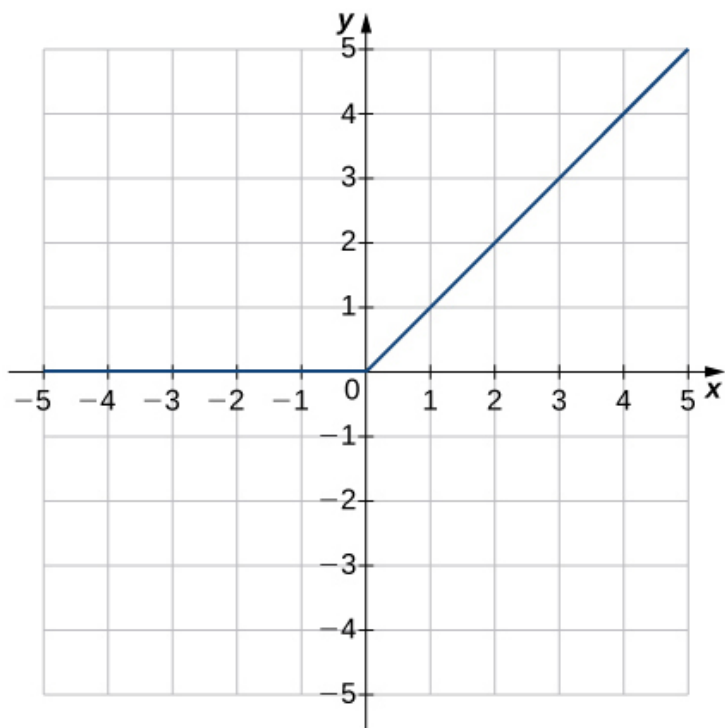


Solution:

Function; a. Domain: $-\infty < x < \infty$, range: $-2 \leq y \leq 2$ b. $x = 0$ c. $y = 0$ d. $-2 < x < 2$ e. Not decreasing f. $-\infty < x < -2$ and $2 < x < \infty$ g. Origin h. Odd

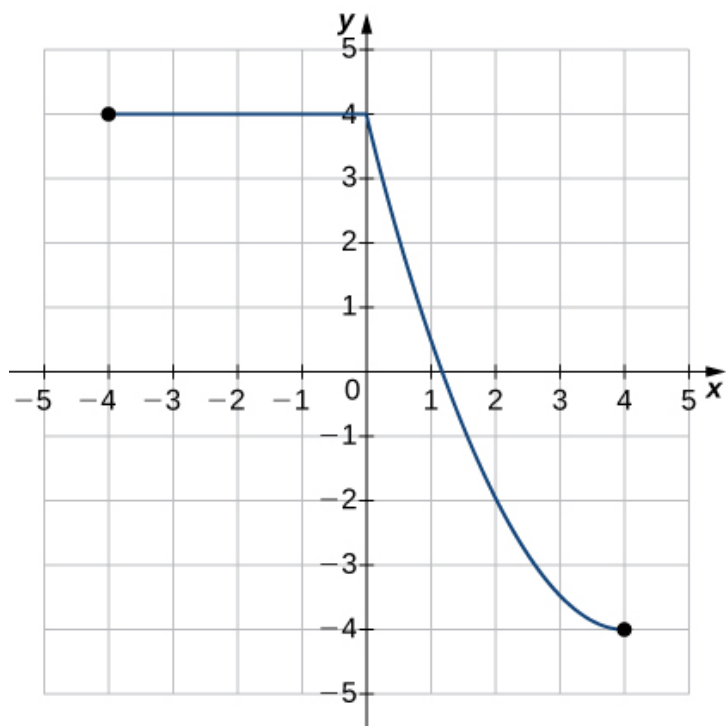
Exercise:

Problem:



Exercise:

Problem:



Solution:

Function; a. Domain: $-4 \leq x \leq 4$, range: $-4 \leq y \leq 4$ b. $x = 1.2$ c. $y = 4$ d. Not increasing e. $0 < x < 4$ f. $-4 < x < 0$ g. No Symmetry h. Neither

For the following exercises, for each pair of functions, find a. $f + g$ b. $f - g$ c. $f \cdot g$ d. f/g . Determine the domain of each of these new functions.

Exercise:

Problem: $f(x) = 3x + 4, g(x) = x - 2$

Exercise:

Problem: $f(x) = x - 8, g(x) = 5x^2$

Solution:

a. $5x^2 + x - 8$; all real numbers b. $-5x^2 + x - 8$; all real numbers c. $5x^3 - 40x^2$; all real numbers d. $\frac{x-8}{5x^2}; x \neq 0$

Exercise:

Problem: $f(x) = 3x^2 + 4x + 1, g(x) = x + 1$

Exercise:

Problem: $f(x) = 9 - x^2, g(x) = x^2 - 2x - 3$

Solution:

a. $-2x + 6$; all real numbers b. $-2x^2 + 2x + 12$; all real numbers c. $-x^4 + 2x^3 + 12x^2 - 18x - 27$; all real numbers d. $-\frac{x+3}{x+1}; x \neq -1, 3$

Exercise:

Problem: $f(x) = \sqrt{x}, g(x) = x - 2$

Exercise:

Problem: $f(x) = 6 + \frac{1}{x}, g(x) = \frac{1}{x}$

Solution:

a. $6 + \frac{2}{x}; x \neq 0$ b. $6; x \neq 0$ c. $\frac{6}{x} + \frac{1}{x^2}; x \neq 0$ d. $6x + 1; x \neq 0$

For the following exercises, for each pair of functions, find a. $(f \circ g)(x)$ and b. $(g \circ f)(x)$ Simplify the results. Find the domain of each of the results.

Exercise:

Problem: $f(x) = 3x, g(x) = x + 5$

Exercise:

Problem: $f(x) = x + 4, g(x) = 4x - 1$

Solution:

a. $4x + 3$; all real numbers b. $4x + 15$; all real numbers

Exercise:

Problem: $f(x) = 2x + 4, g(x) = x^2 - 2$

Exercise:

Problem: $f(x) = x^2 + 7, g(x) = x^2 - 3$

Solution:

a. $x^4 - 6x^2 + 16$; all real numbers b. $x^4 + 14x^2 + 46$; all real numbers

Exercise:

Problem: $f(x) = \sqrt{x}, g(x) = x + 9$

Exercise:

Problem: $f(x) = \frac{3}{2x+1}, g(x) = \frac{2}{x}$

Solution:

a. $\frac{3x}{4+x}; x \neq 0, -4$ b. $\frac{4x+2}{3}; x \neq -\frac{1}{2}$

Exercise:

Problem: $f(x) = |x + 1|, g(x) = x^2 + x - 4$

Exercise:**Problem:**

The table below lists the NBA championship winners for the years 2001 to 2012.

Year	Winner
2001	LA Lakers
2002	LA Lakers
2003	San Antonio Spurs
2004	Detroit Pistons
2005	San Antonio Spurs
2006	Miami Heat
2007	San Antonio Spurs
2008	Boston Celtics
2009	LA Lakers
2010	LA Lakers
2011	Dallas Mavericks
2012	Miami Heat

- Consider the relation in which the domain values are the years 2001 to 2012 and the range is the corresponding winner. Is this relation a function? Explain why or why not.
- Consider the relation where the domain values are the winners and the range is the corresponding years. Is this relation a function? Explain why or why not.

Solution:

a. Yes, because there is only one winner for each year. b. No, because there are three teams that won more than once during the years 2001 to 2012.

Exercise:

Problem: [T] The area A of a square depends on the length of the side s .

- Write a function $A(s)$ for the area of a square.
- Find and interpret $A(6.5)$.
- Find the exact and the two-significant-digit approximation to the length of the sides of a square with area 56 square units.

Exercise:

Problem: [T] The volume of a cube depends on the length of the sides s .

- Write a function $V(s)$ for the area of a square.
- Find and interpret $V(11.8)$.

Solution:

a. $V(s) = s^3$ b. $V(11.8) \approx 1643$; a cube of side length 11.8 each has a volume of approximately 1643 cubic units.

Exercise:**Problem:**

[T] A rental car company rents cars for a flat fee of \$20 and an hourly charge of \$10.25. Therefore, the total cost C to rent a car is a function of the hours t the car is rented plus the flat fee.

- Write the formula for the function that models this situation.
- Find the total cost to rent a car for 2 days and 7 hours.
- Determine how long the car was rented if the bill is \$432.73.

Exercise:

Problem:

[T] A vehicle has a 20-gal tank and gets 15 mpg. The number of miles N that can be driven depends on the amount of gas x in the tank.

- Write a formula that models this situation.
- Determine the number of miles the vehicle can travel on (i) a full tank of gas and (ii) $3/4$ of a tank of gas.
- Determine the domain and range of the function.
- Determine how many times the driver had to stop for gas if she has driven a total of 578 mi.

Solution:

a. $N(x) = 15x$ b. i. $N(20) = 15(20) = 300$; therefore, the vehicle can travel 300 mi on a full tank of gas. ii. $N(15) = 225$; therefore, the vehicle can travel 225 mi on $3/4$ of a tank of gas. c. Domain: $0 \leq x \leq 20$; range: $[0, 300]$ d. The driver had to stop at least once, given that it takes approximately 39 gal of gas to drive a total of 578 mi.

Exercise:**Problem:**

[T] The volume V of a sphere depends on the length of its radius as $V = (4/3)\pi r^3$. Because Earth is not a perfect sphere, we can use the *mean radius* when measuring from the center to its surface. The mean radius is the average distance from the physical center to the surface, based on a large number of samples. Find the volume of Earth with mean radius 6.371×10^6 m.

Exercise:**Problem:**

[T] A certain bacterium grows in culture in a circular region. The radius of the circle, measured in centimeters, is given by $r(t) = 6 - [5/(t^2 + 1)]$, where t is time measured in hours since a circle of a 1-cm radius of the bacterium was put into the culture.

- Express the area of the bacteria as a function of time.
- Find the exact and approximate area of the bacterial culture in 3 hours.
- Express the circumference of the bacteria as a function of time.
- Find the exact and approximate circumference of the bacteria in 3 hours.

Solution:

a. $A(t) = A(r(t)) = \pi \cdot \left(6 - \frac{5}{t^2+1}\right)^2$ b. Exact: $\frac{121\pi}{4}$; approximately 95 cm² c.
 $C(t) = C(r(t)) = 2\pi \left(6 - \frac{5}{t^2+1}\right)$ d. Exact: 11π ; approximately 35 cm

Exercise:**Problem:**

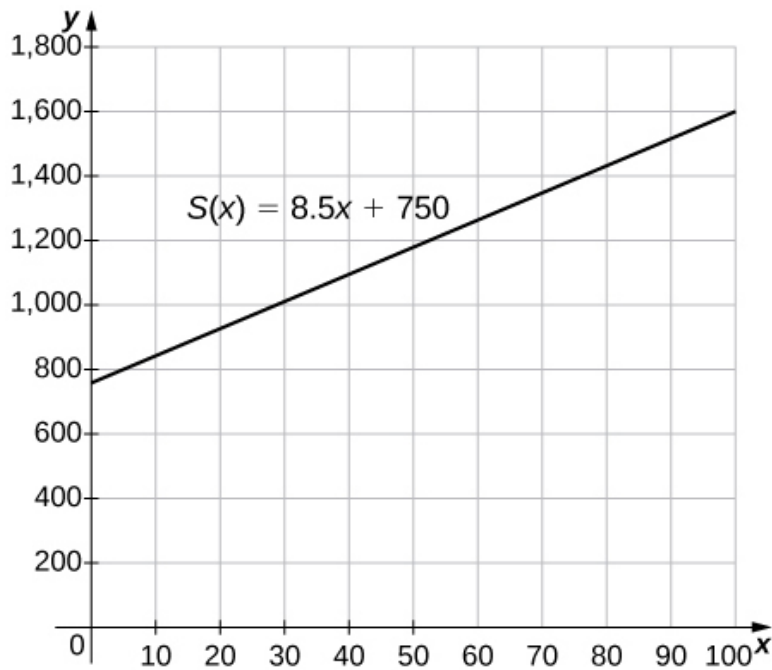
[T] An American tourist visits Paris and must convert U.S. dollars to Euros, which can be done using the function $E(x) = 0.79x$, where x is the number of U.S. dollars and $E(x)$ is the equivalent number of Euros. Since conversion rates fluctuate, when the tourist returns to the United States 2 weeks later, the conversion from Euros to U.S. dollars is $D(x) = 1.245x$, where x is the number of Euros and $D(x)$ is the equivalent number of U.S. dollars.

- Find the composite function that converts directly from U.S. dollars to U.S. dollars via Euros. Did this tourist lose value in the conversion process?
- Use (a) to determine how many U.S. dollars the tourist would get back at the end of her trip if she converted an extra \$200 when she arrived in Paris.

Exercise:**Problem:**

[T] The manager at a skateboard shop pays his workers a monthly salary S of \$750 plus a commission of \$8.50 for each skateboard they sell.

- Write a function $y = S(x)$ that models a worker's monthly salary based on the number of skateboards x he or she sells.
- Find the approximate monthly salary when a worker sells 25, 40, or 55 skateboards.
- Use the INTERSECT feature on a graphing calculator to determine the number of skateboards that must be sold for a worker to earn a monthly income of \$1400. (*Hint:* Find the intersection of the function and the line $y = 1400$.)



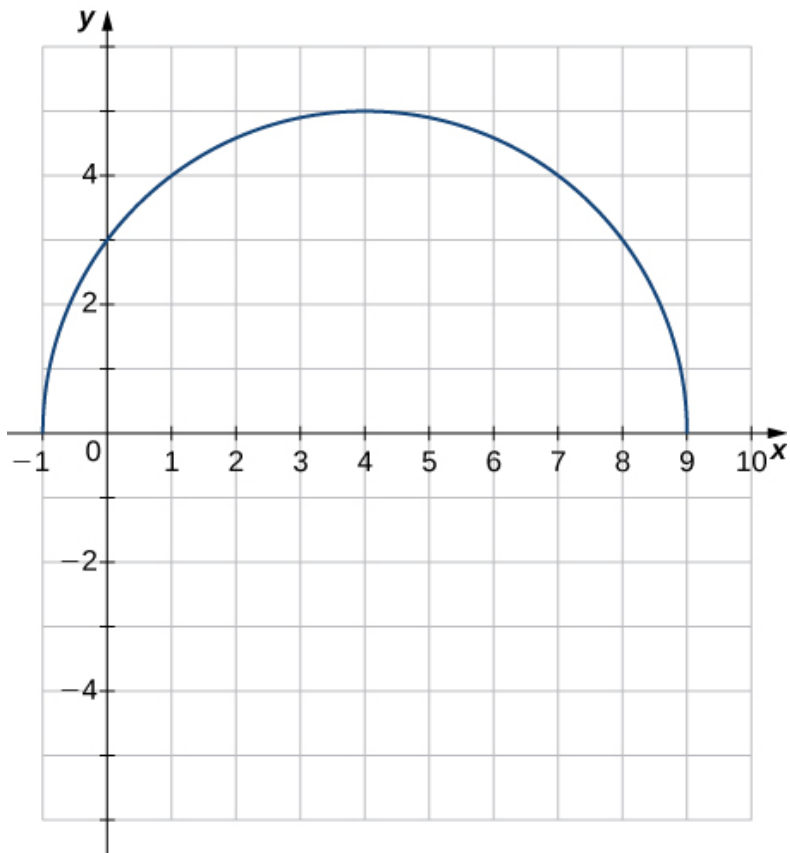
Solution:

a. $S(x) = 8.5x + 750$ b. \$962.50, \$1090, \$1217.50 c. 77 skateboards

Exercise:

Problem:

[T] Use a graphing calculator to graph the half-circle $y = \sqrt{25 - (x - 4)^2}$. Then, use the INTERCEPT feature to find the value of both the x - and y -intercepts.



Glossary

absolute value function

$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

composite function

given two functions f and g , a new function, denoted $g \circ f$, such that

$$(g \circ f)(x) = g(f(x))$$

decreasing on the interval I

a function decreasing on the interval I if, for all $x_1, x_2 \in I$, $f(x_1) \geq f(x_2)$ if $x_1 < x_2$

dependent variable

the output variable for a function

domain

the set of inputs for a function

even function

a function is even if $f(-x) = f(x)$ for all x in the domain of f

function

a set of inputs, a set of outputs, and a rule for mapping each input to exactly one output

graph of a function

the set of points (x, y) such that x is in the domain of f and $y = f(x)$

increasing on the interval I

a function increasing on the interval I if for all $x_1, x_2 \in I$, $f(x_1) \leq f(x_2)$ if $x_1 < x_2$

independent variable

the input variable for a function

odd function

a function is odd if $f(-x) = -f(x)$ for all x in the domain of f

range

the set of outputs for a function

symmetry about the origin

the graph of a function f is symmetric about the origin if $(-x, -y)$ is on the graph of f whenever (x, y) is on the graph

symmetry about the y -axis

the graph of a function f is symmetric about the y -axis if $(-x, y)$ is on the graph of f whenever (x, y) is on the graph

table of values

a table containing a list of inputs and their corresponding outputs

vertical line test

given the graph of a function, every vertical line intersects the graph, at most, once

zeros of a function

when a real number x is a zero of a function f , $f(x) = 0$

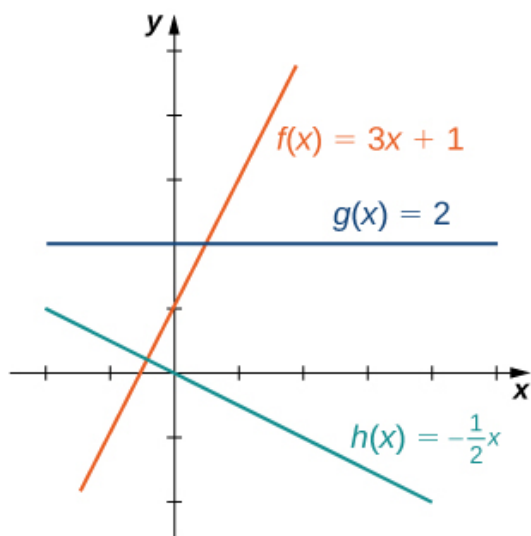
Basic Classes of Functions

- Calculate the slope of a linear function and interpret its meaning.
- Recognize the degree of a polynomial.
- Find the roots of a quadratic polynomial.
- Describe the graphs of basic odd and even polynomial functions.
- Identify a rational function.
- Describe the graphs of power and root functions.
- Explain the difference between algebraic and transcendental functions.
- Graph a piecewise-defined function.
- Sketch the graph of a function that has been shifted, stretched, or reflected from its initial graph position.

We have studied the general characteristics of functions, so now let's examine some specific classes of functions. We begin by reviewing the basic properties of linear and quadratic functions, and then generalize to include higher-degree polynomials. By combining root functions with polynomials, we can define general algebraic functions and distinguish them from the transcendental functions we examine later in this chapter. We finish the section with examples of piecewise-defined functions and take a look at how to sketch the graph of a function that has been shifted, stretched, or reflected from its initial form.

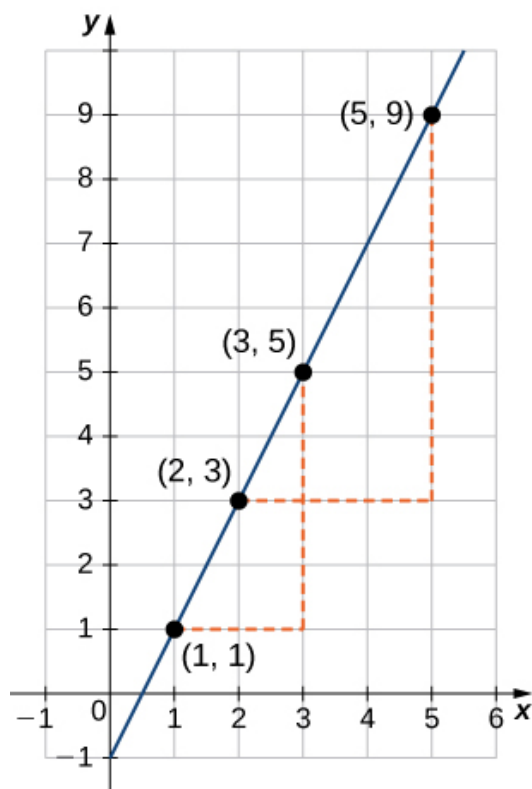
Linear Functions and Slope

The easiest type of function to consider is a **linear function**. Linear functions have the form $f(x) = ax + b$, where a and b are constants. In [\[link\]](#), we see examples of linear functions when a is positive, negative, and zero. Note that if $a > 0$, the graph of the line rises as x increases. In other words, $f(x) = ax + b$ is increasing on $(-\infty, \infty)$. If $a < 0$, the graph of the line falls as x increases. In this case, $f(x) = ax + b$ is decreasing on $(-\infty, \infty)$. If $a = 0$, the line is horizontal.



These linear functions are increasing or decreasing on $(-\infty, \infty)$ and one function is a horizontal line.

As suggested by [\[link\]](#), the graph of any linear function is a line. One of the distinguishing features of a line is its slope. The **slope** is the change in y for each unit change in x . The slope measures both the steepness and the direction of a line. If the slope is positive, the line points upward when moving from left to right. If the slope is negative, the line points downward when moving from left to right. If the slope is zero, the line is horizontal. To calculate the slope of a line, we need to determine the ratio of the change in y versus the change in x . To do so, we choose any two points (x_1, y_1) and (x_2, y_2) on the line and calculate $\frac{y_2 - y_1}{x_2 - x_1}$. In [\[link\]](#), we see this ratio is independent of the points chosen.



$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 3}{5 - 2} = 2$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{3 - 1} = 2$$

For any linear function, the slope $(y_2 - y_1)/(x_2 - x_1)$ is independent of the choice of points (x_1, y_1) and (x_2, y_2) on the line.

Note:

Definition

Consider line L passing through points (x_1, y_1) and (x_2, y_2) . Let $\Delta y = y_2 - y_1$ and $\Delta x = x_2 - x_1$ denote the changes in y and x , respectively. The **slope** of the line is

Equation:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}.$$

We now examine the relationship between slope and the formula for a linear function. Consider the linear function given by the formula $f(x) = ax + b$. As discussed earlier, we know the graph of a linear function is given by a line. We can use our definition of

slope to calculate the slope of this line. As shown, we can determine the slope by calculating $(y_2 - y_1)/(x_2 - x_1)$ for any points (x_1, y_1) and (x_2, y_2) on the line. Evaluating the function f at $x = 0$, we see that $(0, b)$ is a point on this line. Evaluating this function at $x = 1$, we see that $(1, a + b)$ is also a point on this line. Therefore, the slope of this line is

Equation:

$$\frac{(a + b) - b}{1 - 0} = a.$$

We have shown that the coefficient a is the slope of the line. We can conclude that the formula $f(x) = ax + b$ describes a line with slope a . Furthermore, because this line intersects the y -axis at the point $(0, b)$, we see that the y -intercept for this linear function is $(0, b)$. We conclude that the formula $f(x) = ax + b$ tells us the slope, a , and the y -intercept, $(0, b)$, for this line. Since we often use the symbol m to denote the slope of a line, we can write

Equation:

$$f(x) = mx + b$$

to denote the **slope-intercept form** of a linear function.

Sometimes it is convenient to express a linear function in different ways. For example, suppose the graph of a linear function passes through the point (x_1, y_1) and the slope of the line is m . Since any other point $(x, f(x))$ on the graph of f must satisfy the equation

Equation:

$$m = \frac{f(x) - y_1}{x - x_1},$$

this linear function can be expressed by writing

Equation:

$$f(x) - y_1 = m(x - x_1).$$

We call this equation the **point-slope equation** for that linear function.

Since every nonvertical line is the graph of a linear function, the points on a nonvertical line can be described using the slope-intercept or point-slope equations. However, a vertical line does not represent the graph of a function and cannot be expressed in either of these forms. Instead, a vertical line is described by the equation $x = k$ for some

constant k . Since neither the slope-intercept form nor the point-slope form allows for vertical lines, we use the notation

Equation:

$$ax + by = c,$$

where a, b are both not zero, to denote the **standard form of a line**.

Note:

Definition

Consider a line passing through the point (x_1, y_1) with slope m . The equation

Equation:

$$y - y_1 = m(x - x_1)$$

is the **point-slope equation** for that line.

Consider a line with slope m and y -intercept $(0, b)$. The equation

Equation:

$$y = mx + b$$

is an equation for that line in **slope-intercept form**.

The **standard form of a line** is given by the equation

Equation:

$$ax + by = c,$$

where a and b are both not zero. This form is more general because it allows for a vertical line, $x = k$.

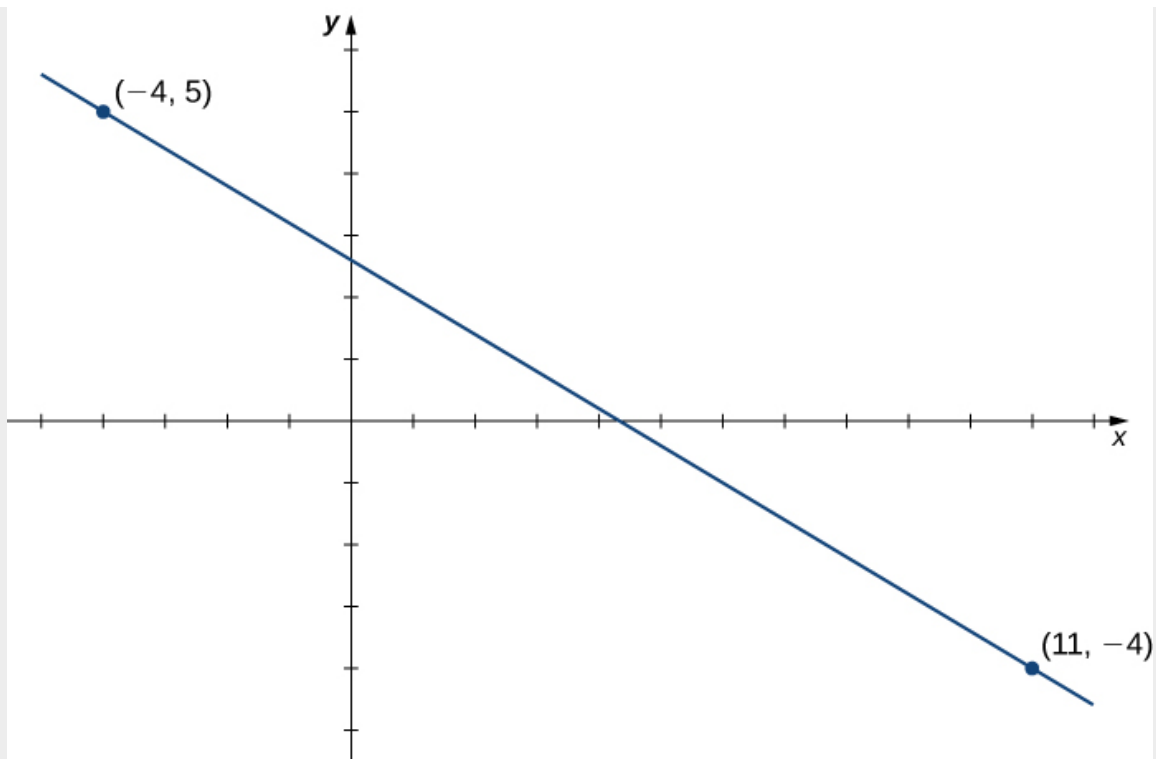
Example:

Exercise:

Problem:

Finding the Slope and Equations of Lines

Consider the line passing through the points $(11, -4)$ and $(-4, 5)$, as shown in [\[link\]](#).



Finding the equation of a linear function with a graph that is a line between two given points.

- Find the slope of the line.
- Find an equation for this linear function in point-slope form.
- Find an equation for this linear function in slope-intercept form.

Solution:

- The slope of the line is

Equation:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-4)}{-4 - 11} = -\frac{9}{15} = -\frac{3}{5}.$$

- To find an equation for the linear function in point-slope form, use the slope $m = -3/5$ and choose any point on the line. If we choose the point $(11, -4)$, we get the equation

Equation:

$$f(x) + 4 = -\frac{3}{5}(x - 11).$$

- c. To find an equation for the linear function in slope-intercept form, solve the equation in part b. for $f(x)$. When we do this, we get the equation

Equation:

$$f(x) = -\frac{3}{5}x + \frac{13}{5}.$$

Note:

Exercise:

Problem:

Consider the line passing through points $(-3, 2)$ and $(1, 4)$. Find the slope of the line.

Find an equation of that line in point-slope form. Find an equation of that line in slope-intercept form.

Solution:

$m = 1/2$. The point-slope form is

$$y - 4 = \frac{1}{2}(x - 1).$$

The slope-intercept form is

$$y = \frac{1}{2}x + \frac{7}{2}.$$

Hint

The slope $m = \Delta y / \Delta x$.

Example:

Exercise:

Problem:

A Linear Distance Function

Jessica leaves her house at 5:50 a.m. and goes for a 9-mile run. She returns to her house at 7:08 a.m. Answer the following questions, assuming Jessica runs at a constant pace.

- Describe the distance D (in miles) Jessica runs as a linear function of her run time t (in minutes).
- Sketch a graph of D .
- Interpret the meaning of the slope.

Solution:

- At time $t = 0$, Jessica is at her house, so $D(0) = 0$. At time $t = 78$ minutes, Jessica has finished running 9 mi, so $D(78) = 9$. The slope of the linear function is

Equation:

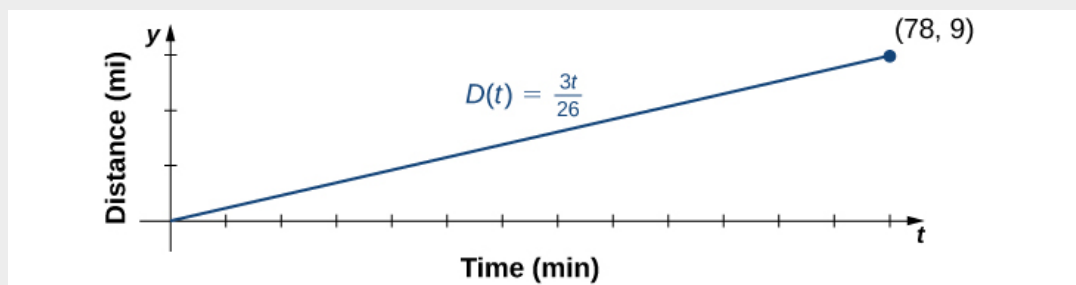
$$m = \frac{9 - 0}{78 - 0} = \frac{3}{26}.$$

The y -intercept is $(0, 0)$, so the equation for this linear function is

Equation:

$$D(t) = \frac{3}{26}t.$$

- To graph D , use the fact that the graph passes through the origin and has slope $m = 3/26$.



- The slope $m = 3/26 \approx 0.115$ describes the distance (in miles) Jessica runs per minute, or her average velocity.

Polynomials

A linear function is a special type of a more general class of functions: polynomials. A **polynomial function** is any function that can be written in the form

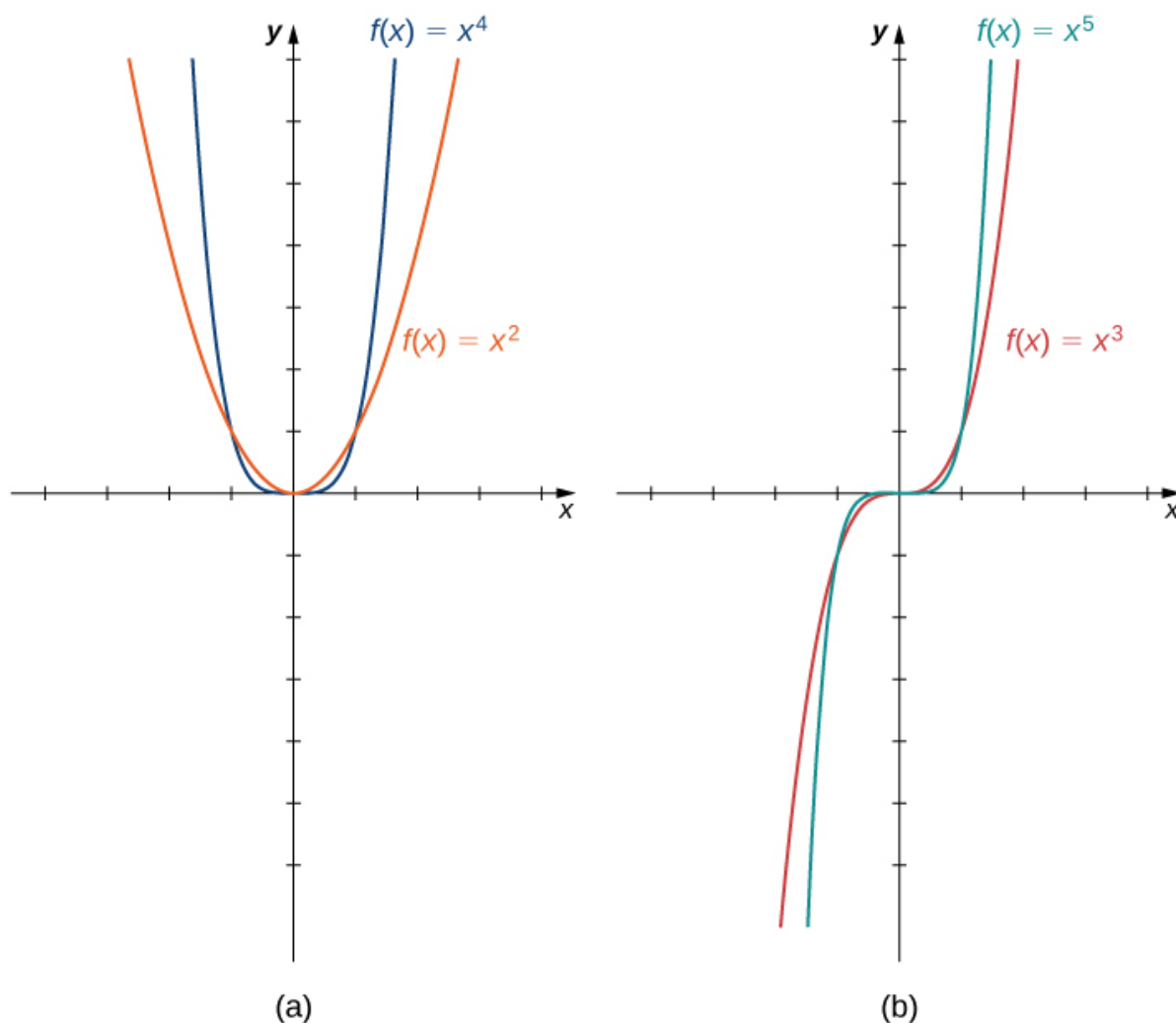
Equation:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

for some integer $n \geq 0$ and constants a_n, a_{n-1}, \dots, a_0 , where $a_n \neq 0$. In the case when $n = 0$, we allow for $a_0 = 0$; if $a_0 = 0$, the function $f(x) = 0$ is called the *zero function*. The value n is called the **degree** of the polynomial; the constant a_n is called the *leading coefficient*. A linear function of the form $f(x) = mx + b$ is a polynomial of degree 1 if $m \neq 0$ and degree 0 if $m = 0$. A polynomial of degree 0 is also called a *constant function*. A polynomial function of degree 2 is called a **quadratic function**. In particular, a quadratic function has the form $f(x) = ax^2 + bx + c$, where $a \neq 0$. A polynomial function of degree 3 is called a **cubic function**.

Power Functions

Some polynomial functions are power functions. A **power function** is any function of the form $f(x) = ax^b$, where a and b are any real numbers. The exponent in a power function can be any real number, but here we consider the case when the exponent is a positive integer. (We consider other cases later.) If the exponent is a positive integer, then $f(x) = ax^n$ is a polynomial. If n is even, then $f(x) = ax^n$ is an even function because $f(-x) = a(-x)^n = ax^n$ if n is even. If n is odd, then $f(x) = ax^n$ is an odd function because $f(-x) = a(-x)^n = -ax^n$ if n is odd ([link](#)).



(a) For any even integer n , $f(x) = ax^n$ is an even function. (b) For any odd integer n , $f(x) = ax^n$ is an odd function.

Behavior at Infinity

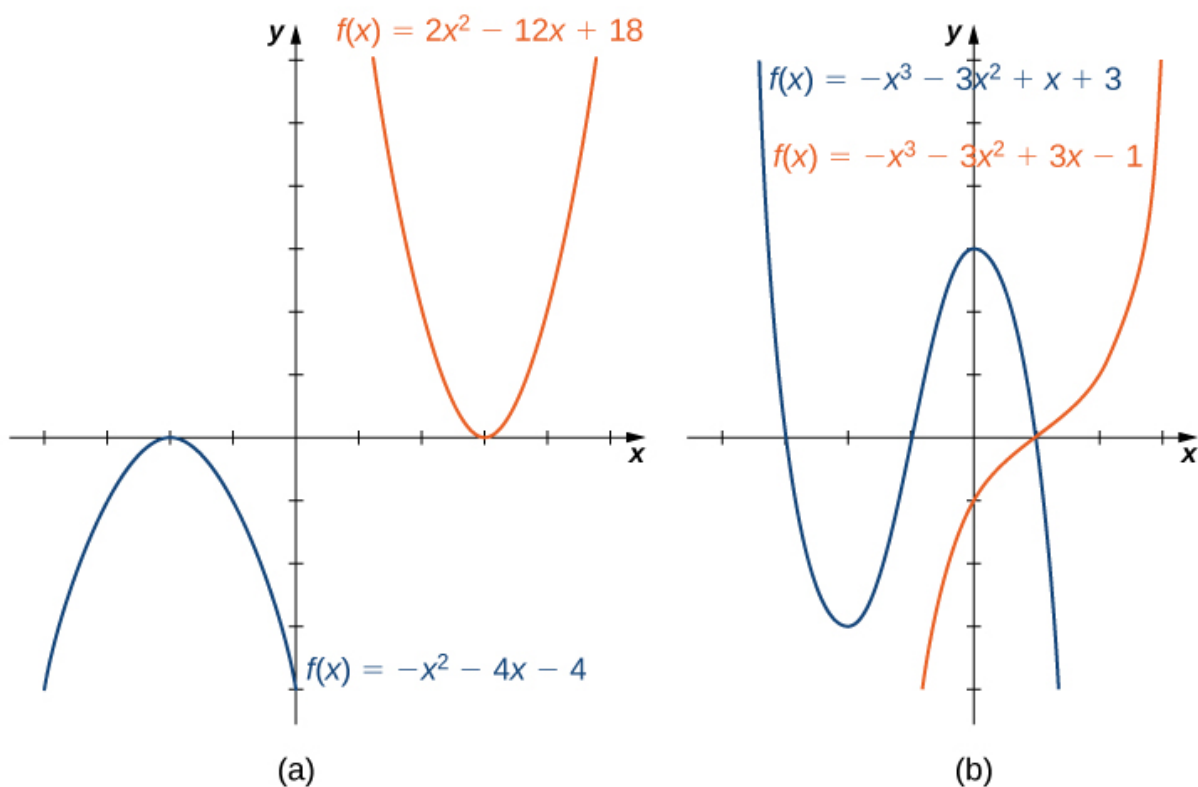
To determine the behavior of a function f as the inputs approach infinity, we look at the values $f(x)$ as the inputs, x , become larger. For some functions, the values of $f(x)$ approach a finite number. For example, for the function $f(x) = 2 + 1/x$, the values $1/x$ become closer and closer to zero for all values of x as they get larger and larger. For this function, we say “ $f(x)$ approaches two as x goes to infinity,” and we write $f(x) \rightarrow 2$ as $x \rightarrow \infty$. The line $y = 2$ is a horizontal asymptote for the function

$f(x) = 2 + 1/x$ because the graph of the function gets closer to the line as x gets larger.

For other functions, the values $f(x)$ may not approach a finite number but instead may become larger for all values of x as they get larger. In that case, we say “ $f(x)$ approaches infinity as x approaches infinity,” and we write $f(x) \rightarrow \infty$ as $x \rightarrow \infty$. For example, for the function $f(x) = 3x^2$, the outputs $f(x)$ become larger as the inputs x get larger. We can conclude that the function $f(x) = 3x^2$ approaches infinity as x approaches infinity, and we write $3x^2 \rightarrow \infty$ as $x \rightarrow \infty$. The behavior as $x \rightarrow -\infty$ and the meaning of $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$ or $x \rightarrow -\infty$ can be defined similarly. We can describe what happens to the values of $f(x)$ as $x \rightarrow \infty$ and as $x \rightarrow -\infty$ as the *end behavior* of the function.

To understand the end behavior for polynomial functions, we can focus on quadratic and cubic functions. The behavior for higher-degree polynomials can be analyzed similarly. Consider a quadratic function $f(x) = ax^2 + bx + c$. If $a > 0$, the values $f(x) \rightarrow \infty$ as $x \rightarrow \pm\infty$. If $a < 0$, the values $f(x) \rightarrow -\infty$ as $x \rightarrow \pm\infty$. Since the graph of a quadratic function is a parabola, the parabola opens upward if $a > 0$; the parabola opens downward if $a < 0$. (See [\[link\]](#)(a).)

Now consider a cubic function $f(x) = ax^3 + bx^2 + cx + d$. If $a > 0$, then $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$. If $a < 0$, then $f(x) \rightarrow -\infty$ as $x \rightarrow \infty$ and $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$. As we can see from both of these graphs, the leading term of the polynomial determines the end behavior. (See [\[link\]](#)(b).)



(a) For a quadratic function, if the leading coefficient $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. (b) For a cubic function f , if the leading coefficient $a > 0$, the values $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ and the values $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$. If the leading coefficient $a < 0$, the opposite is true.

Zeros of Polynomial Functions

Another characteristic of the graph of a polynomial function is where it intersects the x -axis. To determine where a function f intersects the x -axis, we need to solve the equation $f(x) = 0$ for x . In the case of the linear function $f(x) = mx + b$, the x -intercept is given by solving the equation $mx + b = 0$. In this case, we see that the x -intercept is given by $(-b/m, 0)$. In the case of a quadratic function, finding the x -intercept(s) requires finding the zeros of a quadratic equation: $ax^2 + bx + c = 0$. In some cases, it is easy to factor the polynomial $ax^2 + bx + c$ to find the zeros. If not, we make use of the quadratic formula.

Note:

Rule: The Quadratic Formula

Consider the quadratic equation

Equation:

$$ax^2 + bx + c = 0,$$

where $a \neq 0$. The solutions of this equation are given by the quadratic formula

Equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the discriminant $b^2 - 4ac > 0$, this formula tells us there are two real numbers that satisfy the quadratic equation. If $b^2 - 4ac = 0$, this formula tells us there is only one solution, and it is a real number. If $b^2 - 4ac < 0$, no real numbers satisfy the quadratic equation.

In the case of higher-degree polynomials, it may be more complicated to determine where the graph intersects the x -axis. In some instances, it is possible to find the x -intercepts by factoring the polynomial to find its zeros. In other cases, it is impossible to calculate the exact values of the x -intercepts. However, as we see later in the text, in cases such as this, we can use analytical tools to approximate (to a very high degree) where the x -intercepts are located. Here we focus on the graphs of polynomials for which we can calculate their zeros explicitly.

Example:**Exercise:****Problem:****Graphing Polynomial Functions**

For the following functions a. and b., i. describe the behavior of $f(x)$ as $x \rightarrow \pm\infty$, ii. find all zeros of f , and iii. sketch a graph of f .

a. $f(x) = -2x^2 + 4x - 1$

b. $f(x) = x^3 - 3x^2 - 4x$

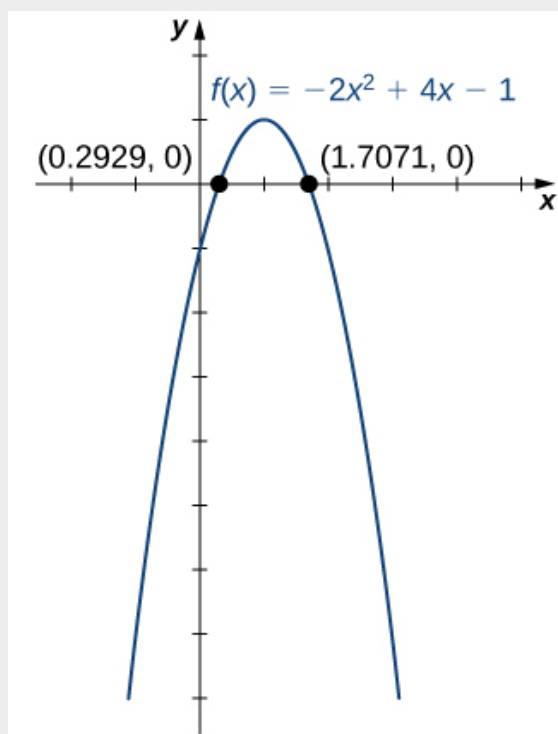
Solution:

a. The function $f(x) = -2x^2 + 4x - 1$ is a quadratic function.

- i. Because $a = -2 < 0$, as $x \rightarrow \pm\infty$, $f(x) \rightarrow -\infty$.
- ii. To find the zeros of f , use the quadratic formula. The zeros are
Equation:

$$x = \frac{-4 \pm \sqrt{4^2 - 4(-2)(-1)}}{2(-2)} = \frac{-4 \pm \sqrt{8}}{-4} = \frac{-4 \pm 2\sqrt{2}}{-4} = \frac{2 \pm \sqrt{2}}{2}.$$

- iii. To sketch the graph of f , use the information from your previous answers and combine it with the fact that the graph is a parabola opening downward.



- b. The function $f(x) = x^3 - 3x^2 - 4x$ is a cubic function.

- i. Because $a = 1 > 0$, as $x \rightarrow \infty$, $f(x) \rightarrow \infty$. As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$.
- ii. To find the zeros of f , we need to factor the polynomial. First, when we factor x out of all the terms, we find

Equation:

$$f(x) = x(x^2 - 3x - 4).$$

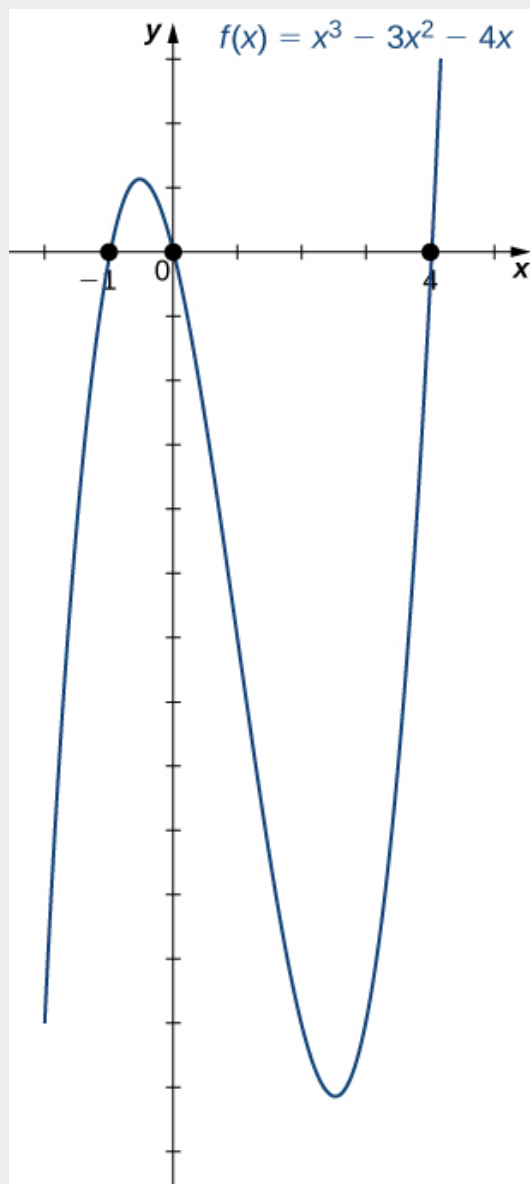
Then, when we factor the quadratic function $x^2 - 3x - 4$, we find

Equation:

$$f(x) = x(x - 4)(x + 1).$$

Therefore, the zeros of f are $x = 0, 4, -1$.

iii. Combining the results from parts i. and ii., draw a rough sketch of f .



Note:

Exercise:

Problem:

Consider the quadratic function $f(x) = 3x^2 - 6x + 2$. Find the zeros of f . Does the parabola open upward or downward?

Solution:

The zeros are $x = 1 \pm \sqrt{3}/3$. The parabola opens upward.

Hint

Use the quadratic formula.

Mathematical Models

A large variety of real-world situations can be described using **mathematical models**. A mathematical model is a method of simulating real-life situations with mathematical equations. Physicists, engineers, economists, and other researchers develop models by combining observation with quantitative data to develop equations, functions, graphs, and other mathematical tools to describe the behavior of various systems accurately. Models are useful because they help predict future outcomes. Examples of mathematical models include the study of population dynamics, investigations of weather patterns, and predictions of product sales.

As an example, let's consider a mathematical model that a company could use to describe its revenue for the sale of a particular item. The amount of revenue R a company receives for the sale of n items sold at a price of p dollars per item is described by the equation $R = p \cdot n$. The company is interested in how the sales change as the price of the item changes. Suppose the data in [\[link\]](#) show the number of units a company sells as a function of the price per item.

p	6	8	10	12	14
n	19.4	18.5	16.2	13.8	12.2

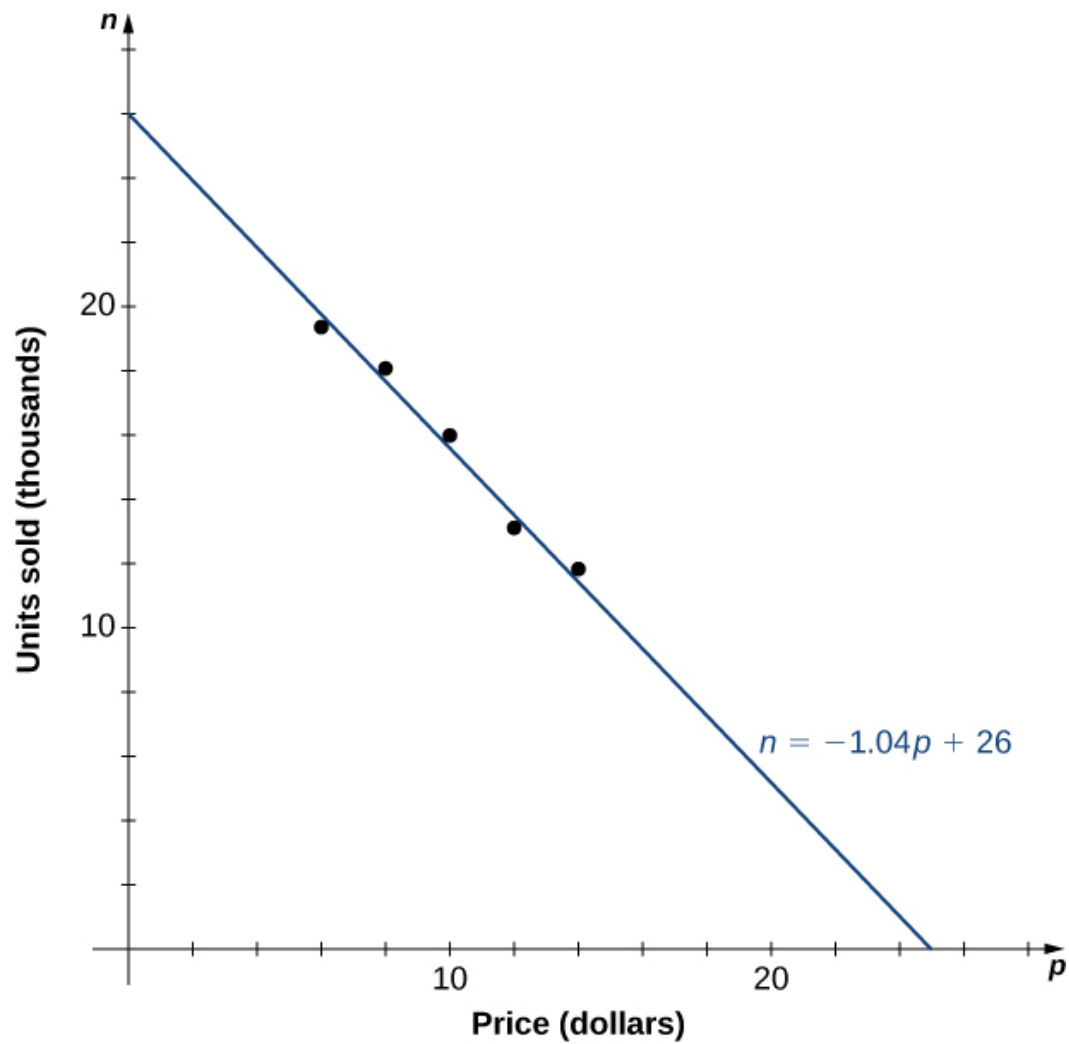
Number of Units Sold n (in Thousands) as a Function of Price per Unit p (in Dollars)

In [\[link\]](#), we see the graph the number of units sold (in thousands) as a function of price (in dollars). We note from the shape of the graph that the number of units sold is likely a linear function of price per item, and the data can be closely approximated by the linear function $n = -1.04p + 26$ for $0 \leq p \leq 25$, where n predicts the number of units sold in thousands. Using this linear function, the revenue (in thousands of dollars) can be estimated by the quadratic function

Equation:

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p$$

for $0 \leq p \leq 25$. In [\[link\]](#), we use this quadratic function to predict the amount of revenue the company receives depending on the price the company charges per item. Note that we cannot conclude definitively the actual number of units sold for values of p , for which no data are collected. However, given the other data values and the graph shown, it seems reasonable that the number of units sold (in thousands) if the price charged is p dollars may be close to the values predicted by the linear function $n = -1.04p + 26$.



The data collected for the number of items sold as a function of price is roughly linear. We use the linear function $n = -1.04p + 26$ to estimate this function.

Example:

Exercise:

Problem:

Maximizing Revenue

A company is interested in predicting the amount of revenue it will receive depending on the price it charges for a particular item. Using the data from [\[link\]](#), the company arrives at the following quadratic function to model revenue R as a function of price per item p :

Equation:

$$R(p) = p \cdot (-1.04p + 26) = -1.04p^2 + 26p$$

for $0 \leq p \leq 25$.

- Predict the revenue if the company sells the item at a price of $p = \$5$ and $p = \$17$.
- Find the zeros of this function and interpret the meaning of the zeros.
- Sketch a graph of R .
- Use the graph to determine the value of p that maximizes revenue. Find the maximum revenue.

Solution:

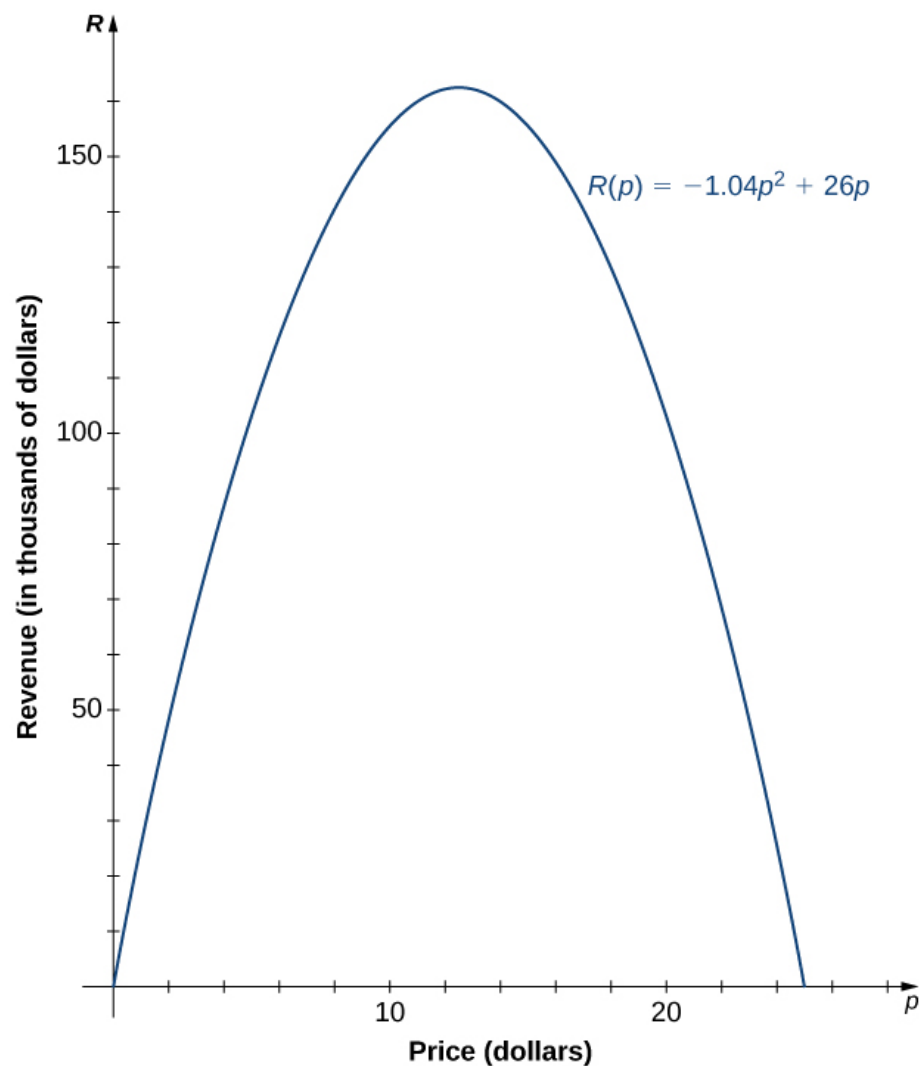
- Evaluating the revenue function at $p = 5$ and $p = 17$, we can conclude that

Equation:

$$R(5) = -1.04(5)^2 + 26(5) = 104, \text{ so revenue} = \$104,000;$$

$$R(17) = -1.04(17)^2 + 26(17) = 141.44, \text{ so revenue} = \$144,440.$$

- The zeros of this function can be found by solving the equation $-1.04p^2 + 26p = 0$. When we factor the quadratic expression, we get $p(-1.04p + 26) = 0$. The solutions to this equation are given by $p = 0, 25$. For these values of p , the revenue is zero. When $p = \$0$, the revenue is zero because the company is giving away its merchandise for free. When $p = \$25$, the revenue is zero because the price is too high, and no one will buy any items.
- Knowing the fact that the function is quadratic, we also know the graph is a parabola. Since the leading coefficient is negative, the parabola opens downward. One property of parabolas is that they are symmetric about the axis, so since the zeros are at $p = 0$ and $p = 25$, the parabola must be symmetric about the line halfway between them, or $p = 12.5$.



- d. The function is a parabola with zeros at $p = 0$ and $p = 25$, and it is symmetric about the line $p = 12.5$, so the maximum revenue occurs at a price of $p = \$12.50$ per item. At that price, the revenue is $R(p) = -1.04(12.5)^2 + 26(12.5) = \$162,500$.

Algebraic Functions

By allowing for quotients and fractional powers in polynomial functions, we create a larger class of functions. An **algebraic function** is one that involves addition, subtraction, multiplication, division, rational powers, and roots. Two types of algebraic functions are rational functions and root functions.

Just as rational numbers are quotients of integers, rational functions are quotients of polynomials. In particular, a **rational function** is any function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials. For example,

Equation:

$$f(x) = \frac{3x - 1}{5x + 2} \quad \text{and} \quad g(x) = \frac{4}{x^2 + 1}$$

are rational functions. A **root function** is a power function of the form $f(x) = x^{1/n}$, where n is a positive integer greater than one. For example, $f(x) = x^{1/2} = \sqrt{x}$ is the square-root function and $g(x) = x^{1/3} = \sqrt[3]{x}$ is the cube-root function. By allowing for compositions of root functions and rational functions, we can create other algebraic functions. For example, $f(x) = \sqrt{4 - x^2}$ is an algebraic function.

Example:

Exercise:

Problem:

Finding Domain and Range for Algebraic Functions

For each of the following functions, find the domain and range.

- a. $f(x) = \frac{3x-1}{5x+2}$
- b. $f(x) = \sqrt{4 - x^2}$

Solution:

- a. It is not possible to divide by zero, so the domain is the set of real numbers x such that $x \neq -2/5$. To find the range, we need to find the values y for which there exists a real number x such that

Equation:

$$y = \frac{3x - 1}{5x + 2}.$$

When we multiply both sides of this equation by $5x + 2$, we see that x must satisfy the equation

Equation:

$$5xy + 2y = 3x - 1.$$

From this equation, we can see that x must satisfy

Equation:

$$2y + 1 = x(3 - 5y).$$

If $y = 3/5$, this equation has no solution. On the other hand, as long as $y \neq 3/5$,

Equation:

$$x = \frac{2y + 1}{3 - 5y}$$

satisfies this equation. We can conclude that the range of f is $\{y | y \neq 3/5\}$.

- b. To find the domain of f , we need $4 - x^2 \geq 0$. When we factor, we write $4 - x^2 = (2 - x)(2 + x) \geq 0$. This inequality holds if and only if both terms are positive or both terms are negative. For both terms to be positive, we need to find x such that

Equation:

$$2 - x \geq 0 \quad \text{and} \quad 2 + x \geq 0.$$

These two inequalities reduce to $2 \geq x$ and $x \geq -2$. Therefore, the set $\{x | -2 \leq x \leq 2\}$ must be part of the domain. For both terms to be negative, we need

Equation:

$$2 - x \leq 0 \quad \text{and} \quad 2 + x \leq 0.$$

These two inequalities also reduce to $2 \leq x$ and $x \leq -2$. There are no values of x that satisfy both of these inequalities. Thus, we can conclude the domain of this function is $\{x | -2 \leq x \leq 2\}$.

If $-2 \leq x \leq 2$, then $0 \leq 4 - x^2 \leq 4$. Therefore, $0 \leq \sqrt{4 - x^2} \leq 2$, and the range of f is $\{y | 0 \leq y \leq 2\}$.

Note:

Exercise:

Problem: Find the domain and range for the function $f(x) = (5x + 2)/(2x - 1)$.

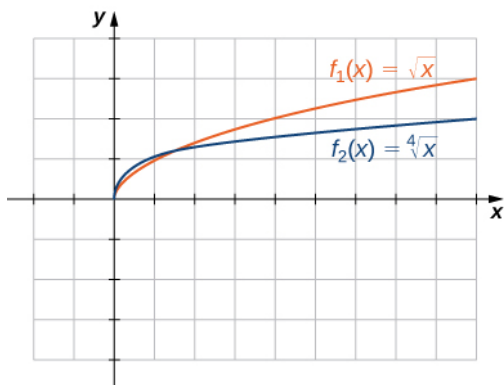
Solution:

The domain is the set of real numbers x such that $x \neq 1/2$. The range is the set $\{y | y \neq 5/2\}$.

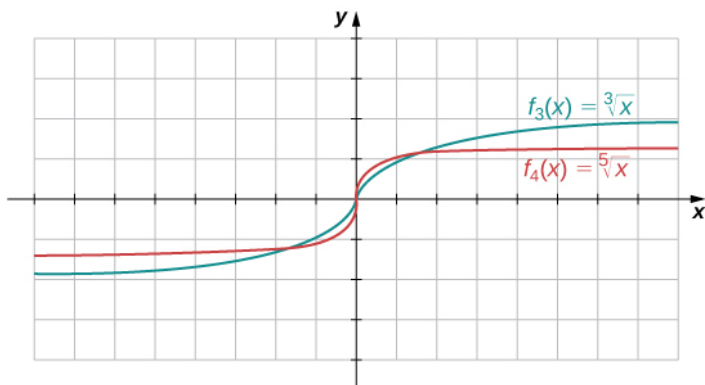
Hint

The denominator cannot be zero. Solve the equation $y = (5x + 2)/(2x - 1)$ for x to find the range.

The root functions $f(x) = x^{1/n}$ have defining characteristics depending on whether n is odd or even. For all even integers $n \geq 2$, the domain of $f(x) = x^{1/n}$ is the interval $[0, \infty)$. For all odd integers $n \geq 1$, the domain of $f(x) = x^{1/n}$ is the set of all real numbers. Since $x^{1/n} = (-x)^{1/n}$ for odd integers n , $f(x) = x^{1/n}$ is an odd function if n is odd. See the graphs of root functions for different values of n in [\[link\]](#).



(a)



(b)

(a) If n is even, the domain of $f(x) = \sqrt[n]{x}$ is $[0, \infty)$. (b) If n is odd, the domain of $f(x) = \sqrt[n]{x}$ is $(-\infty, \infty)$ and the function $f(x) = \sqrt[n]{x}$ is an odd function.

Example:

Exercise:

Problem:

Finding Domains for Algebraic Functions

For each of the following functions, determine the domain of the function.

- a. $f(x) = \frac{3}{x^2-1}$
- b. $f(x) = \frac{2x+5}{3x^2+4}$
- c. $f(x) = \sqrt{4-3x}$
- d. $f(x) = \sqrt[3]{2x-1}$

Solution:

- a. You cannot divide by zero, so the domain is the set of values x such that $x^2 - 1 \neq 0$. Therefore, the domain is $\{x|x \neq \pm 1\}$.
- b. You need to determine the values of x for which the denominator is zero. Since $3x^2 + 4 \geq 4$ for all real numbers x , the denominator is never zero. Therefore, the domain is $(-\infty, \infty)$.
- c. Since the square root of a negative number is not a real number, the domain is the set of values x for which $4 - 3x \geq 0$. Therefore, the domain is $\{x|x \leq 4/3\}$.
- d. The cube root is defined for all real numbers, so the domain is the interval $(-\infty, \infty)$.

Note:

Exercise:

Problem:

Find the domain for each of the following functions: $f(x) = (5 - 2x)/(x^2 + 2)$ and $g(x) = \sqrt{5x - 1}$.

Solution:

The domain of f is $(-\infty, \infty)$. The domain of g is $\{x|x \geq 1/5\}$.

Hint

Determine the values of x when the expression in the denominator of f is nonzero, and find the values of x when the expression inside the radical of g is nonnegative.

Transcendental Functions

Thus far, we have discussed algebraic functions. Some functions, however, cannot be described by basic algebraic operations. These functions are known as **transcendental functions** because they are said to “transcend,” or go beyond, algebra. The most common transcendental functions are trigonometric, exponential, and logarithmic functions. A *trigonometric function* relates the ratios of two sides of a right triangle. They are $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$. (We discuss trigonometric functions later in the chapter.) An exponential function is a function of the form $f(x) = b^x$, where the base $b > 0$, $b \neq 1$. A **logarithmic function** is a function of the form $f(x) = \log_b(x)$ for some constant $b > 0$, $b \neq 1$, where $\log_b(x) = y$ if and only if $b^y = x$. (We also discuss exponential and logarithmic functions later in the chapter.)

Example:

Exercise:

Problem:

Classifying Algebraic and Transcendental Functions

Classify each of the following functions, a. through c., as algebraic or transcendental.

- a. $f(x) = \frac{\sqrt{x^3+1}}{4x+2}$
- b. $f(x) = 2^{x^2}$
- c. $f(x) = \sin(2x)$

Solution:

- a. Since this function involves basic algebraic operations only, it is an algebraic function.
- b. This function cannot be written as a formula that involves only basic algebraic operations, so it is transcendental. (Note that algebraic functions can only have powers that are rational numbers.)
- c. As in part b., this function cannot be written using a formula involving basic algebraic operations only; therefore, this function is transcendental.

Note:

Exercise:

Problem: Is $f(x) = x/2$ an algebraic or a transcendental function?

Solution:

Algebraic

Piecewise-Defined Functions

Sometimes a function is defined by different formulas on different parts of its domain. A function with this property is known as a **piecewise-defined function**. The absolute value function is an example of a piecewise-defined function because the formula changes with the sign of x :

Equation:

$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}.$$

Other piecewise-defined functions may be represented by completely different formulas, depending on the part of the domain in which a point falls. To graph a piecewise-defined function, we graph each part of the function in its respective domain, on the same coordinate system. If the formula for a function is different for $x < a$ and $x > a$, we need to pay special attention to what happens at $x = a$ when we graph the function. Sometimes the graph needs to include an open or closed circle to indicate the value of the function at $x = a$. We examine this in the next example.

Example:

Exercise:

Problem:

Graphing a Piecewise-Defined Function

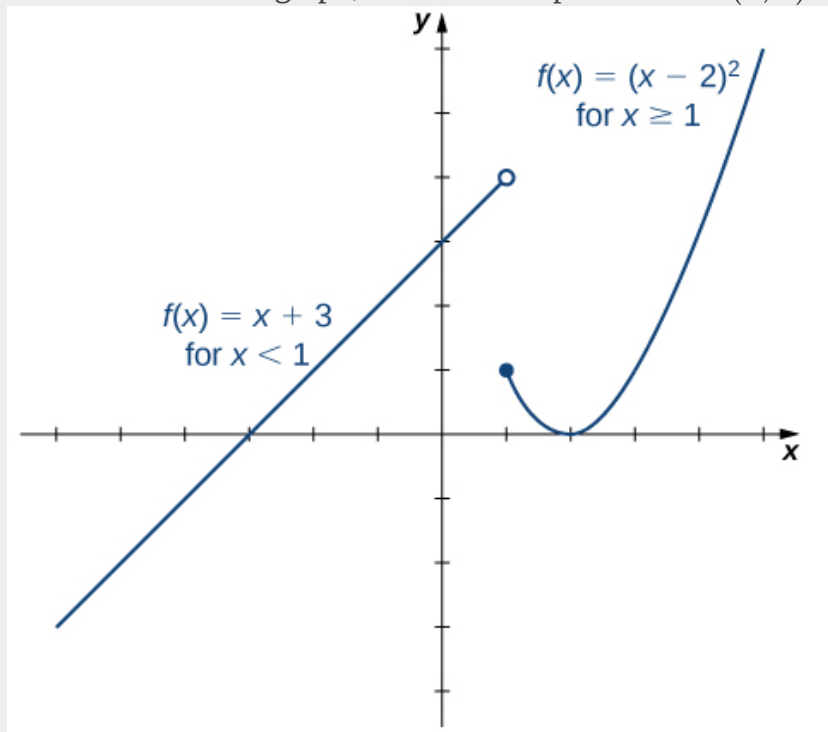
Sketch a graph of the following piecewise-defined function:

Equation:

$$f(x) = \begin{cases} x + 3, & x < 1 \\ (x - 2)^2, & x \geq 1 \end{cases}.$$

Solution:

Graph the linear function $y = x + 3$ on the interval $(-\infty, 1)$ and graph the quadratic function $y = (x - 2)^2$ on the interval $[1, \infty)$. Since the value of the function at $x = 1$ is given by the formula $f(x) = (x - 2)^2$, we see that $f(1) = 1$. To indicate this on the graph, we draw a closed circle at the point $(1, 1)$. The value of the function is given by $f(x) = x + 2$ for all $x < 1$, but not at $x = 1$. To indicate this on the graph, we draw an open circle at $(1, 4)$.



This piecewise-defined function is linear for $x < 1$ and quadratic for $x \geq 1$.

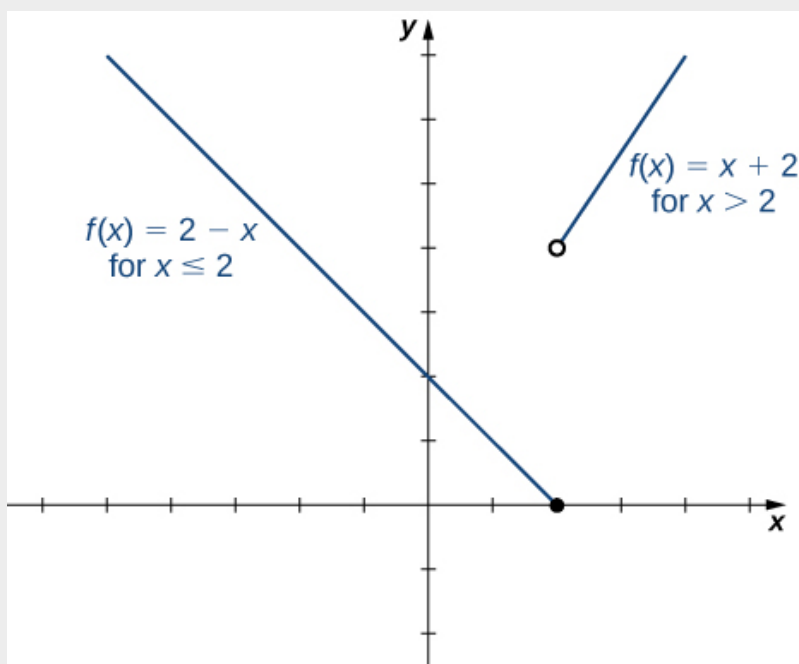
Note:**Exercise:**

Problem: Sketch a graph of the function

Equation:

$$f(x) = \begin{cases} 2 - x, & x \leq 2 \\ x + 2, & x > 2 \end{cases}$$

Solution:



Hint

Graph one linear function for $x \leq 2$ and then graph a different linear function for $x > 2$.

Example:

Exercise:

Problem:

Parking Fees Described by a Piecewise-Defined Function

In a big city, drivers are charged variable rates for parking in a parking garage. They are charged \$10 for the first hour or any part of the first hour and an additional \$2 for each hour or part thereof up to a maximum of \$30 for the day. The parking garage is open from 6 a.m. to 12 midnight.

- Write a piecewise-defined function that describes the cost C to park in the parking garage as a function of hours parked x .
- Sketch a graph of this function $C(x)$.

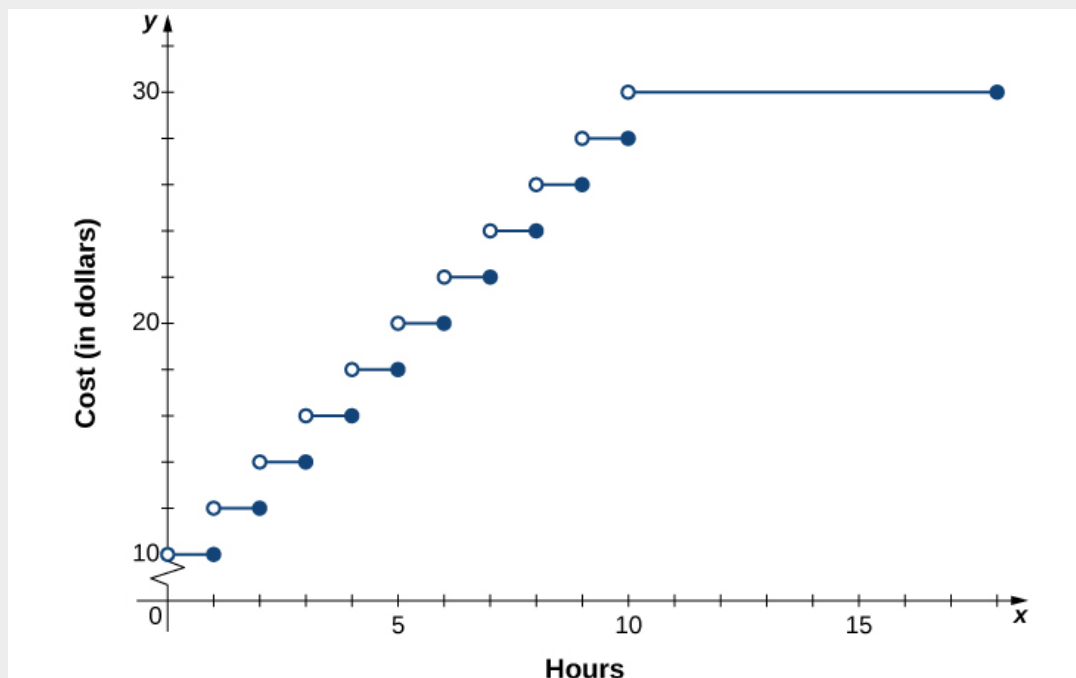
Solution:

- Since the parking garage is open 18 hours each day, the domain for this function is $\{x | 0 < x \leq 18\}$. The cost to park a car at this parking garage can be described piecewise by the function

Equation:

$$C(x) = \begin{cases} 10, & 0 < x \leq 1 \\ 12, & 1 < x \leq 2 \\ 14, & 2 < x \leq 3 \\ 16, & 3 < x \leq 4 \\ \vdots & \\ 30, & 10 < x \leq 18 \end{cases}$$

- The graph of the function consists of several horizontal line segments.



Note:**Exercise:****Problem:**

The cost of mailing a letter is a function of the weight of the letter. Suppose the cost of mailing a letter is 49¢ for the first ounce and 21¢ for each additional ounce. Write a piecewise-defined function describing the cost C as a function of the weight x for $0 < x \leq 3$, where C is measured in cents and x is measured in ounces.

Solution:

$$C(x) = \begin{cases} 49, & 0 < x \leq 1 \\ 70, & 1 < x \leq 2 \\ 91, & 2 < x \leq 3 \end{cases}$$

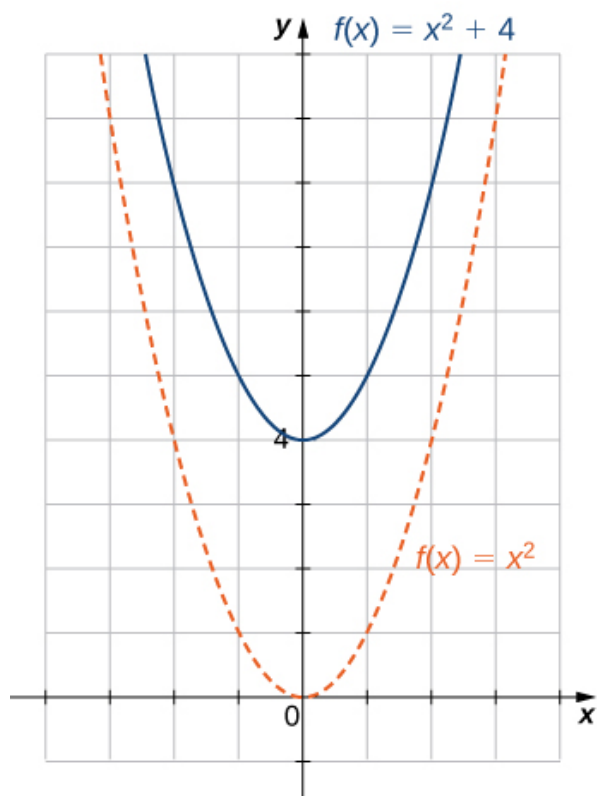
Hint

The piecewise-defined function is constant on the intervals $(0, 1]$, $(1, 2]$, \dots

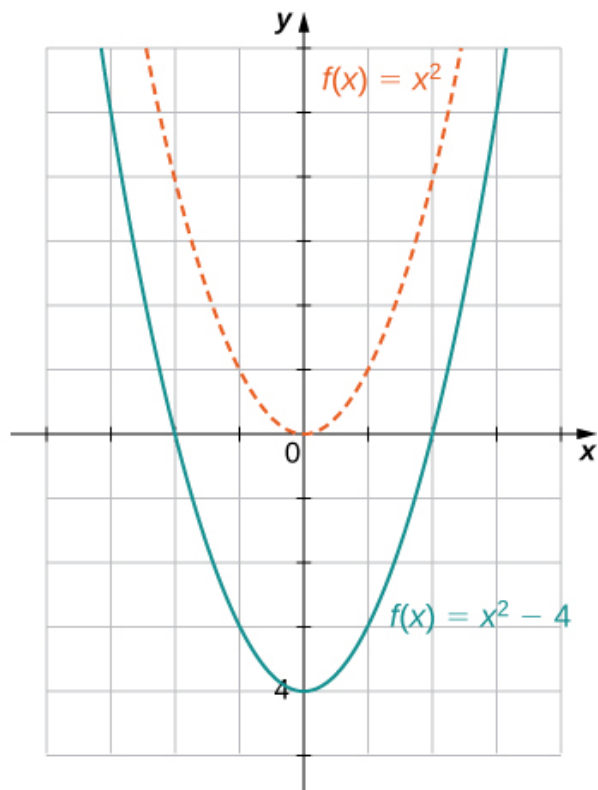
Transformations of Functions

We have seen several cases in which we have added, subtracted, or multiplied constants to form variations of simple functions. In the previous example, for instance, we subtracted 2 from the argument of the function $y = x^2$ to get the function $f(x) = (x - 2)^2$. This subtraction represents a shift of the function $y = x^2$ two units to the right. A shift, horizontally or vertically, is a type of **transformation of a function**. Other transformations include horizontal and vertical scalings, and reflections about the axes.

A vertical shift of a function occurs if we add or subtract the same constant to each output y . For $c > 0$, the graph of $f(x) + c$ is a shift of the graph of $f(x)$ up c units, whereas the graph of $f(x) - c$ is a shift of the graph of $f(x)$ down c units. For example, the graph of the function $f(x) = x^3 + 4$ is the graph of $y = x^3$ shifted up 4 units; the graph of the function $f(x) = x^3 - 4$ is the graph of $y = x^3$ shifted down 4 units ([link](#)).



(a)

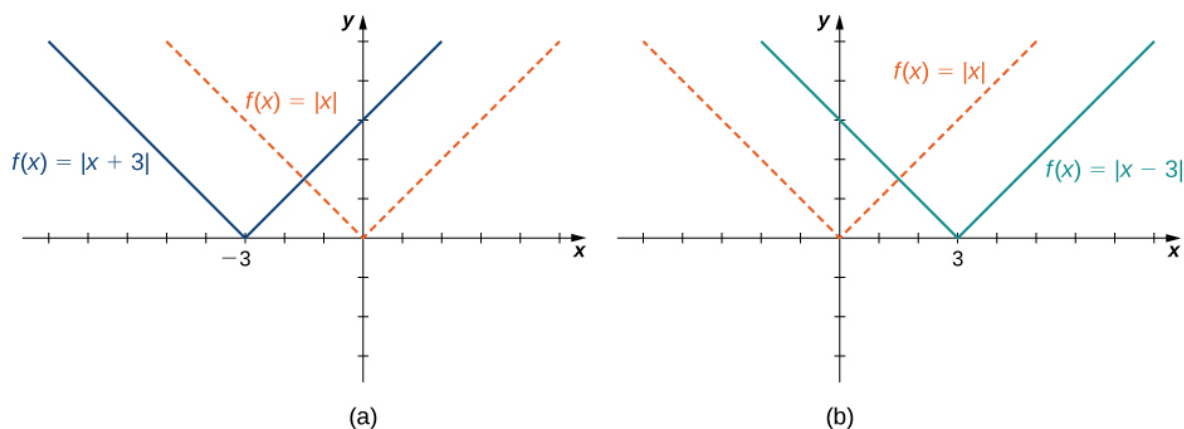


(b)

(a) For $c > 0$, the graph of $y = f(x) + c$ is a vertical shift up c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x) - c$ is a vertical shift down c units of the graph of $y = f(x)$.

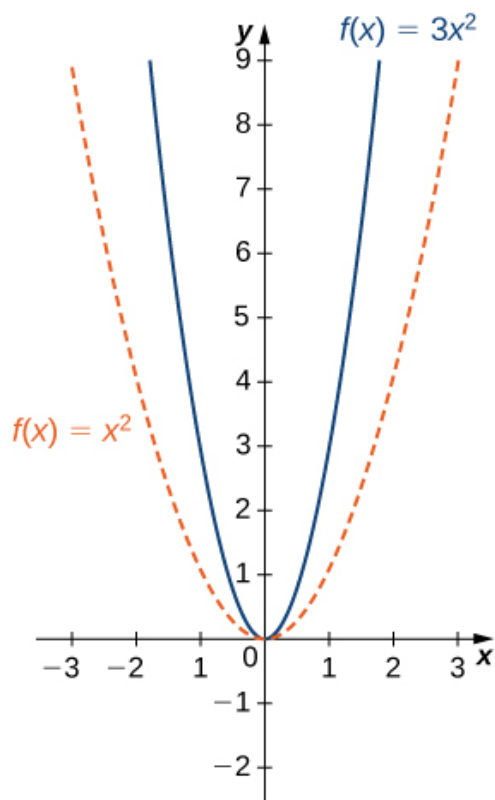
A horizontal shift of a function occurs if we add or subtract the same constant to each input x . For $c > 0$, the graph of $f(x + c)$ is a shift of the graph of $f(x)$ to the left c units; the graph of $f(x - c)$ is a shift of the graph of $f(x)$ to the right c units. Why does the graph shift left when adding a constant and shift right when subtracting a constant? To answer this question, let's look at an example.

Consider the function $f(x) = |x + 3|$ and evaluate this function at $x - 3$. Since $f(x - 3) = |x|$ and $x - 3 < x$, the graph of $f(x) = |x + 3|$ is the graph of $y = |x|$ shifted left 3 units. Similarly, the graph of $f(x) = |x - 3|$ is the graph of $y = |x|$ shifted right 3 units ([link](#)).

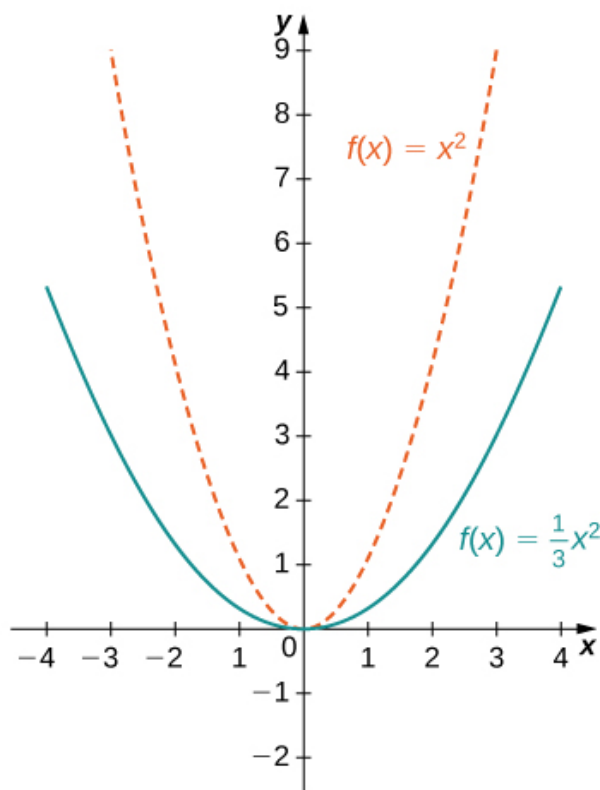


(a) For $c > 0$, the graph of $y = f(x + c)$ is a horizontal shift left c units of the graph of $y = f(x)$. (b) For $c > 0$, the graph of $y = f(x - c)$ is a horizontal shift right c units of the graph of $y = f(x)$.

A vertical scaling of a graph occurs if we multiply all outputs y of a function by the same positive constant. For $c > 0$, the graph of the function $cf(x)$ is the graph of $f(x)$ scaled vertically by a factor of c . If $c > 1$, the values of the outputs for the function $cf(x)$ are larger than the values of the outputs for the function $f(x)$; therefore, the graph has been stretched vertically. If $0 < c < 1$, then the outputs of the function $cf(x)$ are smaller, so the graph has been compressed. For example, the graph of the function $f(x) = 3x^2$ is the graph of $y = x^2$ stretched vertically by a factor of 3, whereas the graph of $f(x) = x^2/3$ is the graph of $y = x^2$ compressed vertically by a factor of 3 ([\[link\]](#)).



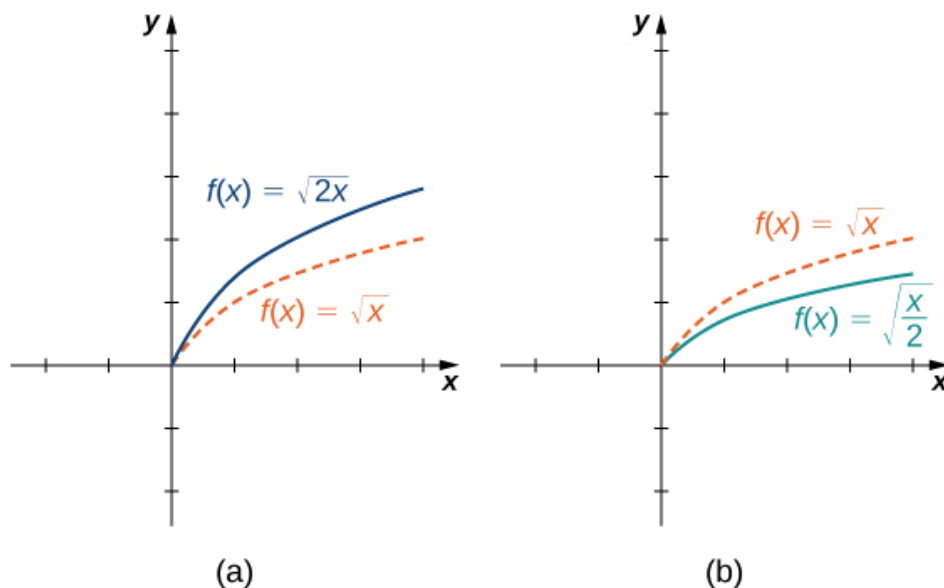
(a)



(b)

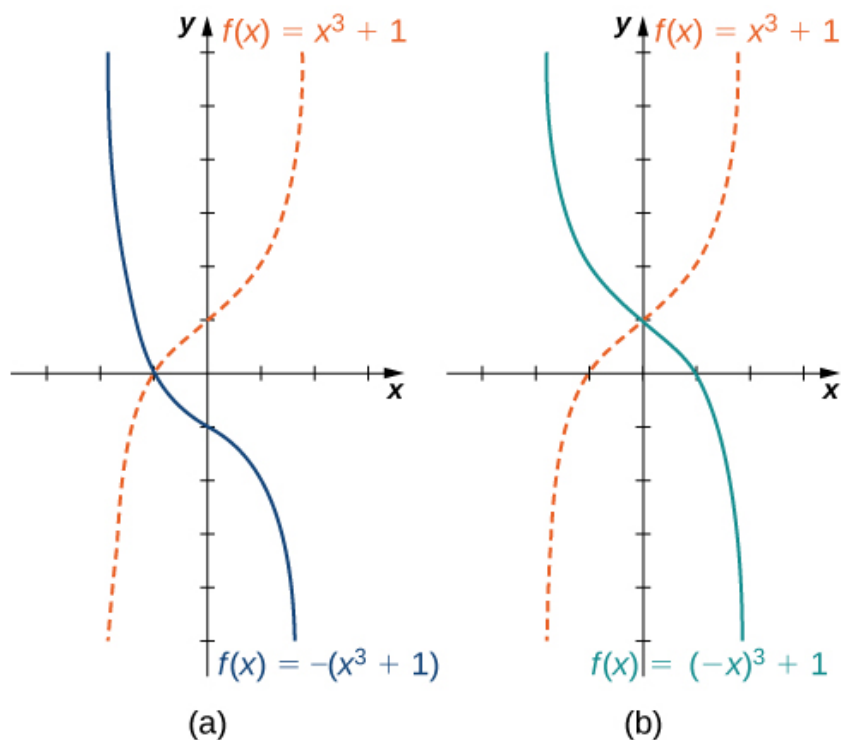
(a) If $c > 1$, the graph of $y = cf(x)$ is a vertical stretch of the graph of $y = f(x)$. (b) If $0 < c < 1$, the graph of $y = cf(x)$ is a vertical compression of the graph of $y = f(x)$.

The horizontal scaling of a function occurs if we multiply the inputs x by the same positive constant. For $c > 0$, the graph of the function $f(cx)$ is the graph of $f(x)$ scaled horizontally by a factor of c . If $c > 1$, the graph of $f(cx)$ is the graph of $f(x)$ compressed horizontally. If $0 < c < 1$, the graph of $f(cx)$ is the graph of $f(x)$ stretched horizontally. For example, consider the function $f(x) = \sqrt{2x}$ and evaluate f at $x/2$. Since $f(x/2) = \sqrt{x}$, the graph of $f(x) = \sqrt{2x}$ is the graph of $y = \sqrt{x}$ compressed horizontally. The graph of $y = \sqrt{x/2}$ is a horizontal stretch of the graph of $y = \sqrt{x}$ ([link](#)).



(a) If $c > 1$, the graph of $y = f(cx)$ is a horizontal compression of the graph of $y = f(x)$. (b) If $0 < c < 1$, the graph of $y = f(cx)$ is a horizontal stretch of the graph of $y = f(x)$.

We have explored what happens to the graph of a function f when we multiply f by a constant $c > 0$ to get a new function $cf(x)$. We have also discussed what happens to the graph of a function f when we multiply the independent variable x by $c > 0$ to get a new function $f(cx)$. However, we have not addressed what happens to the graph of the function if the constant c is negative. If we have a constant $c < 0$, we can write c as a positive number multiplied by -1 ; but, what kind of transformation do we get when we multiply the function or its argument by -1 ? When we multiply all the outputs by -1 , we get a reflection about the x -axis. When we multiply all inputs by -1 , we get a reflection about the y -axis. For example, the graph of $f(x) = -(x^3 + 1)$ is the graph of $y = (x^3 + 1)$ reflected about the x -axis. The graph of $f(x) = (-x)^3 + 1$ is the graph of $y = x^3 + 1$ reflected about the y -axis ([\[link\]](#)).



(a) The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about the x -axis. (b) The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about the y -axis.

If the graph of a function consists of more than one transformation of another graph, it is important to transform the graph in the correct order. Given a function $f(x)$, the graph of the related function $y = cf(a(x+b)) + d$ can be obtained from the graph of $y = f(x)$ by performing the transformations in the following order.

1. Horizontal shift of the graph of $y = f(x)$. If $b > 0$, shift left. If $b < 0$, shift right.
2. Horizontal scaling of the graph of $y = f(x+b)$ by a factor of $|a|$. If $a < 0$, reflect the graph about the y -axis.
3. Vertical scaling of the graph of $y = f(a(x+b))$ by a factor of $|c|$. If $c < 0$, reflect the graph about the x -axis.
4. Vertical shift of the graph of $y = cf(a(x+b))$. If $d > 0$, shift up. If $d < 0$, shift down.

We can summarize the different transformations and their related effects on the graph of a function in the following table.

Transformation of f ($c > 0$)	Effect on the graph of f
$f(x) + c$	Vertical shift up c units
$f(x) - c$	Vertical shift down c units
$f(x + c)$	Shift left by c units
$f(x - c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c > 1$; vertical compression if $0 < c < 1$
$f(cx)$	Horizontal stretch if $0 < c < 1$; horizontal compression if $c > 1$
$-f(x)$	Reflection about the x -axis
$f(-x)$	Reflection about the y -axis

Transformations of Functions

Example:

Exercise:

Problem:

Transforming a Function

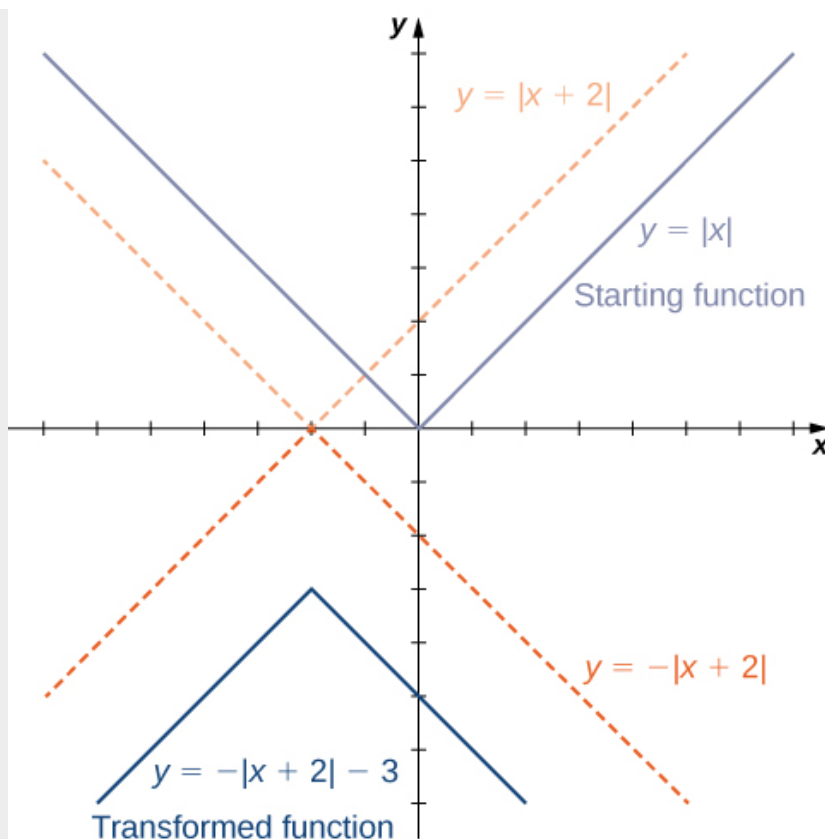
For each of the following functions, a. and b., sketch a graph by using a sequence of transformations of a well-known function.

a. $f(x) = -|x + 2| - 3$

b. $f(x) = 3\sqrt{-x} + 1$

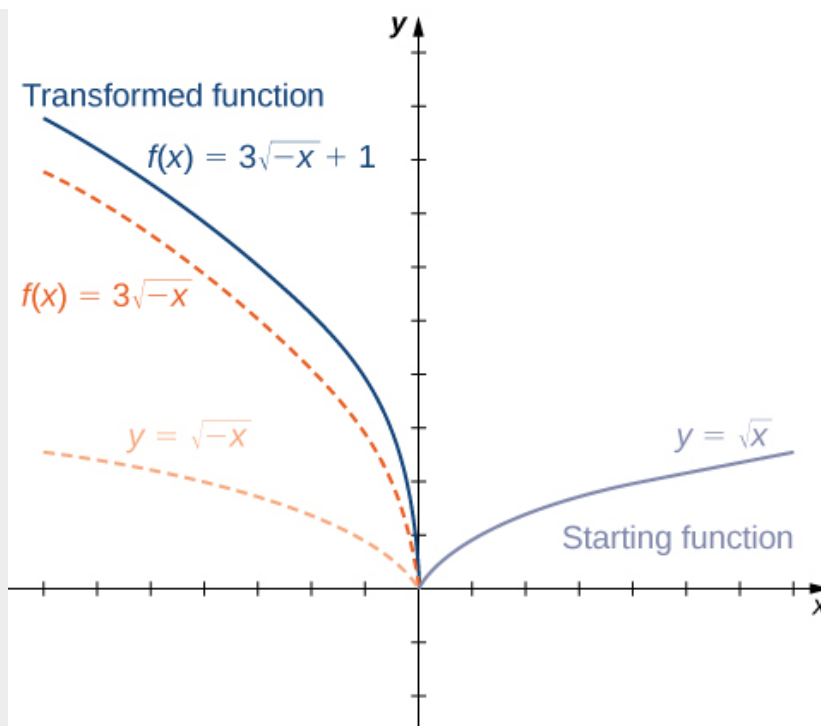
Solution:

- a. Starting with the graph of $y = |x|$, shift 2 units to the left, reflect about the x -axis, and then shift down 3 units.



The function $f(x) = -|x + 2| - 3$ can be viewed as a sequence of three transformations of the function $y = |x|$.

- b. Starting with the graph of $y = \sqrt{x}$, reflect about the y -axis, stretch the graph vertically by a factor of 3, and move up 1 unit.



The function $f(x) = 3\sqrt{-x} + 1$ can be viewed as a sequence of three transformations of the function $y = \sqrt{x}$.

Note:

Exercise:

Problem:

Describe how the function $f(x) = -(x + 1)^2 - 4$ can be graphed using the graph of $y = x^2$ and a sequence of transformations.

Solution:

Shift the graph $y = x^2$ to the left 1 unit, reflect about the x -axis, then shift down 4 units.

Hint

Use [\[link\]](#).

Key Concepts

- The power function $f(x) = x^n$ is an even function if n is even and $n \neq 0$, and it is an odd function if n is odd.
- The root function $f(x) = x^{1/n}$ has the domain $[0, \infty)$ if n is even and the domain $(-\infty, \infty)$ if n is odd. If n is odd, then $f(x) = x^{1/n}$ is an odd function.
- The domain of the rational function $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomial functions, is the set of x such that $q(x) \neq 0$.
- Functions that involve the basic operations of addition, subtraction, multiplication, division, and powers are algebraic functions. All other functions are transcendental. Trigonometric, exponential, and logarithmic functions are examples of transcendental functions.
- A polynomial function f with degree $n \geq 1$ satisfies $f(x) \rightarrow \pm\infty$ as $x \rightarrow \pm\infty$. The sign of the output as $x \rightarrow \infty$ depends on the sign of the leading coefficient only and on whether n is even or odd.
- Vertical and horizontal shifts, vertical and horizontal scalings, and reflections about the x - and y -axes are examples of transformations of functions.

Key Equations

- **Point-slope equation of a line**
 $y - y_1 = m(x - x_1)$
- **Slope-intercept form of a line**
 $y = mx + b$
- **Standard form of a line**
 $ax + by = c$
- **Polynomial function**
 $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

For the following exercises, for each pair of points, a. find the slope of the line passing through the points and b. indicate whether the line is increasing, decreasing, horizontal, or vertical.

Exercise:

Problem: $(-2, 4)$ and $(1, 1)$

Solution:

a. -1 b. Decreasing

Exercise:

Problem: $(-1, 4)$ and $(3, -1)$

Exercise:

Problem: $(3, 5)$ and $(-1, 2)$

Solution:

a. $3/4$ b. Increasing

Exercise:

Problem: $(6, 4)$ and $(4, -3)$

Exercise:

Problem: $(2, 3)$ and $(5, 7)$

Solution:

a. $4/3$ b. Increasing

Exercise:

Problem: $(1, 9)$ and $(-8, 5)$

Exercise:

Problem: $(2, 4)$ and $(1, 4)$

Solution:

a. 0 b. Horizontal

Exercise:

Problem: $(1, 4)$ and $(1, 0)$

For the following exercises, write the equation of the line satisfying the given conditions in slope-intercept form.

Exercise:

Problem: Slope = -6 , passes through $(1, 3)$

Solution:

$$y = -6x + 9$$

Exercise:

Problem: Slope = 3, passes through $(-3, 2)$

Exercise:

Problem: Slope = $\frac{1}{3}$, passes through $(0, 4)$

Solution:

$$y = \frac{1}{3}x + 4$$

Exercise:

Problem: Slope = $\frac{2}{5}$, x -intercept = 8

Exercise:

Problem: Passing through $(2, 1)$ and $(-2, -1)$

Solution:

$$y = \frac{1}{2}x$$

Exercise:

Problem: Passing through $(-3, 7)$ and $(1, 2)$

Exercise:

Problem: x -intercept = 5 and y -intercept = -3

Solution:

$$y = \frac{3}{5}x - 3$$

Exercise:

Problem: x -intercept = -6 and y -intercept = 9

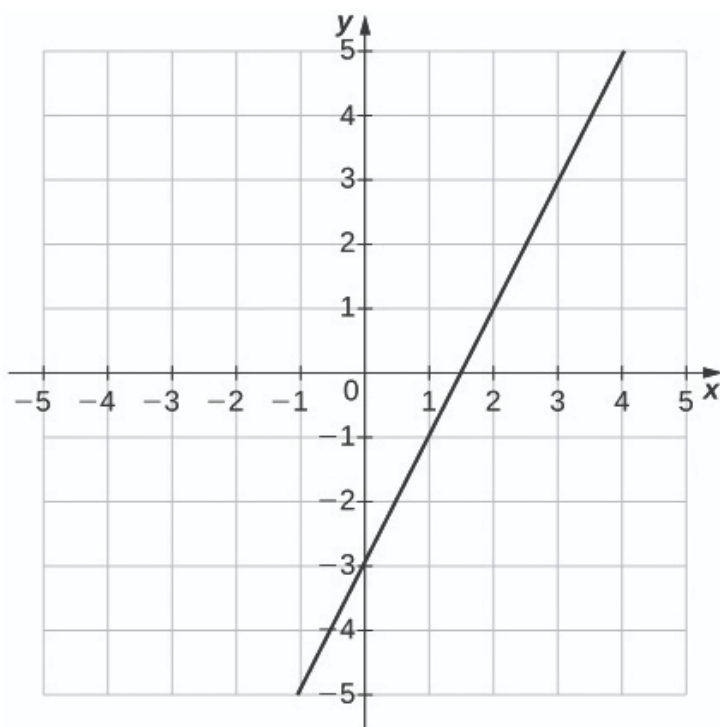
For the following exercises, for each linear equation, a. give the slope m and y -intercept b , if any, and b. graph the line.

Exercise:

Problem: $y = 2x - 3$

Solution:

a. $(m = 2, b = -3)$ b.



Exercise:

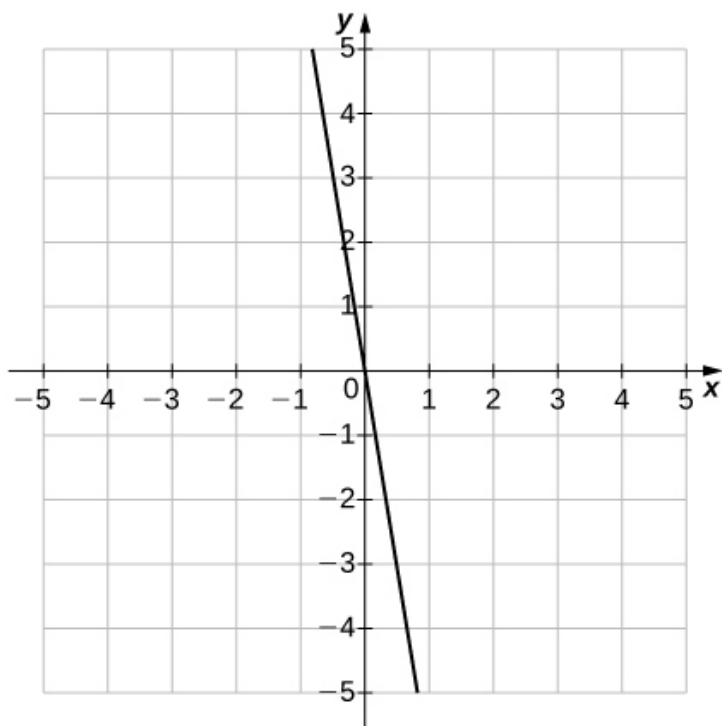
Problem: $y = -\frac{1}{7}x + 1$

Exercise:

Problem: $f(x) = -6x$

Solution:

a. $(m = -6, b = 0)$ b.



Exercise:

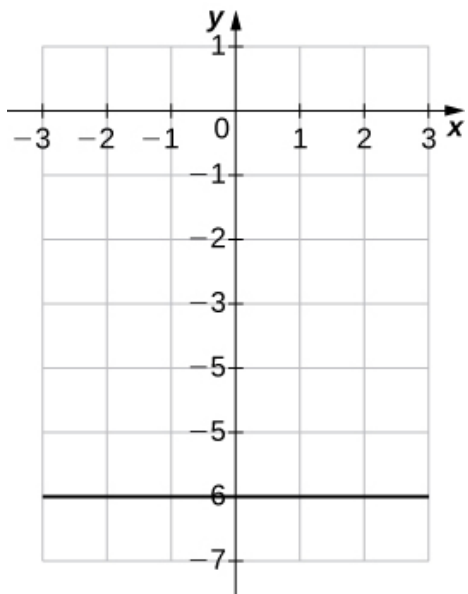
Problem: $f(x) = -5x + 4$

Exercise:

Problem: $4y + 24 = 0$

Solution:

a. $(m = 0, b = -6)$ b.



Exercise:

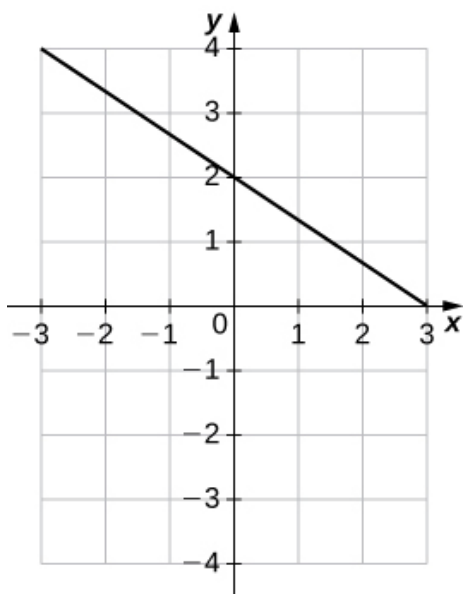
Problem: $8x - 4 = 0$

Exercise:

Problem: $2x + 3y = 6$

Solution:

a. $(m = -\frac{2}{3}, b = 2)$ b.



Exercise:

Problem: $6x - 5y + 15 = 0$

For the following exercises, for each polynomial, a. find the degree; b. find the zeros, if any; c. find the y -intercept(s), if any; d. use the leading coefficient to determine the graph's end behavior; and e. determine algebraically whether the polynomial is even, odd, or neither.

Exercise:

Problem: $f(x) = 2x^2 - 3x - 5$

Solution:

a. 2 b. $\frac{5}{2}, -1$; c. -5 d. Both ends rise e. Neither

Exercise:

Problem: $f(x) = -3x^2 + 6x$

Exercise:

Problem: $f(x) = \frac{1}{2}x^2 - 1$

Solution:

a. 2 b. $\pm\sqrt{2}$ c. -1 d. Both ends rise e. Even

Exercise:

Problem: $f(x) = x^3 + 3x^2 - x - 3$

Exercise:

Problem: $f(x) = 3x - x^3$

Solution:

a. 3 b. $0, \pm\sqrt{3}$ c. 0 d. Left end rises, right end falls e. Odd

For the following exercises, use the graph of $f(x) = x^2$ to graph each transformed function g .

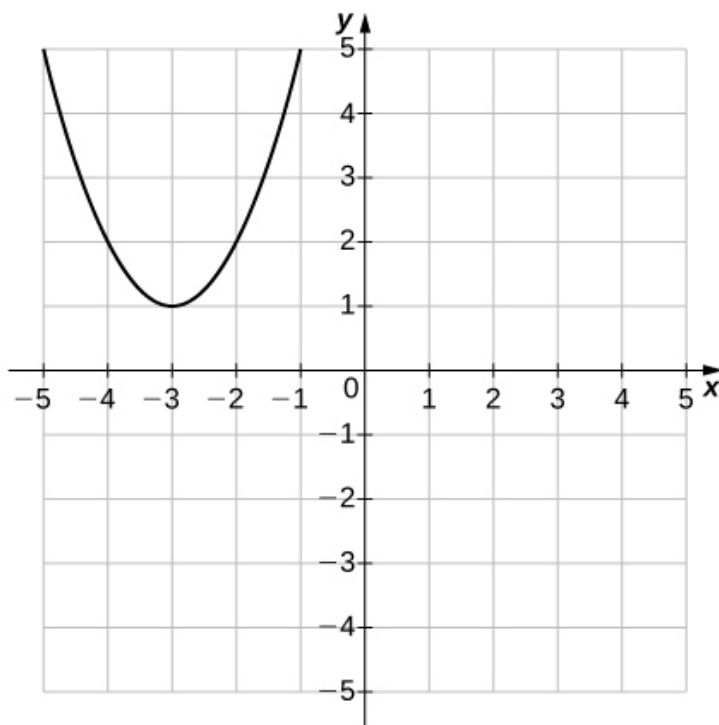
Exercise:

Problem: $g(x) = x^2 - 1$

Exercise:

Problem: $g(x) = (x + 3)^2 + 1$

Solution:



For the following exercises, use the graph of $f(x) = \sqrt{x}$ to graph each transformed function g .

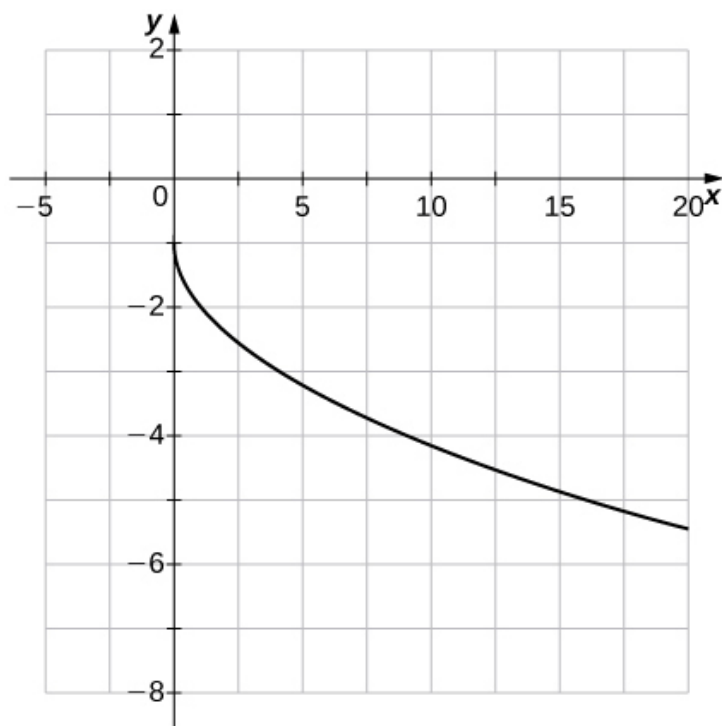
Exercise:

Problem: $g(x) = \sqrt{x + 2}$

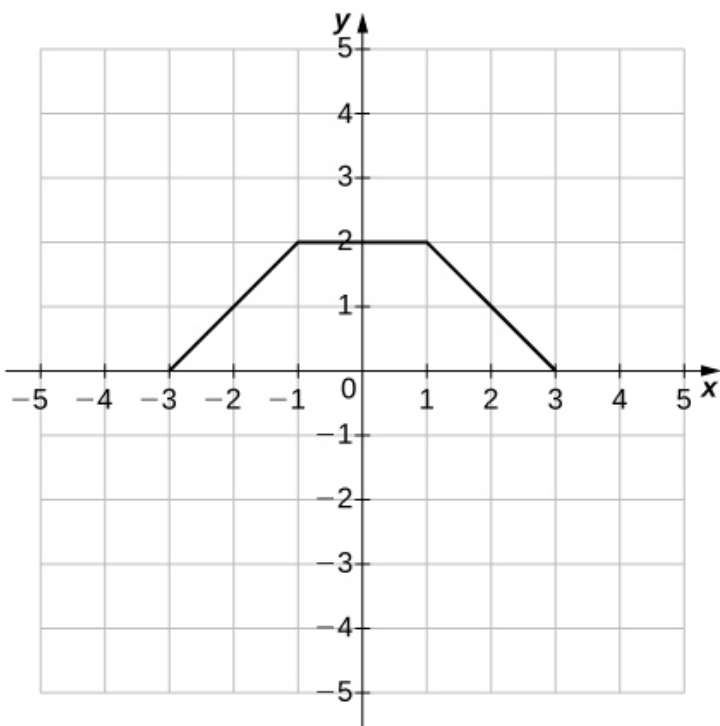
Exercise:

Problem: $g(x) = -\sqrt{x} - 1$

Solution:



For the following exercises, use the graph of $y = f(x)$ to graph each transformed function g .



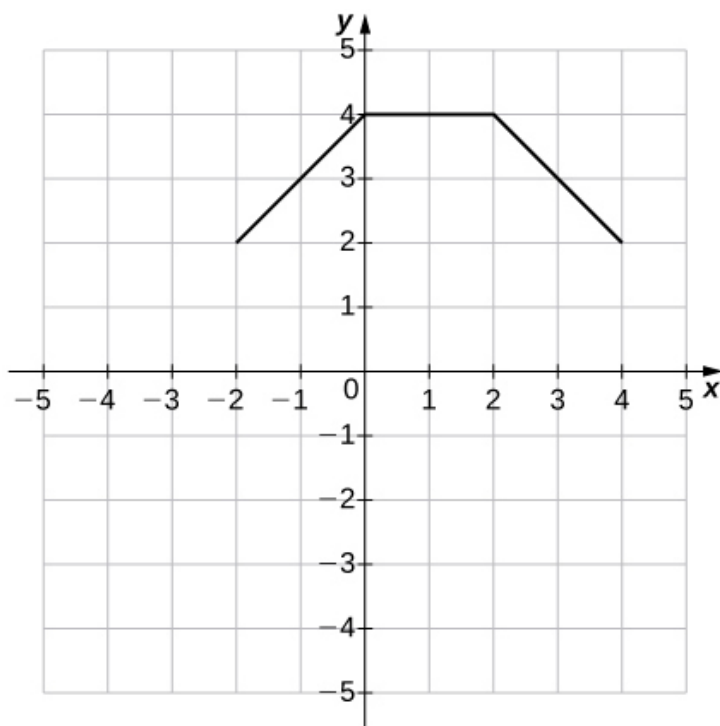
Exercise:

Problem: $g(x) = f(x) + 1$

Exercise:

Problem: $g(x) = f(x - 1) + 2$

Solution:



For the following exercises, for each of the piecewise-defined functions, a. evaluate at the given values of the independent variable and b. sketch the graph.

Exercise:

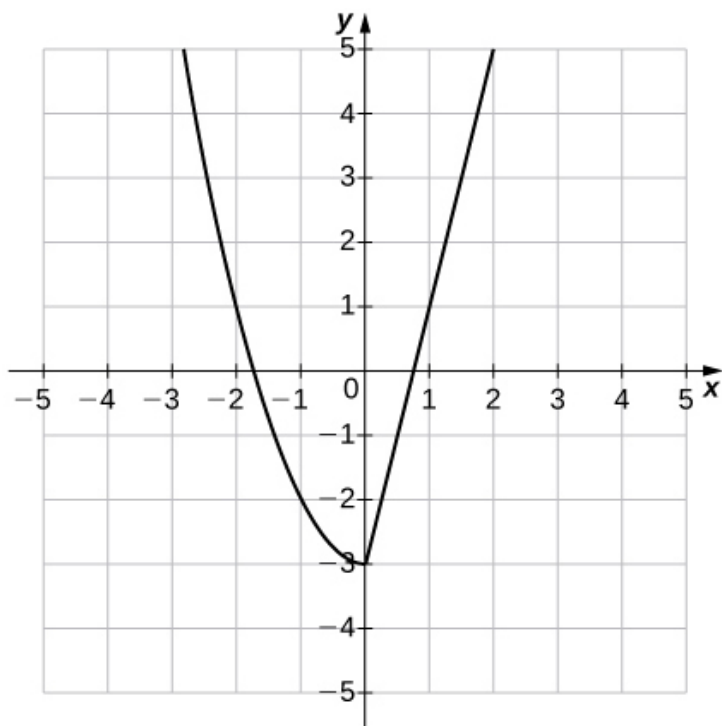
Problem: $f(x) = \begin{cases} 4x + 3, & x \leq 0 \\ -x + 1, & x > 0 \end{cases}; f(-3); f(0); f(2)$

Exercise:

Problem: $f(x) = \begin{cases} x^2 - 3, & x < 0 \\ 4x - 3, & x \geq 0 \end{cases}; f(-4); f(0); f(2)$

Solution:

a. 13, -3, 5 b.



Exercise:

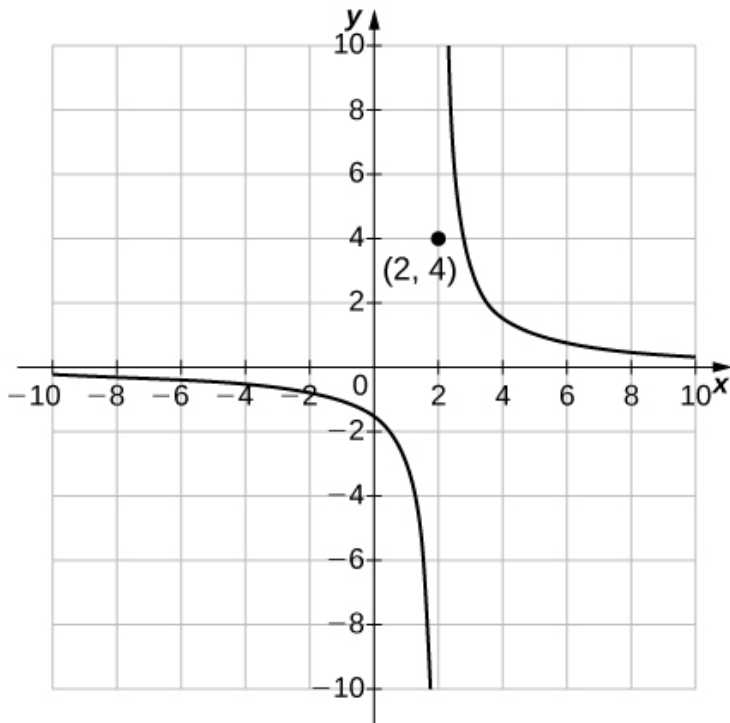
Problem: $h(x) = \begin{cases} x + 1, & x \leq 5 \\ 4, & x > 5 \end{cases}; h(0); h(\pi); h(5)$

Exercise:

Problem: $g(x) = \begin{cases} \frac{3}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}; g(0); g(-4); g(2)$

Solution:

a. $\frac{-3}{2}, \frac{-1}{2}, 4$ b.



For the following exercises, determine whether the statement is *true or false*. Explain why.

Exercise:

Problem: $f(x) = (4x + 1)/(7x - 2)$ is a transcendental function.

Exercise:

Problem: $g(x) = \sqrt[3]{x}$ is an odd root function

Solution:

True; $n = 3$

Exercise:

Problem: A logarithmic function is an algebraic function.

Exercise:

Problem:

A function of the form $f(x) = x^b$, where b is a real valued constant, is an exponential function.

Solution:

False; $f(x) = x^b$, where b is a real-valued constant, is a power function

Exercise:

Problem: The domain of an even root function is all real numbers.

Exercise:**Problem:**

[T] A company purchases some computer equipment for \$20,500. At the end of a 3-year period, the value of the equipment has decreased linearly to \$12,300.

- Find a function $y = V(t)$ that determines the value V of the equipment at the end of t years.
- Find and interpret the meaning of the x - and y -intercepts for this situation.
- What is the value of the equipment at the end of 5 years?
- When will the value of the equipment be \$3000?

Solution:

a. $V(t) = -2733t + 20500$ b. $(0, 20,500)$ means that the initial purchase price of the equipment is \$20,500; $(7.5, 0)$ means that in 7.5 years the computer equipment has no value. c. \$6835 d. In approximately 6.4 years

Exercise:**Problem:**

[T] Total online shopping during the Christmas holidays has increased dramatically during the past 5 years. In 2012 ($t = 0$), total online holiday sales were \$42.3 billion, whereas in 2013 they were \$48.1 billion.

- Find a linear function S that estimates the total online holiday sales in the year t .
- Interpret the slope of the graph of S .
- Use part a. to predict the year when online shopping during Christmas will reach \$60 billion.

Exercise:

Problem:

[T] A family bakery makes cupcakes and sells them at local outdoor festivals. For a music festival, there is a fixed cost of \$125 to set up a cupcake stand. The owner estimates that it costs \$0.75 to make each cupcake. The owner is interested in determining the total cost C as a function of number of cupcakes made.

- Find a linear function that relates cost C to x , the number of cupcakes made.
- Find the cost to bake 160 cupcakes.
- If the owner sells the cupcakes for \$1.50 apiece, how many cupcakes does she need to sell to start making profit? (*Hint: Use the INTERSECTION function on a calculator to find this number.*)

Solution:

- a. $C = 0.75x + 125$ b. \$245 c. 167 cupcakes

Exercise:**Problem:**

[T] A house purchased for \$250,000 is expected to be worth twice its purchase price in 18 years.

- Find a linear function that models the price P of the house versus the number of years t since the original purchase.
- Interpret the slope of the graph of P .
- Find the price of the house 15 years from when it was originally purchased.

Exercise:**Problem:**

[T] A car was purchased for \$26,000. The value of the car depreciates by \$1500 per year.

- Find a linear function that models the value V of the car after t years.
- Find and interpret $V(4)$.

Solution:

- a. $V(t) = -1500t + 26,000$ b. In 4 years, the value of the car is \$20,000.

Exercise:

Problem:

[T] A condominium in an upscale part of the city was purchased for \$432,000. In 35 years it is worth \$60,500. Find the rate of depreciation.

Exercise:**Problem:**

[T] The total cost C (in thousands of dollars) to produce a certain item is modeled by the function $C(x) = 10.50x + 28,500$, where x is the number of items produced. Determine the cost to produce 175 items.

Solution:

\$30,337.50

Exercise:**Problem:**

[T] A professor asks her class to report the amount of time t they spent writing two assignments. Most students report that it takes them about 45 minutes to type a four-page assignment and about 1.5 hours to type a nine-page assignment.

- Find the linear function $y = N(t)$ that models this situation, where N is the number of pages typed and t is the time in minutes.
- Use part a. to determine how many pages can be typed in 2 hours.
- Use part a. to determine how long it takes to type a 20-page assignment.

Exercise:**Problem:**

[T] The output (as a percent of total capacity) of nuclear power plants in the United States can be modeled by the function $P(t) = 1.8576t + 68.052$, where t is time in years and $t = 0$ corresponds to the beginning of 2000. Use the model to predict the percentage output in 2015.

Solution:

96% of the total capacity

Exercise:

Problem:

[T] The admissions office at a public university estimates that 65% of the students offered admission to the class of 2019 will actually enroll.

- a. Find the linear function $y = N(x)$, where N is the number of students that actually enroll and x is the number of all students offered admission to the class of 2019.
- b. If the university wants the 2019 freshman class size to be 1350, determine how many students should be admitted.

Glossary

algebraic function

a function involving any combination of only the basic operations of addition, subtraction, multiplication, division, powers, and roots applied to an input variable x

cubic function

a polynomial of degree 3; that is, a function of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$

degree

for a polynomial function, the value of the largest exponent of any term

linear function

a function that can be written in the form $f(x) = mx + b$

logarithmic function

a function of the form $f(x) = \log_b(x)$ for some base $b > 0, b \neq 1$ such that $y = \log_b(x)$ if and only if $b^y = x$

mathematical model

A method of simulating real-life situations with mathematical equations

piecewise-defined function

a function that is defined differently on different parts of its domain

point-slope equation

equation of a linear function indicating its slope and a point on the graph of the function

polynomial function

a function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

power function

a function of the form $f(x) = x^n$ for any positive integer $n \geq 1$

quadratic function

a polynomial of degree 2; that is, a function of the form $f(x) = ax^2 + bx + c$
where $a \neq 0$

rational function

a function of the form $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials

root function

a function of the form $f(x) = x^{1/n}$ for any integer $n \geq 2$

slope

the change in y for each unit change in x

slope-intercept form

equation of a linear function indicating its slope and y -intercept

transcendental function

a function that cannot be expressed by a combination of basic arithmetic operations

transformation of a function

a shift, scaling, or reflection of a function

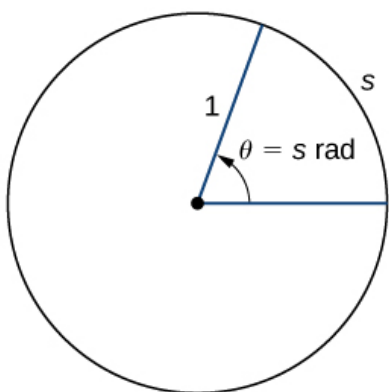
Trigonometric Functions

- Convert angle measures between degrees and radians.
- Recognize the triangular and circular definitions of the basic trigonometric functions.
- Write the basic trigonometric identities.
- Identify the graphs and periods of the trigonometric functions.
- Describe the shift of a sine or cosine graph from the equation of the function.

Trigonometric functions are used to model many phenomena, including sound waves, vibrations of strings, alternating electrical current, and the motion of pendulums. In fact, almost any repetitive, or cyclical, motion can be modeled by some combination of trigonometric functions. In this section, we define the six basic trigonometric functions and look at some of the main identities involving these functions.

Radian Measure

To use trigonometric functions, we first must understand how to measure the angles. Although we can use both radians and degrees, **radians** are a more natural measurement because they are related directly to the unit circle, a circle with radius 1. The radian measure of an angle is defined as follows. Given an angle θ , let s be the length of the corresponding arc on the unit circle ([link](#)). We say the angle corresponding to the arc of length 1 has radian measure 1.



The radian measure of an angle θ is the arc length s of the associated arc on the unit circle.

Since an angle of 360° corresponds to the circumference of a circle, or an arc of length 2π , we conclude that an angle with a degree measure of 360° has a radian measure of 2π . Similarly, we see that 180° is equivalent to π radians. [link](#) shows the relationship between common degree and radian values.

Degrees	Radians	Degrees	Radians
0	0	120	$2\pi/3$
30	$\pi/6$	135	$3\pi/4$
45	$\pi/4$	150	$5\pi/6$
60	$\pi/3$	180	π
90	$\pi/2$		

Common Angles Expressed in Degrees and Radians

Example:

Exercise:

Problem:

Converting between Radians and Degrees

- Express 225° using radians.
- Express $5\pi/3$ rad using degrees.

Solution:

Use the fact that 180° is equivalent to π radians as a conversion factor: $1 = \frac{\pi \text{ rad}}{180^\circ} = \frac{180^\circ}{\pi \text{ rad}}$.

- $225^\circ = 225^\circ \cdot \frac{\pi}{180^\circ} = \frac{5\pi}{4} \text{ rad}$
- $\frac{5\pi}{3} \text{ rad} = \frac{5\pi}{3} \cdot \frac{180^\circ}{\pi} = 300^\circ$

Note:

Exercise:

Problem: Express 210° using radians. Express $11\pi/6$ rad using degrees.

Solution:

$7\pi/6$; 330°

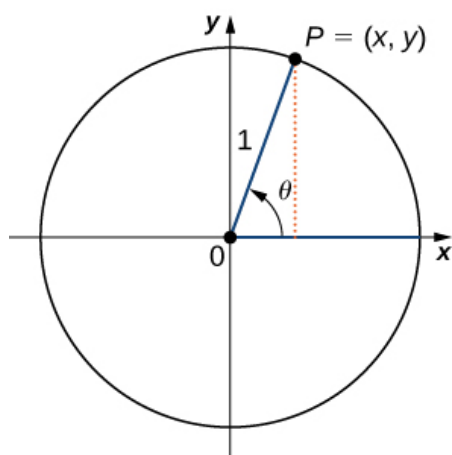
Hint

π radians is equal to 180° .

The Six Basic Trigonometric Functions

Trigonometric functions allow us to use angle measures, in radians or degrees, to find the coordinates of a point on any circle—not only on a unit circle—or to find an angle given a point on a circle. They also define the relationship among the sides and angles of a triangle.

To define the trigonometric functions, first consider the unit circle centered at the origin and a point $P = (x, y)$ on the unit circle. Let θ be an angle with an initial side that lies along the positive x -axis and with a terminal side that is the line segment OP . An angle in this position is said to be in *standard position* ([link](#)). We can then define the values of the six trigonometric functions for θ in terms of the coordinates x and y .



The angle θ is in standard position. The values of the trigonometric functions for θ are defined in terms of the coordinates x and y .

Note:

Definition

Let $P = (x, y)$ be a point on the unit circle centered at the origin O . Let θ be an angle with an initial side along the positive x -axis and a terminal side given by the line segment OP . The **trigonometric functions** are then defined as

Equation:

$$\begin{array}{ll} \sin \theta = y & \csc \theta = \frac{1}{y} \\ \cos \theta = x & \sec \theta = \frac{1}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$

If $x = 0$, $\sec \theta$ and $\tan \theta$ are undefined. If $y = 0$, then $\cot \theta$ and $\csc \theta$ are undefined.

We can see that for a point $P = (x, y)$ on a circle of radius r with a corresponding angle θ , the coordinates x and y satisfy

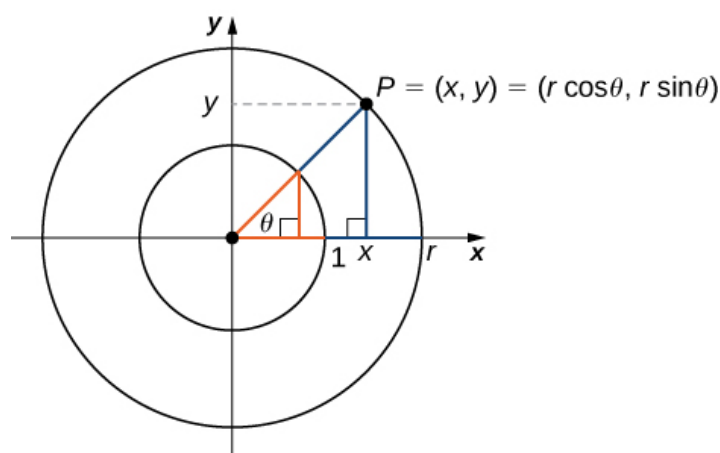
Equation:

$$\cos \theta = \frac{x}{r}$$
$$x = r \cos \theta$$

Equation:

$$\sin \theta = \frac{y}{r}$$
$$y = r \sin \theta.$$

The values of the other trigonometric functions can be expressed in terms of x , y , and r ([link](#)).



For a point $P = (x, y)$ on a circle of radius r , the coordinates x and y satisfy $x = r \cos \theta$ and $y = r \sin \theta$.

[link](#) shows the values of sine and cosine at the major angles in the first quadrant. From this table, we can determine the values of sine and cosine at the corresponding angles in the other quadrants. The values of the other trigonometric functions are calculated easily from the values of $\sin \theta$ and $\cos \theta$.

θ	$\sin \theta$	$\cos \theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Values of $\sin \theta$ and $\cos \theta$ at Major Angles θ in the First Quadrant

Example:

Exercise:

Problem:

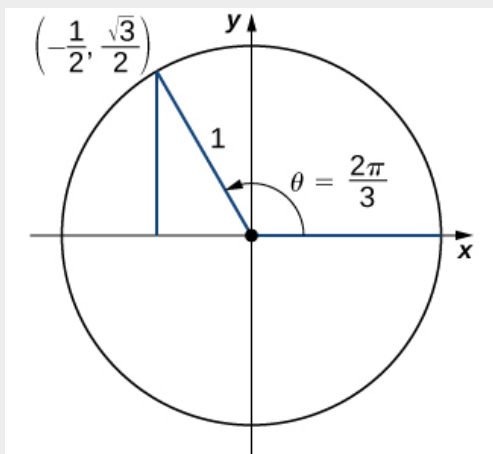
Evaluating Trigonometric Functions

Evaluate each of the following expressions.

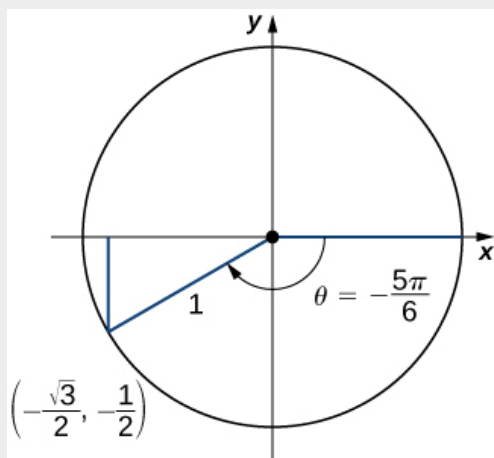
- a. $\sin \left(\frac{2\pi}{3} \right)$
- b. $\cos \left(-\frac{5\pi}{6} \right)$
- c. $\tan \left(\frac{15\pi}{4} \right)$

Solution:

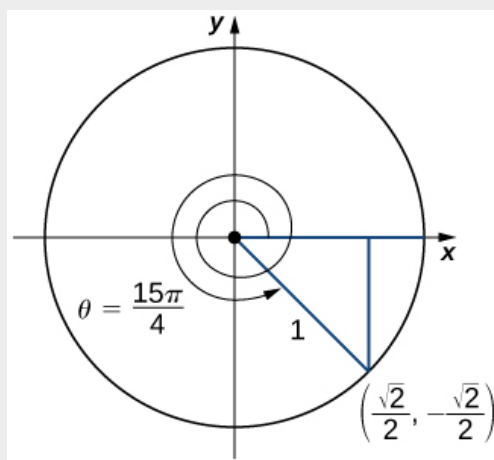
- a. On the unit circle, the angle $\theta = \frac{2\pi}{3}$ corresponds to the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$. Therefore,
- $$\sin \left(\frac{2\pi}{3} \right) = y = \frac{\sqrt{3}}{2}.$$



- b. An angle $\theta = -\frac{5\pi}{6}$ corresponds to a revolution in the negative direction, as shown. Therefore, $\cos\left(-\frac{5\pi}{6}\right) = x = -\frac{\sqrt{3}}{2}$.



- c. An angle $\theta = \frac{15\pi}{4} = 2\pi + \frac{7\pi}{4}$. Therefore, this angle corresponds to more than one revolution, as shown. Knowing the fact that an angle of $\frac{7\pi}{4}$ corresponds to the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, we can conclude that $\tan\left(\frac{15\pi}{4}\right) = \frac{y}{x} = -1$.



Note:

Exercise:

Problem: Evaluate $\cos(3\pi/4)$ and $\sin(-\pi/6)$.

Solution:

$$\cos(3\pi/4) = -\sqrt{2}/2; \sin(-\pi/6) = -1/2$$

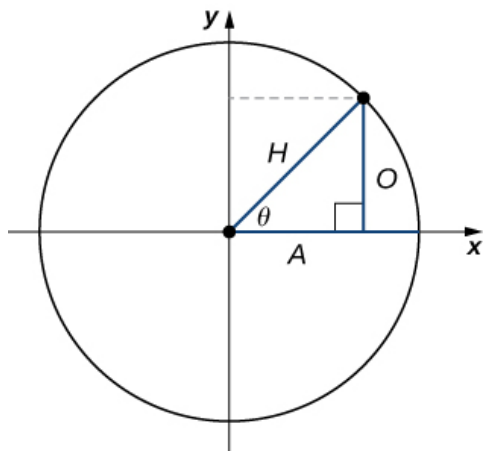
Hint

Look at angles on the unit circle.

As mentioned earlier, the ratios of the side lengths of a right triangle can be expressed in terms of the trigonometric functions evaluated at either of the acute angles of the triangle. Let θ be one of the acute angles. Let A be the length of the adjacent leg, O be the length of the opposite leg, and H be the length of the hypotenuse. By inscribing the triangle into a circle of radius H , as shown in [\[link\]](#), we see that A , H , and O satisfy the following relationships with θ :

Equation:

$$\begin{aligned}\sin \theta &= \frac{O}{H} & \csc \theta &= \frac{H}{O} \\ \cos \theta &= \frac{A}{H} & \sec \theta &= \frac{H}{A} \\ \tan \theta &= \frac{O}{A} & \cot \theta &= \frac{A}{O}\end{aligned}$$



By inscribing a right triangle in a circle, we can express the ratios of the side lengths in terms of the trigonometric functions evaluated at θ .

Example:

Exercise:

Problem:

Constructing a Wooden Ramp

A wooden ramp is to be built with one end on the ground and the other end at the top of a short staircase. If the top of the staircase is 4 ft from the ground and the angle between the ground and the ramp is to be 10° , how long does the ramp need to be?

Solution:

Let x denote the length of the ramp. In the following image, we see that x needs to satisfy the equation $\sin(10^\circ) = 4/x$. Solving this equation for x , we see that $x = 4/\sin(10^\circ) \approx 23.035$ ft.



Note:

Exercise:

Problem:

A house painter wants to lean a 20-ft ladder against a house. If the angle between the base of the ladder and the ground is to be 60° , how far from the house should she place the base of the ladder?

Solution:

10 ft

Hint

Draw a right triangle with hypotenuse 20.

Trigonometric Identities

A **trigonometric identity** is an equation involving trigonometric functions that is true for all angles θ for which the functions are defined. We can use the identities to help us solve or simplify equations. The main trigonometric identities are listed next.

Note:

Rule: Trigonometric Identities

Reciprocal identities

Equation:

$$\begin{aligned}\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta} \\ \csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta}\end{aligned}$$

Pythagorean identities

Equation:

$$\sin^2 \theta + \cos^2 \theta = 1 \quad 1 + \tan^2 \theta = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta$$

Addition and subtraction formulas

Equation:

$$\sin (\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

Equation:

$$\cos (\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

Double-angle formulas

Equation:

$$\sin (2\theta) = 2 \sin \theta \cos \theta$$

Equation:

$$\cos (2\theta) = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta = \cos^2 \theta - \sin^2 \theta$$

Example:

Exercise:

Problem:

Solving Trigonometric Equations

For each of the following equations, use a trigonometric identity to find all solutions.

a. $1 + \cos (2\theta) = \cos \theta$

b. $\sin (2\theta) = \tan \theta$

Solution:

a. Using the double-angle formula for $\cos(2\theta)$, we see that θ is a solution of

Equation:

$$1 + \cos(2\theta) = \cos \theta$$

if and only if

Equation:

$$1 + 2\cos^2\theta - 1 = \cos\theta,$$

which is true if and only if

Equation:

$$2\cos^2\theta - \cos\theta = 0.$$

To solve this equation, it is important to note that we need to factor the left-hand side and not divide both sides of the equation by $\cos\theta$. The problem with dividing by $\cos\theta$ is that it is possible that $\cos\theta$ is zero. In fact, if we did divide both sides of the equation by $\cos\theta$, we would miss some of the solutions of the original equation. Factoring the left-hand side of the equation, we see that θ is a solution of this equation if and only if

Equation:

$$\cos\theta(2\cos\theta - 1) = 0.$$

Since $\cos\theta = 0$ when

Equation:

$$\theta = \frac{\pi}{2}, \frac{\pi}{2} \pm \pi, \frac{\pi}{2} \pm 2\pi, \dots,$$

and $\cos\theta = 1/2$ when

Equation:

$$\theta = \frac{\pi}{3}, \frac{\pi}{3} \pm 2\pi, \dots \text{ or } \theta = -\frac{\pi}{3}, -\frac{\pi}{3} \pm 2\pi, \dots,$$

we conclude that the set of solutions to this equation is

Equation:

$$\theta = \frac{\pi}{2} + n\pi, \theta = \frac{\pi}{3} + 2n\pi, \text{ and } \theta = -\frac{\pi}{3} + 2n\pi, n = 0, \pm 1, \pm 2, \dots$$

- b. Using the double-angle formula for $\sin(2\theta)$ and the reciprocal identity for $\tan(\theta)$, the equation can be written as

Equation:

$$2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}.$$

To solve this equation, we multiply both sides by $\cos\theta$ to eliminate the denominator, and say that if θ satisfies this equation, then θ satisfies the equation

Equation:

$$2\sin\theta\cos^2\theta - \sin\theta = 0.$$

However, we need to be a little careful here. Even if θ satisfies this new equation, it may not satisfy the original equation because, to satisfy the original equation, we would need to be able to divide both sides of the equation by $\cos\theta$. However, if $\cos\theta = 0$, we cannot divide both sides of the equation by $\cos\theta$. Therefore, it is possible that we may arrive at extraneous solutions. So, at the end, it is important to check for extraneous solutions.

Returning to the equation, it is important that we factor $\sin\theta$ out of both terms on the left-hand side instead of dividing both sides of the equation by $\sin\theta$. Factoring the left-hand side of the equation, we can rewrite this equation as

Equation:

$$\sin\theta(2\cos^2\theta - 1) = 0.$$

Therefore, the solutions are given by the angles θ such that $\sin\theta = 0$ or $\cos^2\theta = 1/2$. The solutions of the first equation are $\theta = 0, \pm\pi, \pm2\pi, \dots$. The solutions of the second equation are $\theta = \pi/4, (\pi/4) \pm (\pi/2), (\pi/4) \pm \pi, \dots$. After checking for extraneous solutions, the set of solutions to the equation is

Equation:

$$\theta = n\pi \quad \text{and} \quad \theta = \frac{\pi}{4} + \frac{n\pi}{2}, n = 0, \pm 1, \pm 2, \dots$$

Note:

Exercise:

Problem: Find all solutions to the equation $\cos(2\theta) = \sin\theta$.

Solution:

$$\theta = \frac{3\pi}{2} + 2n\pi, \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi \text{ for } n = 0, \pm 1, \pm 2, \dots$$

Hint

Use the double-angle formula for cosine.

Example:

Exercise:

Problem:

Proving a Trigonometric Identity

Prove the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$.

Solution:

We start with the identity

Equation:

$$\sin^2\theta + \cos^2\theta = 1.$$

Dividing both sides of this equation by $\cos^2\theta$, we obtain

Equation:

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}.$$

Since $\sin\theta/\cos\theta = \tan\theta$ and $1/\cos\theta = \sec\theta$, we conclude that

Equation:

$$\tan^2\theta + 1 = \sec^2\theta.$$

Note:**Exercise:**

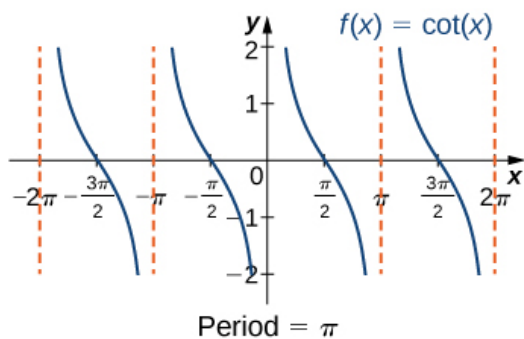
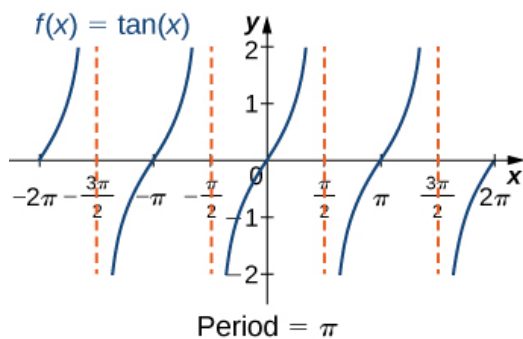
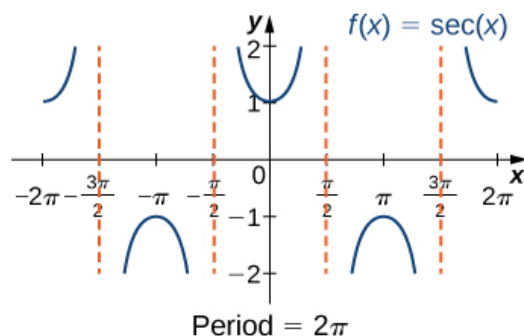
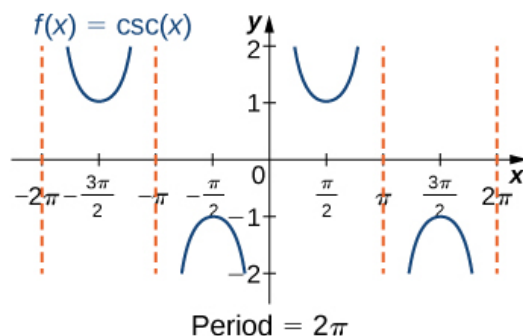
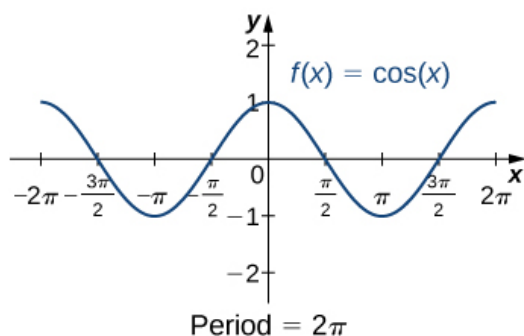
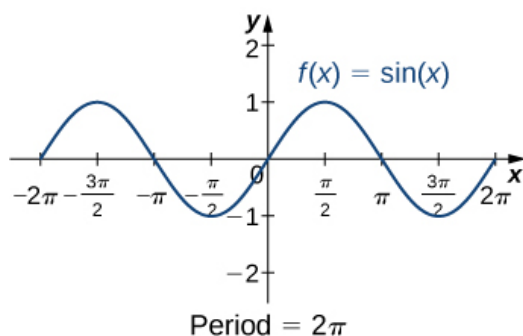
Problem: Prove the trigonometric identity $1 + \cot^2\theta = \csc^2\theta$.

Hint

Divide both sides of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$.

Graphs and Periods of the Trigonometric Functions

We have seen that as we travel around the unit circle, the values of the trigonometric functions repeat. We can see this pattern in the graphs of the functions. Let $P = (x, y)$ be a point on the unit circle and let θ be the corresponding angle. Since the angle θ and $\theta + 2\pi$ correspond to the same point P , the values of the trigonometric functions at θ and at $\theta + 2\pi$ are the same. Consequently, the trigonometric functions are **periodic functions**. The period of a function f is defined to be the smallest positive value p such that $f(x + p) = f(x)$ for all values x in the domain of f . The sine, cosine, secant, and cosecant functions have a period of 2π . Since the tangent and cotangent functions repeat on an interval of length π , their period is π ([link](#)).



The six trigonometric functions are periodic.

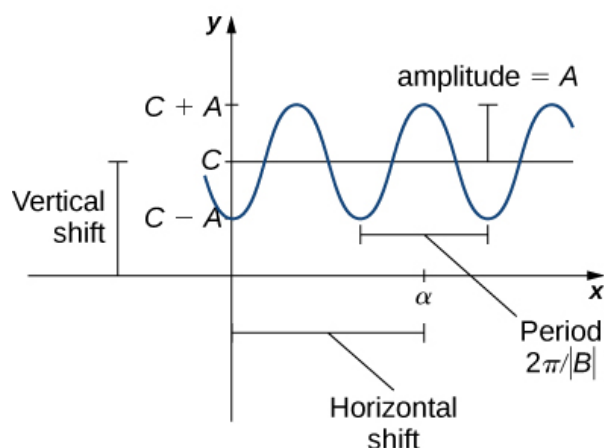
Just as with algebraic functions, we can apply transformations to trigonometric functions. In particular, consider the following function:

Equation:

$$f(x) = A \sin(B(x - \alpha)) + C.$$

In [\[link\]](#), the constant α causes a horizontal or phase shift. The factor B changes the period. This transformed sine function will have a period $2\pi/|B|$. The factor A results in a vertical stretch by a factor of $|A|$. We say $|A|$ is the “amplitude of f .” The constant C causes a vertical shift.

$$f(x) = A\sin(B(x - \alpha)) + C$$



A graph of a general sine function.

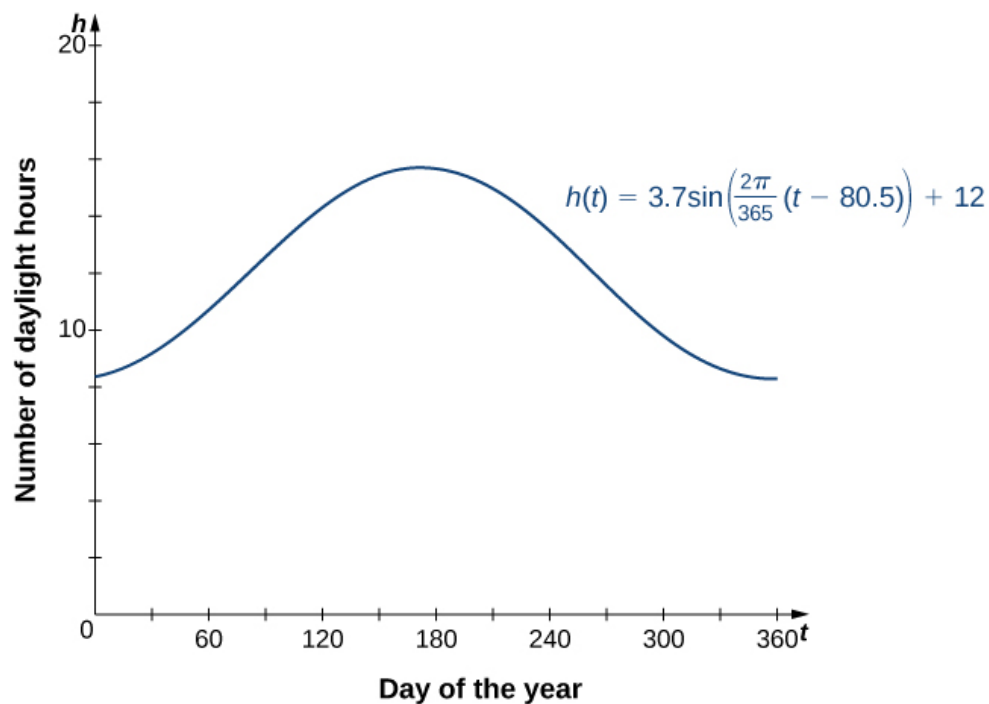
Notice in [\[link\]](#) that the graph of $y = \cos x$ is the graph of $y = \sin x$ shifted to the left $\pi/2$ units. Therefore, we can write $\cos x = \sin(x + \pi/2)$. Similarly, we can view the graph of $y = \sin x$ as the graph of $y = \cos x$ shifted right $\pi/2$ units, and state that $\sin x = \cos(x - \pi/2)$.

A shifted sine curve arises naturally when graphing the number of hours of daylight in a given location as a function of the day of the year. For example, suppose a city reports that June 21 is the longest day of the year with 15.7 hours and December 21 is the shortest day of the year with 8.3 hours. It can be shown that the function

Equation:

$$h(t) = 3.7 \sin\left(\frac{2\pi}{365}(t - 80.5)\right) + 12$$

is a model for the number of hours of daylight h as a function of day of the year t ([\[link\]](#)).



The hours of daylight as a function of day of the year can be modeled by a shifted sine curve.

Example:

Exercise:

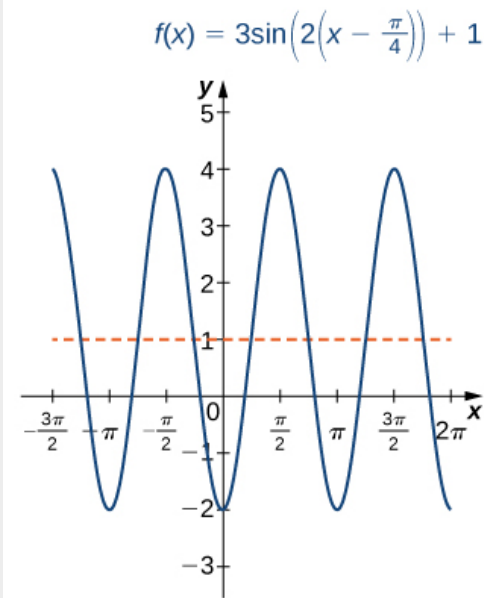
Problem:

Sketching the Graph of a Transformed Sine Curve

Sketch a graph of $f(x) = 3 \sin\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$.

Solution:

This graph is a phase shift of $y = \sin(x)$ to the right by $\pi/4$ units, followed by a horizontal compression by a factor of 2, a vertical stretch by a factor of 3, and then a vertical shift by 1 unit. The period of f is π .



Note:

Exercise:

Problem:

Describe the relationship between the graph of $f(x) = 3\sin(4x) - 5$ and the graph of $y = \sin(x)$.

Solution:

To graph $f(x) = 3\sin(4x) - 5$, the graph of $y = \sin(x)$ needs to be compressed horizontally by a factor of 4, then stretched vertically by a factor of 3, then shifted down 5 units. The function f will have a period of $\pi/2$ and an amplitude of 3.

Hint

The graph of f can be sketched using the graph of $y = \sin(x)$ and a sequence of three transformations.

Key Concepts

- Radian measure is defined such that the angle associated with the arc of length 1 on the unit circle has radian measure 1. An angle with a degree measure of 180° has a radian measure of π rad.
- For acute angles θ , the values of the trigonometric functions are defined as ratios of two sides of a right triangle in which one of the acute angles is θ .

- For a general angle θ , let (x, y) be a point on a circle of radius r corresponding to this angle θ . The trigonometric functions can be written as ratios involving x , y , and r .
- The trigonometric functions are periodic. The sine, cosine, secant, and cosecant functions have period 2π . The tangent and cotangent functions have period π .

Key Equations

- **Generalized sine function**

$$f(x) = A \sin(B(x - \alpha)) + C$$

For the following exercises, convert each angle in degrees to radians. Write the answer as a multiple of π .

Exercise:

Problem: 240°

Solution:

$$\frac{4\pi}{3} \text{ rad}$$

Exercise:

Problem: 15°

Exercise:

Problem: -60°

Solution:

$$\frac{-\pi}{3}$$

Exercise:

Problem: -225°

Exercise:

Problem: 330°

Solution:

$$\frac{11\pi}{6} \text{ rad}$$

For the following exercises, convert each angle in radians to degrees.

Exercise:

Problem: $\frac{\pi}{2} \text{ rad}$

Exercise:

Problem: $\frac{7\pi}{6}$ rad

Solution:

210°

Exercise:

Problem: $\frac{11\pi}{2}$ rad

Exercise:

Problem: -3π rad

Solution:

-540°

Exercise:

Problem: $\frac{5\pi}{12}$ rad

Evaluate the following functional values.

Exercise:

Problem: $\cos\left(\frac{4\pi}{3}\right)$

Solution:

-0.5

Exercise:

Problem: $\tan\left(\frac{19\pi}{4}\right)$

Exercise:

Problem: $\sin\left(-\frac{3\pi}{4}\right)$

Solution:

$-\frac{\sqrt{2}}{2}$

Exercise:

Problem: $\sec\left(\frac{\pi}{6}\right)$

Exercise:

Problem: $\sin\left(\frac{\pi}{12}\right)$

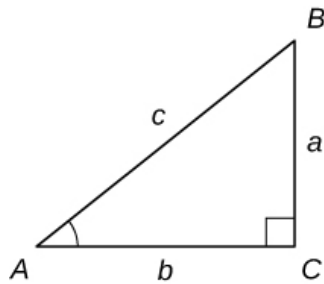
Solution:

$$\frac{\sqrt{3}-1}{2\sqrt{2}}$$

Exercise:

Problem: $\cos\left(\frac{5\pi}{12}\right)$

For the following exercises, consider triangle ABC , a right triangle with a right angle at C . a. Find the missing side of the triangle. b. Find the six trigonometric function values for the angle at A . Where necessary, round to one decimal place.

**Exercise:**

Problem: $a = 4, c = 7$

Solution:

$$\text{a. } b = 5.7 \quad \text{b. } \sin A = \frac{4}{7}, \cos A = \frac{5.7}{7}, \tan A = \frac{4}{5.7}, \csc A = \frac{7}{4}, \sec A = \frac{7}{5.7}, \cot A = \frac{5.7}{4}$$

Exercise:

Problem: $a = 21, c = 29$

Exercise:

Problem: $a = 85.3, b = 125.5$

Solution:

$$\text{a. } c = 151.7 \quad \text{b.}$$

$$\sin A = 0.5623, \cos A = 0.8273, \tan A = 0.6797, \csc A = 1.778, \sec A = 1.209, \cot A = 1.471$$

Exercise:

Problem: $b = 40, c = 41$

Exercise:

Problem: $a = 84, b = 13$

Solution:

a. $c = 85$ b. $\sin A = \frac{84}{85}, \cos A = \frac{13}{85}, \tan A = \frac{84}{13}, \csc A = \frac{85}{84}, \sec A = \frac{85}{13}, \cot A = \frac{13}{84}$

Exercise:

Problem: $b = 28, c = 35$

For the following exercises, P is a point on the unit circle. a. Find the (exact) missing coordinate value of each point and b. find the values of the six trigonometric functions for the angle θ with a terminal side that passes through point P . Rationalize denominators.

Exercise:

Problem: $P\left(\frac{7}{25}, y\right), y > 0$

Solution:

a. $y = \frac{24}{25}$ b. $\sin \theta = \frac{24}{25}, \cos \theta = \frac{7}{25}, \tan \theta = \frac{24}{7}, \csc \theta = \frac{25}{24}, \sec \theta = \frac{25}{7}, \cot \theta = \frac{7}{24}$

Exercise:

Problem: $P\left(\frac{-15}{17}, y\right), y < 0$

Exercise:

Problem: $P\left(x, \frac{\sqrt{7}}{3}\right), x < 0$

Solution:

a. $x = \frac{-\sqrt{2}}{3}$ b.
 $\sin \theta = \frac{\sqrt{7}}{3}, \cos \theta = \frac{-\sqrt{2}}{3}, \tan \theta = \frac{-\sqrt{14}}{2}, \csc \theta = \frac{3\sqrt{7}}{7}, \sec \theta = \frac{-3\sqrt{2}}{2}, \cot \theta = \frac{-\sqrt{14}}{7}$

Exercise:

Problem: $P\left(x, \frac{-\sqrt{15}}{4}\right), x > 0$

For the following exercises, simplify each expression by writing it in terms of sines and cosines, then simplify. The final answer does not have to be in terms of sine and cosine only.

Exercise:

Problem: $\tan^2 x + \sin x \csc x$

Solution:

$$\sec^2 x$$

Exercise:

Problem: $\sec x \sin x \cot x$

Exercise:

Problem: $\frac{\tan^2 x}{\sec^2 x}$

Solution:

$$\sin^2 x$$

Exercise:

Problem: $\sec x - \cos x$

Exercise:

Problem: $(1 + \tan \theta)^2 - 2 \tan \theta$

Solution:

$$\sec^2 \theta$$

Exercise:

Problem: $\sin x (\csc x - \sin x)$

Exercise:

Problem: $\frac{\cos t}{\sin t} + \frac{\sin t}{1 + \cos t}$

Solution:

$$\frac{1}{\sin t} = \csc t$$

Exercise:

Problem: $\frac{1 + \tan^2 \alpha}{1 + \cot^2 \alpha}$

For the following exercises, verify that each equation is an identity.

Exercise:

Problem: $\frac{\tan\theta\cot\theta}{\csc\theta} = \sin\theta$

Exercise:

Problem: $\frac{\sec^2\theta}{\tan\theta} = \sec\theta\csc\theta$

Exercise:

Problem: $\frac{\sin t}{\csc t} + \frac{\cos t}{\sec t} = 1$

Exercise:

Problem: $\frac{\sin x}{\cos x + 1} + \frac{\cos x - 1}{\sin x} = 0$

Exercise:

Problem: $\cot\gamma + \tan\gamma = \sec\gamma\csc\gamma$

Exercise:

Problem: $\sin^2\beta + \tan^2\beta + \cos^2\beta = \sec^2\beta$

Exercise:

Problem: $\frac{1}{1-\sin\alpha} + \frac{1}{1+\sin\alpha} = 2\sec^2\alpha$

Exercise:

Problem: $\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \csc^2\theta$

For the following exercises, solve the trigonometric equations on the interval $0 \leq \theta < 2\pi$.

Exercise:

Problem: $2\sin\theta - 1 = 0$

Solution:

$$\left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\}$$

Exercise:

Problem: $1 + \cos\theta = \frac{1}{2}$

Exercise:

Problem: $2\tan^2\theta = 2$

Solution:

$$\left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}$$

Exercise:

Problem: $4\sin^2\theta - 2 = 0$

Exercise:

Problem: $\sqrt{3}\cot\theta + 1 = 0$

Solution:

$$\left\{ \frac{2\pi}{3}, \frac{5\pi}{3} \right\}$$

Exercise:

Problem: $3\sec\theta - 2\sqrt{3} = 0$

Exercise:

Problem: $2\cos\theta\sin\theta = \sin\theta$

Solution:

$$\left\{ 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3} \right\}$$

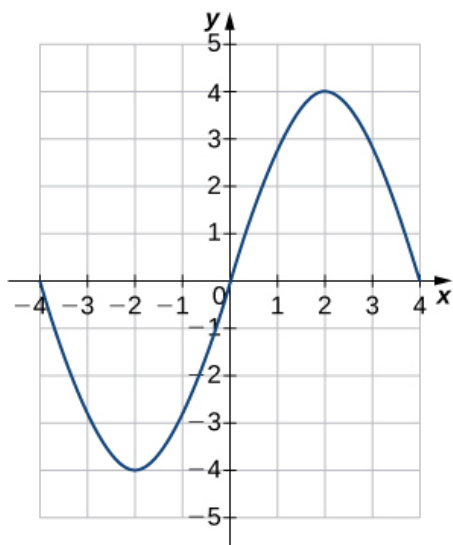
Exercise:

Problem: $\csc^2\theta + 2\csc\theta + 1 = 0$

For the following exercises, each graph is of the form $y = A\sin Bx$ or $y = A\cos Bx$, where $B > 0$. Write the equation of the graph.

Exercise:

Problem:

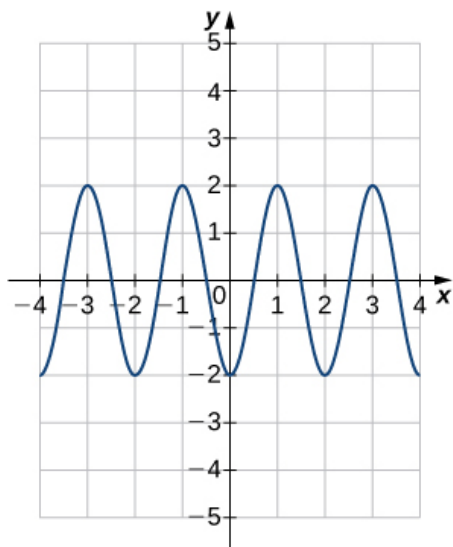


Solution:

$$y = 4 \sin \left(\frac{\pi}{4} x \right)$$

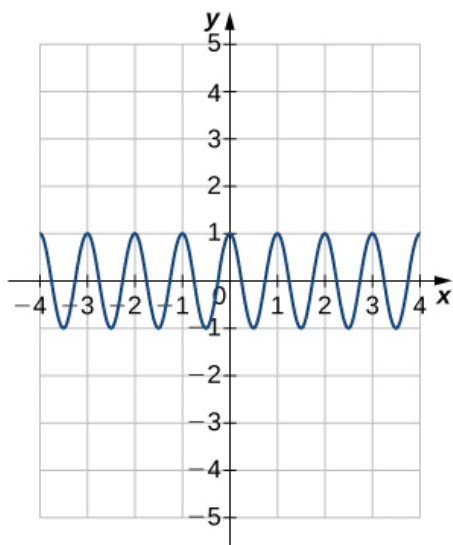
Exercise:

Problem:



Exercise:

Problem:

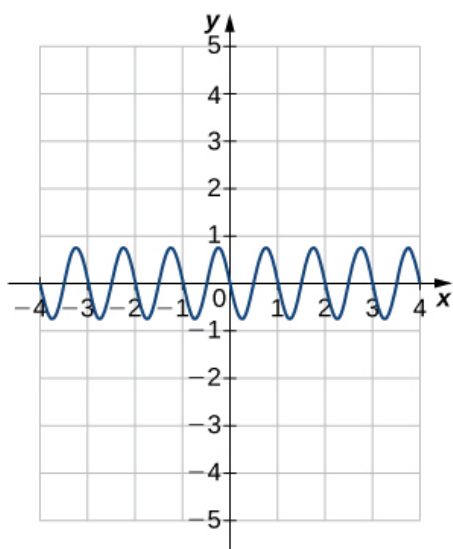


Solution:

$$y = \cos(2\pi x)$$

Exercise:

Problem:



For the following exercises, find a. the amplitude, b. the period, and c. the phase shift with direction for each function.

Exercise:

Problem: $y = \sin\left(x - \frac{\pi}{4}\right)$

Solution:

a. 1 b. 2π c. $\frac{\pi}{4}$ units to the right

Exercise:

Problem: $y = 3 \cos(2x + 3)$

Exercise:

Problem: $y = \frac{-1}{2} \sin\left(\frac{1}{4}x\right)$

Solution:

a. $\frac{1}{2}$ b. 8π c. No phase shift

Exercise:

Problem: $y = 2 \cos\left(x - \frac{\pi}{3}\right)$

Exercise:

Problem: $y = -3 \sin(\pi x + 2)$

Solution:

a. 3 b. 2 c. $\frac{2}{\pi}$ units to the left

Exercise:

Problem: $y = 4 \cos\left(2x - \frac{\pi}{2}\right)$

Exercise:

Problem:

[T] The diameter of a wheel rolling on the ground is 40 in. If the wheel rotates through an angle of 120° , how many inches does it move? Approximate to the nearest whole inch.

Solution:

Approximately 42 in.

Exercise:

Problem:

[T] Find the length of the arc intercepted by central angle θ in a circle of radius r . Round to the nearest hundredth.

a. $r = 12.8$ cm, $\theta = \frac{5\pi}{6}$ rad b. $r = 4.378$ cm, $\theta = \frac{7\pi}{6}$ rad c. $r = 0.964$ cm, $\theta = 50^\circ$ d.
 $r = 8.55$ cm, $\theta = 325^\circ$

Exercise:**Problem:**

[T] As a point P moves around a circle, the measure of the angle changes. The measure of how fast the angle is changing is called *angular speed*, ω , and is given by $\omega = \theta/t$, where θ is in radians and t is time. Find the angular speed for the given data. Round to the nearest thousandth.

a. $\theta = \frac{7\pi}{4}$ rad, $t = 10$ sec b. $\theta = \frac{3\pi}{5}$ rad, $t = 8$ sec c. $\theta = \frac{2\pi}{9}$ rad, $t = 1$ min d.
 $\theta = 23.76$ rad, $t = 14$ min

Solution:

a. 0.550 rad/sec b. 0.236 rad/sec c. 0.698 rad/min d. 1.697 rad/min

Exercise:**Problem:**

[T] A total of 250,000 m² of land is needed to build a nuclear power plant. Suppose it is decided that the area on which the power plant is to be built should be circular.

- Find the radius of the circular land area.
- If the land area is to form a 45° sector of a circle instead of a whole circle, find the length of the curved side.

Exercise:

Problem: [T] The area of an isosceles triangle with equal sides of length x is

$$\frac{1}{2}x^2\sin\theta,$$

where θ is the angle formed by the two sides. Find the area of an isosceles triangle with equal sides of length 8 in. and angle $\theta = 5\pi/12$ rad.

Solution:

$$\approx 30.9 \text{ in}^2$$

Exercise:**Problem:**

[T] A particle travels in a circular path at a constant angular speed ω . The angular speed is modeled by the function $\omega = 9|\cos(\pi t - \pi/12)|$. Determine the angular speed at $t = 9$ sec.

Exercise:

Problem: [T] An alternating current for outlets in a home has voltage given by the function

$$V(t) = 150\cos 368t,$$

where V is the voltage in volts at time t in seconds.

- Find the period of the function and interpret its meaning.
- Determine the number of periods that occur when 1 sec has passed.

Solution:

- a. $\pi/184$; the voltage repeats every $\pi/184$ sec b. Approximately 59 periods

Exercise:

Problem: [T] The number of hours of daylight in a northeast city is modeled by the function

Equation:

$$N(t) = 12 + 3 \sin \left[\frac{2\pi}{365}(t - 79) \right],$$

where t is the number of days after January 1.

- Find the amplitude and period.
- Determine the number of hours of daylight on the longest day of the year.
- Determine the number of hours of daylight on the shortest day of the year.
- Determine the number of hours of daylight 90 days after January 1.
- Sketch the graph of the function for one period starting on January 1.

Exercise:

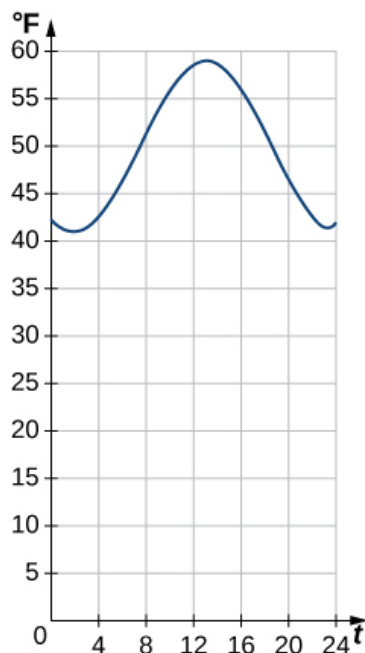
Problem:

[T] Suppose that $T = 50 + 10 \sin \left[\frac{\pi}{12}(t - 8) \right]$ is a mathematical model of the temperature (in degrees Fahrenheit) at t hours after midnight on a certain day of the week.

- Determine the amplitude and period.
- Find the temperature 7 hours after midnight.
- At what time does $T = 60^\circ$?
- Sketch the graph of T over $0 \leq t \leq 24$.

Solution:

- a. Amplitude = 10; period = 24 b. $47.4^\circ F$ c. 14 hours later, or 2 p.m. d.



Exercise:

Problem:

[T] The function $H(t) = 8\sin\left(\frac{\pi}{6}t\right)$ models the height H (in feet) of the tide t hours after midnight. Assume that $t = 0$ is midnight.

- Find the amplitude and period.
- Graph the function over one period.
- What is the height of the tide at 4:30 a.m.?

Glossary

periodic function

a function is periodic if it has a repeating pattern as the values of x move from left to right

radians

for a circular arc of length s on a circle of radius 1, the radian measure of the associated angle θ is s

trigonometric functions

functions of an angle defined as ratios of the lengths of the sides of a right triangle

trigonometric identity

an equation involving trigonometric functions that is true for all angles θ for which the functions in the equation are defined

Inverse Functions

- Determine the conditions for when a function has an inverse.
- Use the horizontal line test to recognize when a function is one-to-one.
- Find the inverse of a given function.
- Draw the graph of an inverse function.
- Evaluate inverse trigonometric functions.

An inverse function reverses the operation done by a particular function. In other words, whatever a function does, the inverse function undoes it. In this section, we define an inverse function formally and state the necessary conditions for an inverse function to exist. We examine how to find an inverse function and study the relationship between the graph of a function and the graph of its inverse. Then we apply these ideas to define and discuss properties of the inverse trigonometric functions.

Existence of an Inverse Function

We begin with an example. Given a function f and an output $y = f(x)$, we are often interested in finding what value or values x were mapped to y by f . For example, consider the function $f(x) = x^3 + 4$. Since any output $y = x^3 + 4$, we can solve this equation for x to find that the input is $x = \sqrt[3]{y - 4}$. This equation defines x as a function of y . Denoting this function as f^{-1} , and writing $x = f^{-1}(y) = \sqrt[3]{y - 4}$, we see that for any x in the domain of f , $f^{-1}(f(x)) = f^{-1}(x^3 + 4) = x$. Thus, this new function, f^{-1} , “undid” what the original function f did. A function with this property is called the inverse function of the original function.

Note:

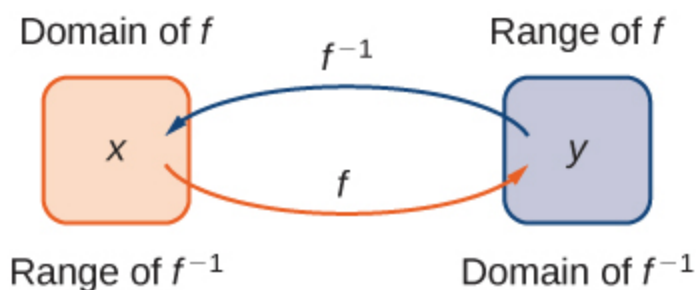
Definition

Given a function f with domain D and range R , its **inverse function** (if it exists) is the function f^{-1} with domain R and range D such that $f^{-1}(y) = x$ if $f(x) = y$. In other words, for a function f and its inverse f^{-1} ,

Equation:

$$f^{-1}(f(x)) = x \text{ for all } x \text{ in } D, \text{ and } f(f^{-1}(y)) = y \text{ for all } y \text{ in } R.$$

Note that f^{-1} is read as “f inverse.” Here, the -1 is not used as an exponent and $f^{-1}(x) \neq 1/f(x)$. [\[link\]](#) shows the relationship between the domain and range of f and the domain and range of f^{-1} .



Given a function f and its inverse f^{-1} , $f^{-1}(y) = x$ if and only if $f(x) = y$. The range of f becomes the domain of f^{-1} and the domain of f becomes the range of f^{-1} .

Recall that a function has exactly one output for each input. Therefore, to define an inverse function, we need to map each input to exactly one output. For example, let's try to find the inverse function for $f(x) = x^2$. Solving the equation $y = x^2$ for x , we arrive at the equation $x = \pm\sqrt{y}$. This equation does not describe x as a function of y because there are two solutions to this equation for every $y > 0$. The problem with trying to find an inverse function for $f(x) = x^2$ is that two inputs are sent to the same output for each output $y > 0$. The function $f(x) = x^3 + 4$ discussed earlier did not have this problem. For that function, each input was sent to a

different output. A function that sends each input to a *different* output is called a one-to-one function.

Note:

Definition

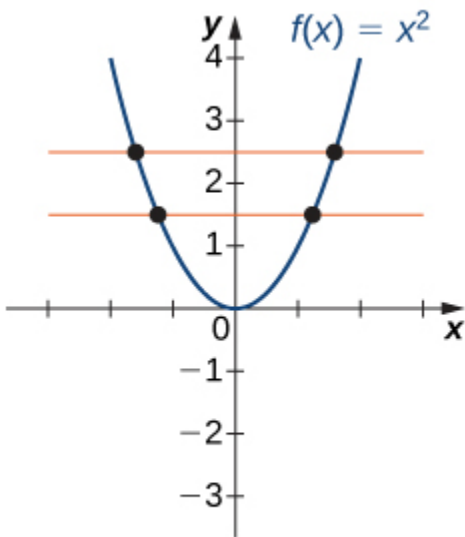
We say a f is a **one-to-one function** if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

One way to determine whether a function is one-to-one is by looking at its graph. If a function is one-to-one, then no two inputs can be sent to the same output. Therefore, if we draw a horizontal line anywhere in the xy -plane, according to the **horizontal line test**, it cannot intersect the graph more than once. We note that the horizontal line test is different from the vertical line test. The vertical line test determines whether a graph is the graph of a function. The horizontal line test determines whether a function is one-to-one ([\[link\]](#)).

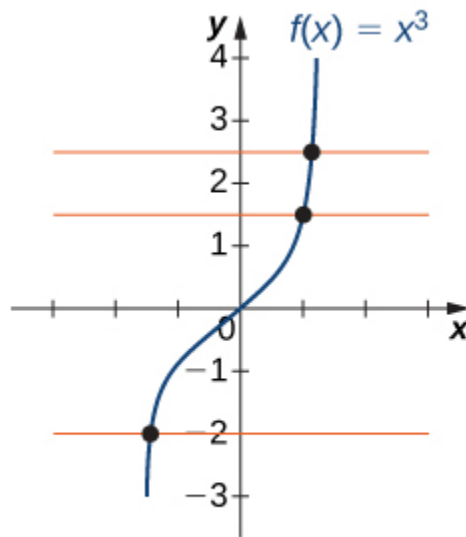
Note:

Rule: Horizontal Line Test

A function f is one-to-one if and only if every horizontal line intersects the graph of f no more than once.



(a)



(b)

- (a) The function $f(x) = x^2$ is not one-to-one because it fails the horizontal line test. (b) The function $f(x) = x^3$ is one-to-one because it passes the horizontal line test.

Example:

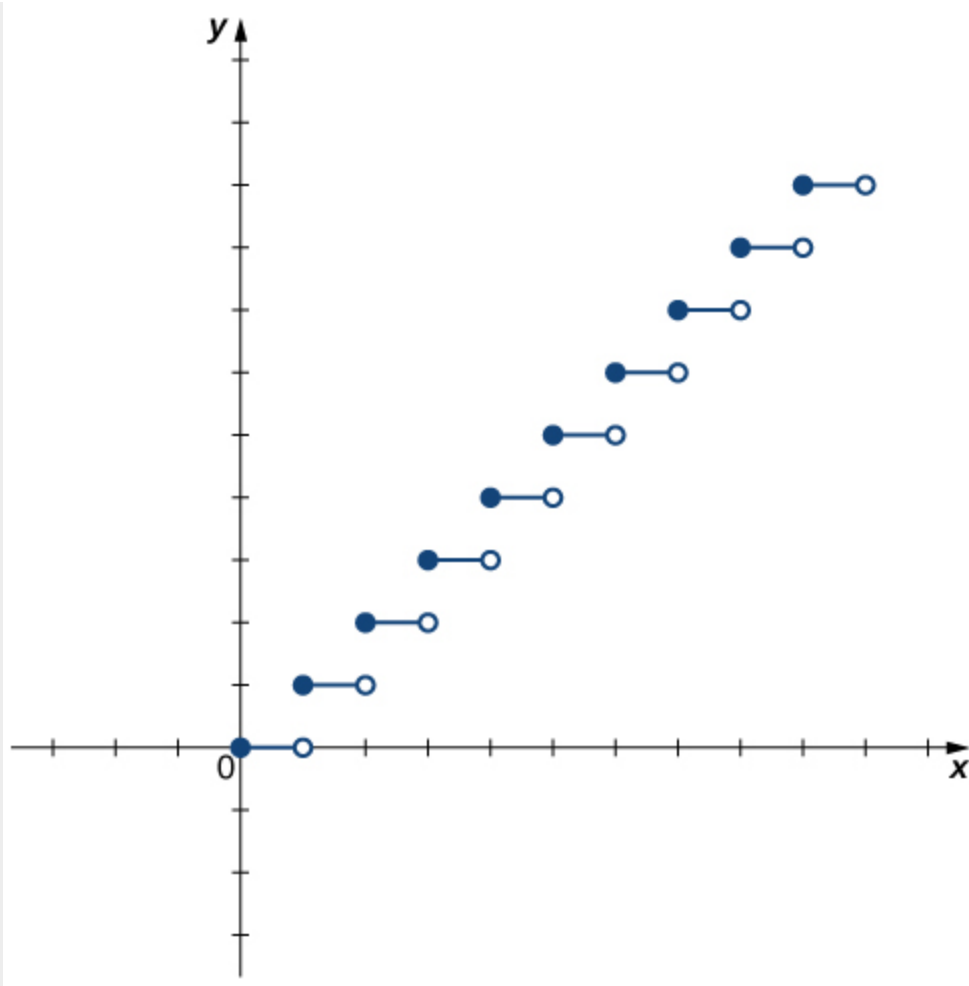
Exercise:

Problem:

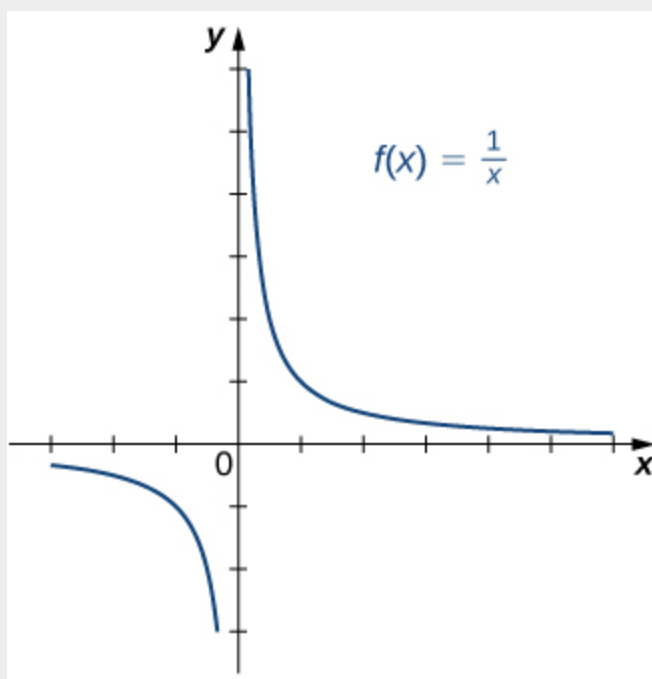
Determining Whether a Function Is One-to-One

For each of the following functions, use the horizontal line test to determine whether it is one-to-one.

a.

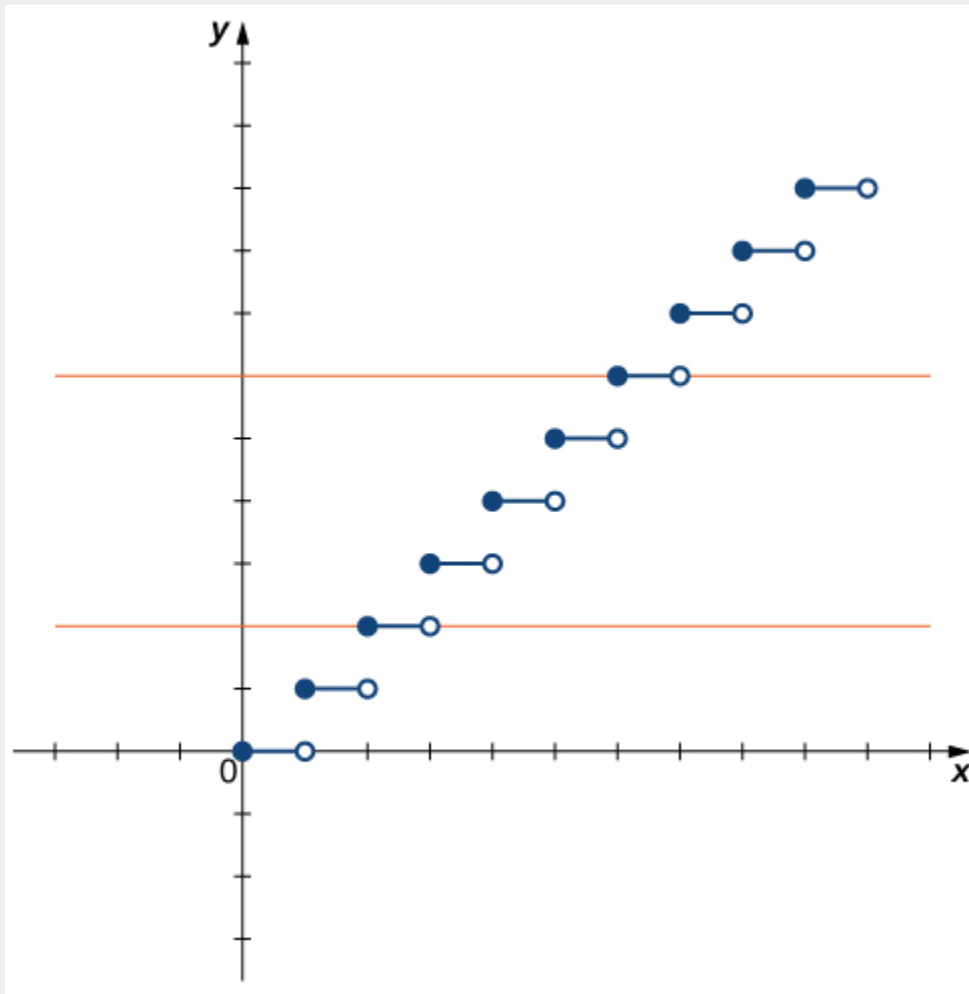


b.

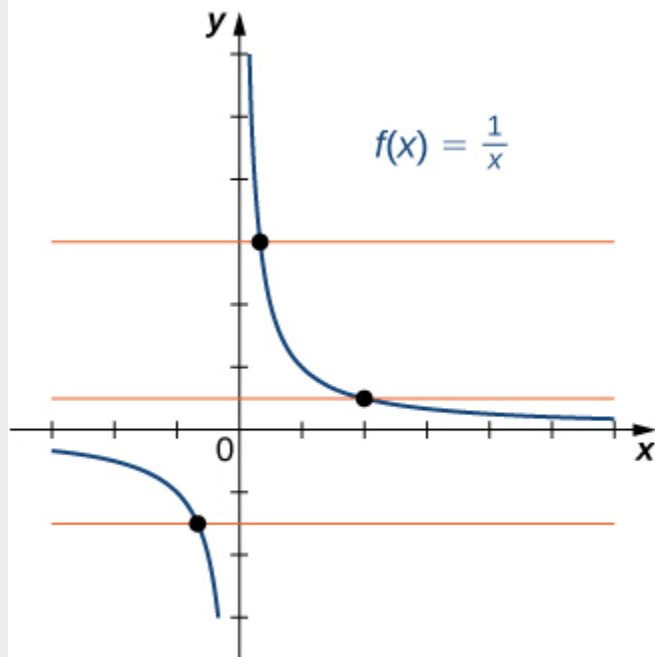


Solution:

- a. Since the horizontal line $y = n$ for any integer $n \geq 0$ intersects the graph more than once, this function is not one-to-one.



- b. Since every horizontal line intersects the graph once (at most), this function is one-to-one.

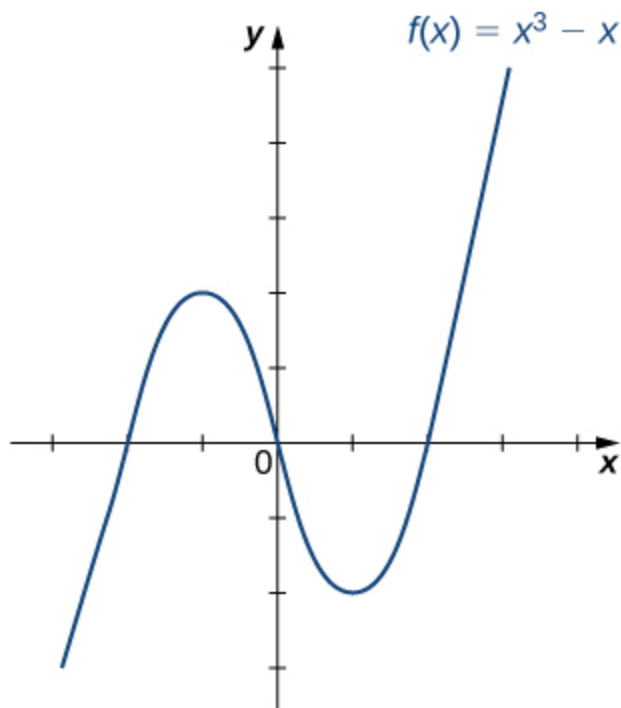


Note:

Exercise:

Problem:

Is the function f graphed in the following image one-to-one?



Solution:

No.

Hint

Use the horizontal line test.

Finding a Function's Inverse

We can now consider one-to-one functions and show how to find their inverses. Recall that a function maps elements in the domain of f to elements in the range of f . The inverse function maps each element from the range of f back to its corresponding element from the domain of f . Therefore, to find the inverse function of a one-to-one function f , given any y in the range of f , we need to determine which x in the domain of f satisfies $f(x) = y$. Since f is one-to-one, there is exactly one such value x . We can find that value x by solving the equation $f(x) = y$ for x . Doing so,

we are able to write x as a function of y where the domain of this function is the range of f and the range of this new function is the domain of f . Consequently, this function is the inverse of f , and we write $x = f^{-1}(y)$. Since we typically use the variable x to denote the independent variable and y to denote the dependent variable, we often interchange the roles of x and y , and write $y = f^{-1}(x)$. Representing the inverse function in this way is also helpful later when we graph a function f and its inverse f^{-1} on the same axes.

Note:**Problem-Solving Strategy: Finding an Inverse Function**

1. Solve the equation $y = f(x)$ for x .
2. Interchange the variables x and y and write $y = f^{-1}(x)$.

Example:**Exercise:****Problem:****Finding an Inverse Function**

Find the inverse for the function $f(x) = 3x - 4$. State the domain and range of the inverse function. Verify that $f^{-1}(f(x)) = x$.

Solution:

Follow the steps outlined in the strategy.

Step 1. If $y = 3x - 4$, then $3x = y + 4$ and $x = \frac{1}{3}y + \frac{4}{3}$.

Step 2. Rewrite as $y = \frac{1}{3}x + \frac{4}{3}$ and let $y = f^{-1}(x)$.

Therefore, $f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$.

Since the domain of f is $(-\infty, \infty)$, the range of f^{-1} is $(-\infty, \infty)$.
Since the range of f is $(-\infty, \infty)$, the domain of f^{-1} is $(-\infty, \infty)$.

You can verify that $f^{-1}(f(x)) = x$ by writing

Equation:

$$f^{-1}(f(x)) = f^{-1}(3x - 4) = \frac{1}{3}(3x - 4) + \frac{4}{3} = x - \frac{4}{3} + \frac{4}{3} = x.$$

Note that for $f^{-1}(x)$ to be the inverse of $f(x)$, both $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for all x in the domain of the inside function.

Note:

Exercise:

Problem:

Find the inverse of the function $f(x) = 3x/(x - 2)$. State the domain and range of the inverse function.

Solution:

$f^{-1}(x) = \frac{2x}{x-3}$. The domain of f^{-1} is $\{x|x \neq 3\}$. The range of f^{-1} is $\{y|y \neq 2\}$.

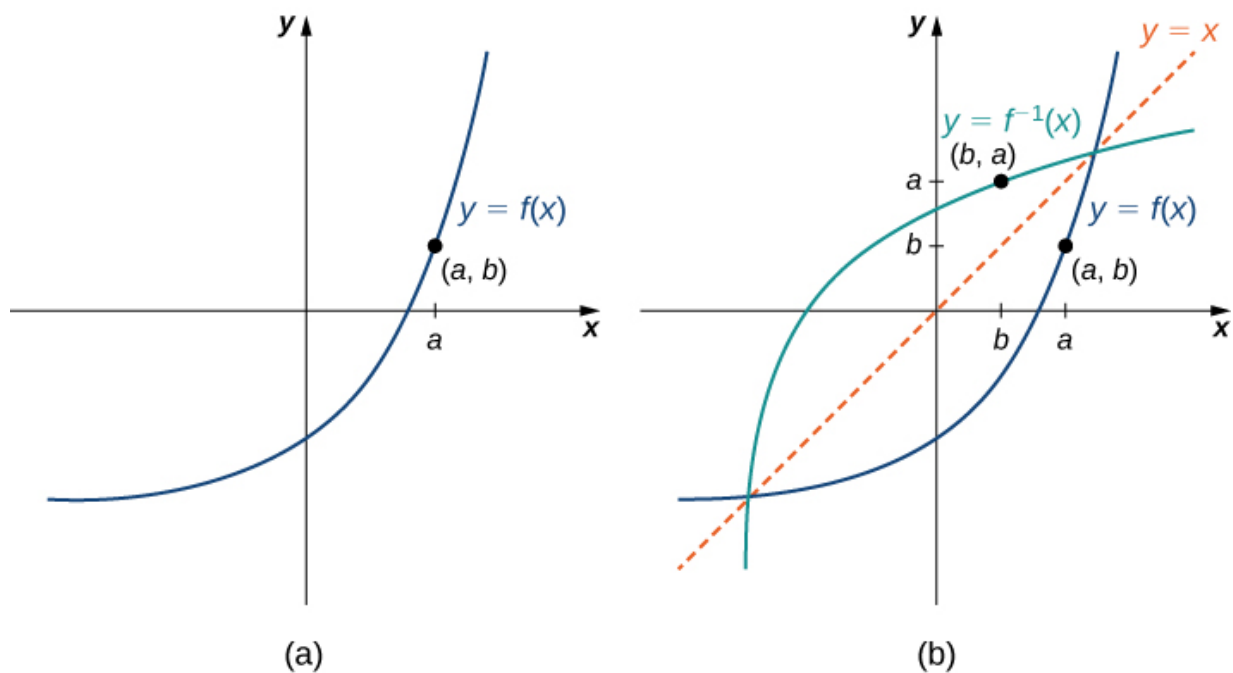
Hint

Use the [\[link\]](#) for finding inverse functions.

Graphing Inverse Functions

Let's consider the relationship between the graph of a function f and the graph of its inverse. Consider the graph of f shown in [\[link\]](#) and a point

(a, b) on the graph. Since $b = f(a)$, then $f^{-1}(b) = a$. Therefore, when we graph f^{-1} , the point (b, a) is on the graph. As a result, the graph of f^{-1} is a reflection of the graph of f about the line $y = x$.



- (a) The graph of this function f shows point (a, b) on the graph of f .
 (b) Since (a, b) is on the graph of f , the point (b, a) is on the graph of f^{-1} . The graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

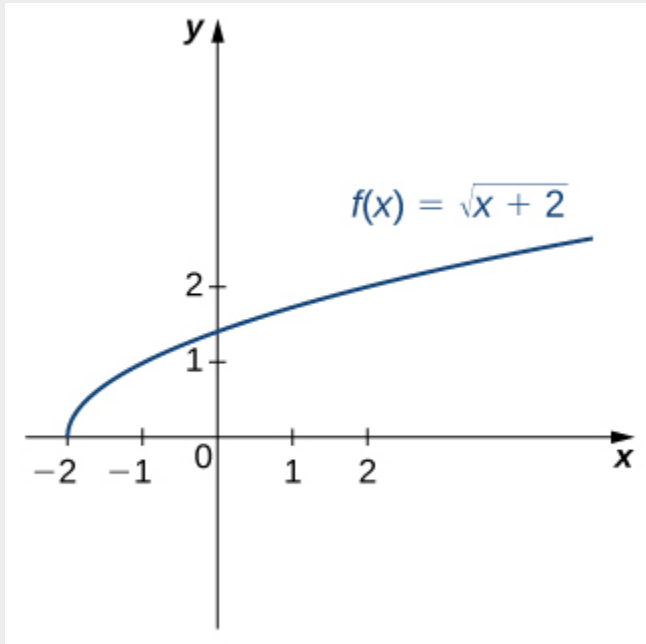
Example:

Exercise:

Problem:

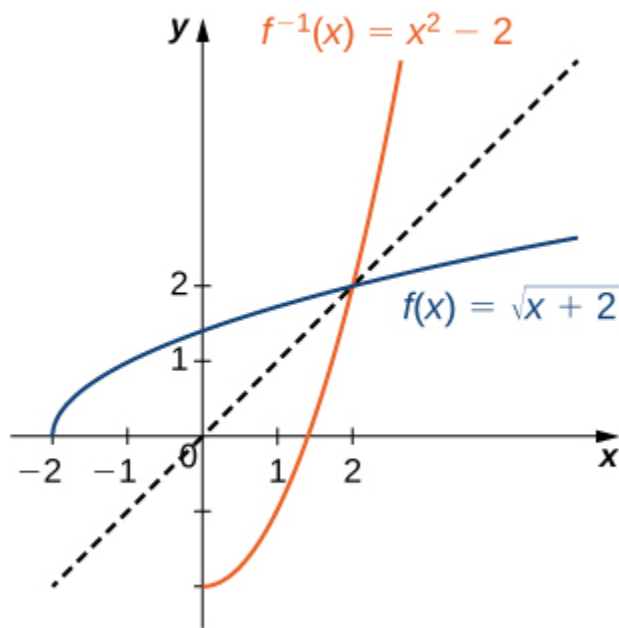
Sketching Graphs of Inverse Functions

For the graph of f in the following image, sketch a graph of f^{-1} by sketching the line $y = x$ and using symmetry. Identify the domain and range of f^{-1} .



Solution:

Reflect the graph about the line $y = x$. The domain of f^{-1} is $[0, \infty)$. The range of f^{-1} is $[-2, \infty)$. By using the preceding strategy for finding inverse functions, we can verify that the inverse function is $f^{-1}(x) = x^2 - 2$, as shown in the graph.



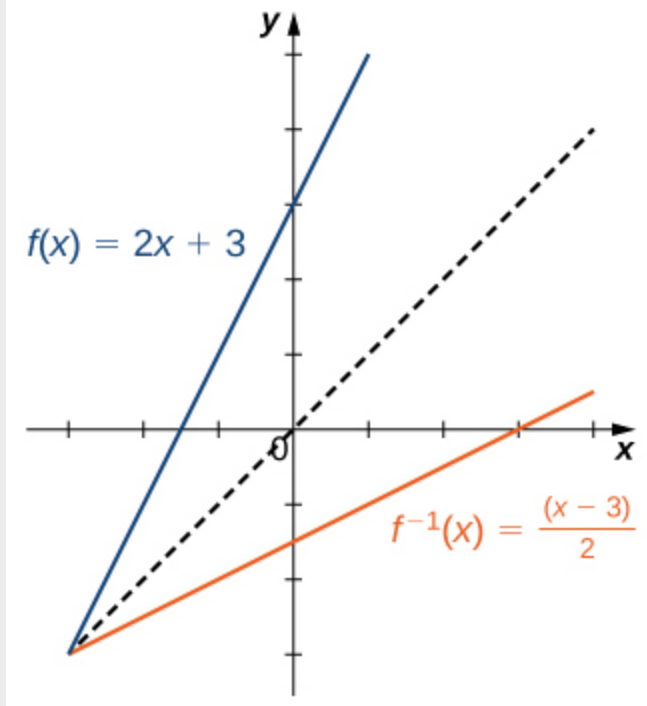
Note:

Exercise:

Problem:

Sketch the graph of $f(x) = 2x + 3$ and the graph of its inverse using the symmetry property of inverse functions.

Solution:



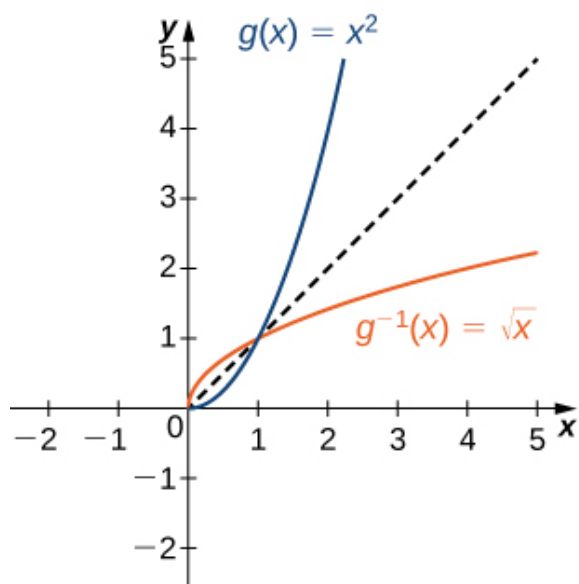
Hint

The graphs are symmetric about the line $y = x$.

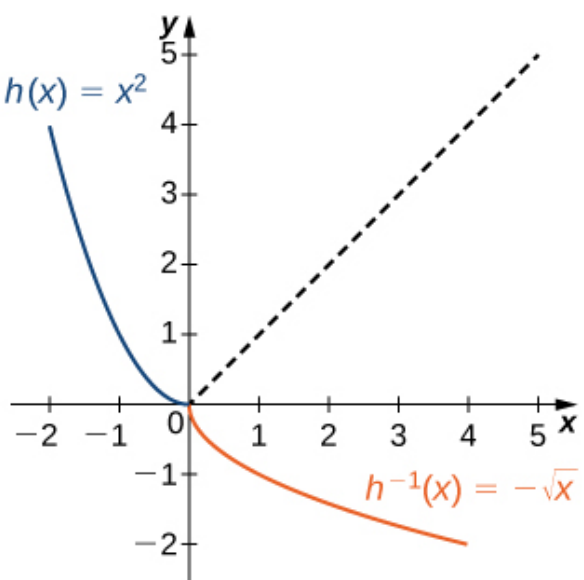
Restricting Domains

As we have seen, $f(x) = x^2$ does not have an inverse function because it is not one-to-one. However, we can choose a subset of the domain of f such that the function is one-to-one. This subset is called a **restricted domain**. By restricting the domain of f , we can define a new function g such that the domain of g is the restricted domain of f and $g(x) = f(x)$ for all x in the domain of g . Then we can define an inverse function for g on that domain. For example, since $f(x) = x^2$ is one-to-one on the interval $[0, \infty)$, we can define a new function g such that the domain of g is $[0, \infty)$ and $g(x) = x^2$ for all x in its domain. Since g is a one-to-one function, it has an inverse function, given by the formula $g^{-1}(x) = \sqrt{x}$. On the other hand, the function $f(x) = x^2$ is also one-to-one on the domain $(-\infty, 0]$. Therefore, we could also define a new function h such that the domain of h is $(-\infty, 0]$

and $h(x) = x^2$ for all x in the domain of h . Then h is a one-to-one function and must also have an inverse. Its inverse is given by the formula $h^{-1}(x) = -\sqrt{x}$ ([link](#)).



(a)



(b)

(a) For $g(x) = x^2$ restricted to $[0, \infty)$, $g^{-1}(x) = \sqrt{x}$. (b) For $h(x) = x^2$ restricted to $(-\infty, 0]$, $h^{-1}(x) = -\sqrt{x}$.

Example:

Exercise:

Problem:

Restricting the Domain

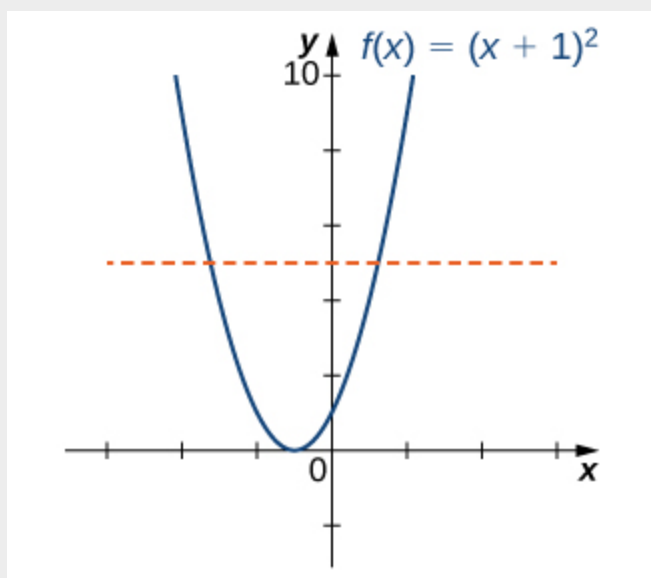
Consider the function $f(x) = (x + 1)^2$.

- Sketch the graph of f and use the horizontal line test to show that f is not one-to-one.

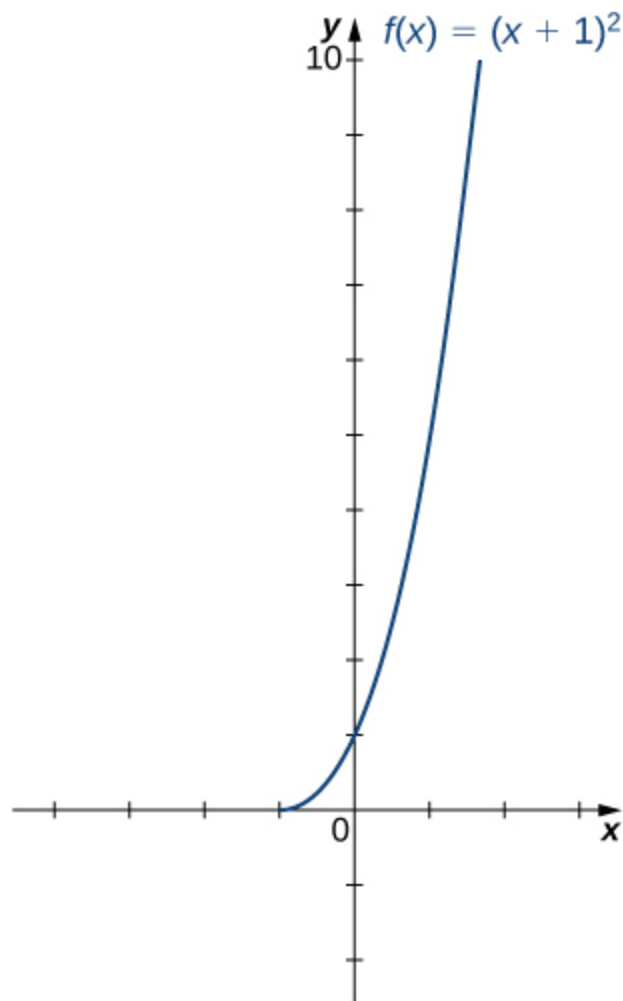
- b. Show that f is one-to-one on the restricted domain $[-1, \infty)$. Determine the domain and range for the inverse of f on this restricted domain and find a formula for f^{-1} .

Solution:

- a. The graph of f is the graph of $y = x^2$ shifted left 1 unit. Since there exists a horizontal line intersecting the graph more than once, f is not one-to-one.



- b. On the interval $[-1, \infty)$, f is one-to-one.



The domain and range of f^{-1} are given by the range and domain of f , respectively. Therefore, the domain of f^{-1} is $[0, \infty)$ and the range of f^{-1} is $[-1, \infty)$. To find a formula for f^{-1} , solve the equation $y = (x + 1)^2$ for x . If $y = (x + 1)^2$, then $x = -1 \pm \sqrt{y}$. Since we are restricting the domain to the interval where $x \geq -1$, we need $\pm\sqrt{y} \geq 0$. Therefore, $x = -1 + \sqrt{y}$. Interchanging x and y , we write $y = -1 + \sqrt{x}$ and conclude that $f^{-1}(x) = -1 + \sqrt{x}$.

Note:

Exercise:

Problem:

Consider $f(x) = 1/x^2$ restricted to the domain $(-\infty, 0)$. Verify that f is one-to-one on this domain. Determine the domain and range of the inverse of f and find a formula for f^{-1} .

Solution:

The domain of f^{-1} is $(0, \infty)$. The range of f^{-1} is $(-\infty, 0)$. The inverse function is given by the formula $f^{-1}(x) = -1/\sqrt{x}$.

Hint

The domain and range of f^{-1} is given by the range and domain of f , respectively. To find f^{-1} , solve $y = 1/x^2$ for x .

Inverse Trigonometric Functions

The six basic trigonometric functions are periodic, and therefore they are not one-to-one. However, if we restrict the domain of a trigonometric function to an interval where it is one-to-one, we can define its inverse. Consider the sine function ([\[link\]](#)). The sine function is one-to-one on an infinite number of intervals, but the standard convention is to restrict the domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. By doing so, we define the inverse sine function on the domain $[-1, 1]$ such that for any x in the interval $[-1, 1]$, the inverse sine function tells us which angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ satisfies $\sin \theta = x$. Similarly, we can restrict the domains of the other trigonometric functions to define **inverse trigonometric functions**, which are functions that tell us which angle in a certain interval has a specified trigonometric value.

Note:**Definition**

The inverse sine function, denoted \sin^{-1} or \arcsin , and the inverse cosine function, denoted \cos^{-1} or \arccos , are defined on the domain

$D = \{x \mid -1 \leq x \leq 1\}$ as follows:

Equation:

$$\sin^{-1}(x) = y \text{ if and only if } \sin(y) = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2};$$

$$\cos^{-1}(x) = y \text{ if and only if } \cos(y) = x \text{ and } 0 \leq y \leq \pi.$$

The inverse tangent function, denoted \tan^{-1} or \arctan , and inverse cotangent function, denoted \cot^{-1} or arccot , are defined on the domain

$D = \{x \mid -\infty < x < \infty\}$ as follows:

Equation:

$$\tan^{-1}(x) = y \text{ if and only if } \tan(y) = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2};$$

$$\cot^{-1}(x) = y \text{ if and only if } \cot(y) = x \text{ and } 0 < y < \pi.$$

The inverse cosecant function, denoted \csc^{-1} or arccsc , and inverse secant function, denoted \sec^{-1} or arcsec , are defined on the domain

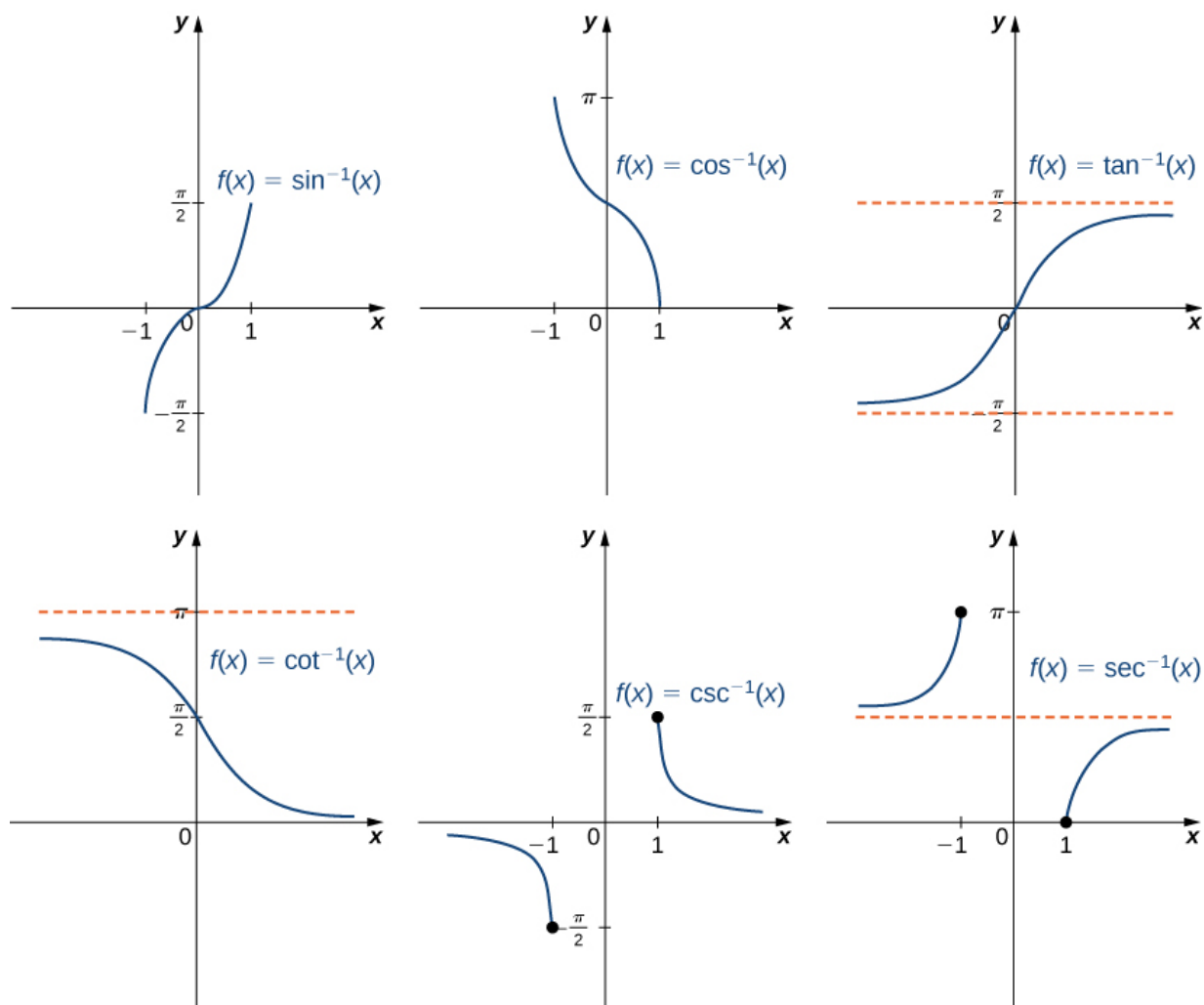
$D = \{x \mid |x| \geq 1\}$ as follows:

Equation:

$$\csc^{-1}(x) = y \text{ if and only if } \csc(y) = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0;$$

$$\sec^{-1}(x) = y \text{ if and only if } \sec(y) = x \text{ and } 0 \leq y \leq \pi, y \neq \pi/2.$$

To graph the inverse trigonometric functions, we use the graphs of the trigonometric functions restricted to the domains defined earlier and reflect the graphs about the line $y = x$ ([link](#)).



The graph of each of the inverse trigonometric functions is a reflection about the line $y = x$ of the corresponding restricted trigonometric function.

Note:

Go to the [following site](#) for more comparisons of functions and their inverses.

When evaluating an inverse trigonometric function, the output is an angle. For example, to evaluate $\cos^{-1}\left(\frac{1}{2}\right)$, we need to find an angle θ such that $\cos\theta = \frac{1}{2}$. Clearly, many angles have this property. However, given the definition of \cos^{-1} , we need the angle θ that not only solves this equation, but also lies in the interval $[0, \pi]$. We conclude that $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

We now consider a composition of a trigonometric function and its inverse. For example, consider the two expressions $\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ and $\sin^{-1}(\sin(\pi))$. For the first one, we simplify as follows:

Equation:

$$\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

For the second one, we have

Equation:

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0.$$

The inverse function is supposed to “undo” the original function, so why isn’t $\sin^{-1}(\sin(\pi)) = \pi$? Recalling our definition of inverse functions, a function f and its inverse f^{-1} satisfy the conditions $f(f^{-1}(y)) = y$ for all y in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f , so what happened here? The issue is that the inverse sine function, \sin^{-1} , is the inverse of the *restricted* sine function defined on the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, for x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it is true that $\sin^{-1}(\sin x) = x$. However, for values of x outside this interval, the equation does not hold, even though $\sin^{-1}(\sin x)$ is defined for all real numbers x .

What about $\sin(\sin^{-1}y)$? Does that have a similar issue? The answer is *no*. Since the domain of \sin^{-1} is the interval $[-1, 1]$, we conclude that $\sin(\sin^{-1}y) = y$ if $-1 \leq y \leq 1$ and the expression is not defined for other values of y . To summarize,

Equation:

$$\sin(\sin^{-1}y) = y \text{ if } -1 \leq y \leq 1$$

and

Equation:

$$\sin^{-1}(\sin x) = x \text{ if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

Similarly, for the cosine function,

Equation:

$$\cos(\cos^{-1}y) = y \text{ if } -1 \leq y \leq 1$$

and

Equation:

$$\cos^{-1}(\cos x) = x \text{ if } 0 \leq x \leq \pi.$$

Similar properties hold for the other trigonometric functions and their inverses.

Example:

Exercise:

Problem:

Evaluating Expressions Involving Inverse Trigonometric Functions

Evaluate each of the following expressions.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

- b. $\tan \left(\tan^{-1} \left(-\frac{1}{\sqrt{3}} \right) \right)$
- c. $\cos^{-1} \left(\cos \left(\frac{5\pi}{4} \right) \right)$
- d. $\sin^{-1} \left(\cos \left(\frac{2\pi}{3} \right) \right)$

Solution:

- a. Evaluating $\sin^{-1} \left(-\sqrt{3}/2 \right)$ is equivalent to finding the angle θ such that $\sin \theta = -\sqrt{3}/2$ and $-\pi/2 \leq \theta \leq \pi/2$. The angle $\theta = -\pi/3$ satisfies these two conditions. Therefore,

$$\sin^{-1} \left(-\sqrt{3}/2 \right) = -\pi/3.$$
- b. First we use the fact that $\tan^{-1} \left(-1/\sqrt{3} \right) = -\pi/6$. Then

$$\tan \left(\tan^{-1} \left(-1/\sqrt{3} \right) \right) = -1/\sqrt{3}.$$
Therefore,

$$\tan \left(\tan^{-1} \left(-1/\sqrt{3} \right) \right) = -1/\sqrt{3}.$$
- c. To evaluate $\cos^{-1} \left(\cos \left(5\pi/4 \right) \right)$, first use the fact that $\cos \left(5\pi/4 \right) = -\sqrt{2}/2$. Then we need to find the angle θ such that $\cos \left(\theta \right) = -\sqrt{2}/2$ and $0 \leq \theta \leq \pi$. Since $3\pi/4$ satisfies both these conditions, we have

$$\cos \left(\cos^{-1} \left(5\pi/4 \right) \right) = \cos \left(\cos^{-1} \left(-\sqrt{2}/2 \right) \right) = 3\pi/4.$$
- d. Since $\cos \left(2\pi/3 \right) = -1/2$, we need to evaluate $\sin^{-1} \left(-1/2 \right)$. That is, we need to find the angle θ such that $\sin \left(\theta \right) = -1/2$ and $-\pi/2 \leq \theta \leq \pi/2$. Since $-\pi/6$ satisfies both these conditions, we can conclude that

$$\sin^{-1} \left(\cos \left(2\pi/3 \right) \right) = \sin^{-1} \left(-1/2 \right) = -\pi/6.$$

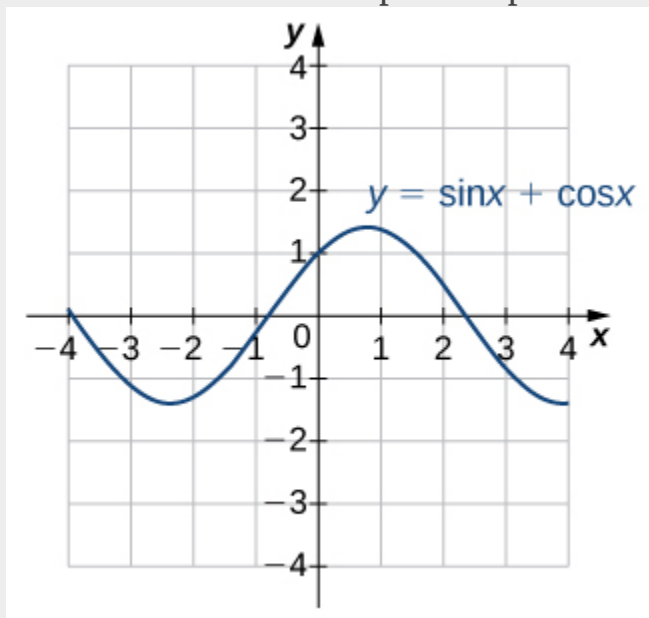
Note:

The Maximum Value of a Function

In many areas of science, engineering, and mathematics, it is useful to know the maximum value a function can obtain, even if we don't know its exact value at a given instant. For instance, if we have a function describing the strength of a roof beam, we would want to know the maximum weight the beam can support without breaking. If we have a function that describes the speed of a train, we would want to know its maximum speed before it jumps off the rails. Safe design often depends on knowing maximum values.

This project describes a simple example of a function with a maximum value that depends on two equation coefficients. We will see that maximum values can depend on several factors other than the independent variable x .

1. Consider the graph in [\[link\]](#) of the function $y = \sin x + \cos x$. Describe its overall shape. Is it periodic? How do you know?



The graph of $y = \sin x + \cos x$.

Using a graphing calculator or other graphing device, estimate the x - and y -values of the maximum point for the graph (the first such point where $x > 0$). It may be helpful to express the x -value as a multiple of π .

2. Now consider other graphs of the form $y = A\sin x + B\cos x$ for various values of A and B . Sketch the graph when $A = 2$ and $B = 1$, and find the x - and y -values for the maximum point. (Remember to express the x -value as a multiple of π , if possible.) Has it moved?
3. Repeat for $A = 1, B = 2$. Is there any relationship to what you found in part (2)?
4. Complete the following table, adding a few choices of your own for A and B :

A	B	x	y		A	B	x	y
0	1				$\sqrt{3}$	1		
1	0				1	$\sqrt{3}$		
1	1				12	5		
1	2				5	12		
2	1							
2	2							
3	4							
4	3							

5. Try to figure out the formula for the y -values.
6. The formula for the x -values is a little harder. The most helpful points from the table are $(1, 1)$, $(1, \sqrt{3})$, $(\sqrt{3}, 1)$. (*Hint: Consider inverse trigonometric functions.*)

7. If you found formulas for parts (5) and (6), show that they work together. That is, substitute the x -value formula you found into $y = A \sin x + B \cos x$ and simplify it to arrive at the y -value formula you found.

Key Concepts

- For a function to have an inverse, the function must be one-to-one. Given the graph of a function, we can determine whether the function is one-to-one by using the horizontal line test.
- If a function is not one-to-one, we can restrict the domain to a smaller domain where the function is one-to-one and then define the inverse of the function on the smaller domain.
- For a function f and its inverse f^{-1} , $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f .
- Since the trigonometric functions are periodic, we need to restrict their domains to define the inverse trigonometric functions.
- The graph of a function f and its inverse f^{-1} are symmetric about the line $y = x$.

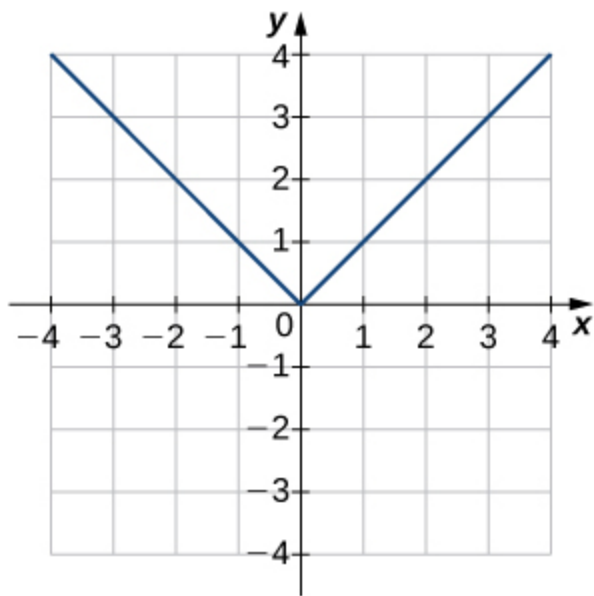
Key Equations

- **Inverse functions**
 $f^{-1}(f(x)) = x$ for all x in D , and $f(f^{-1}(y)) = y$ for all y in R .

For the following exercises, use the horizontal line test to determine whether each of the given graphs is one-to-one.

Exercise:

Problem:

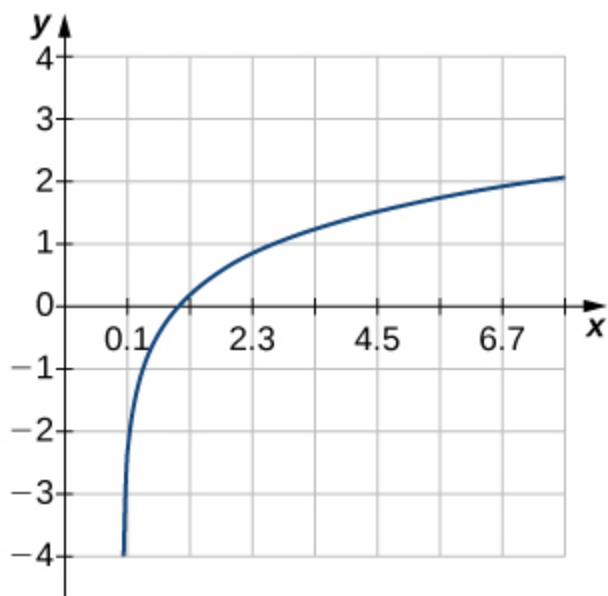


Solution:

Not one-to-one

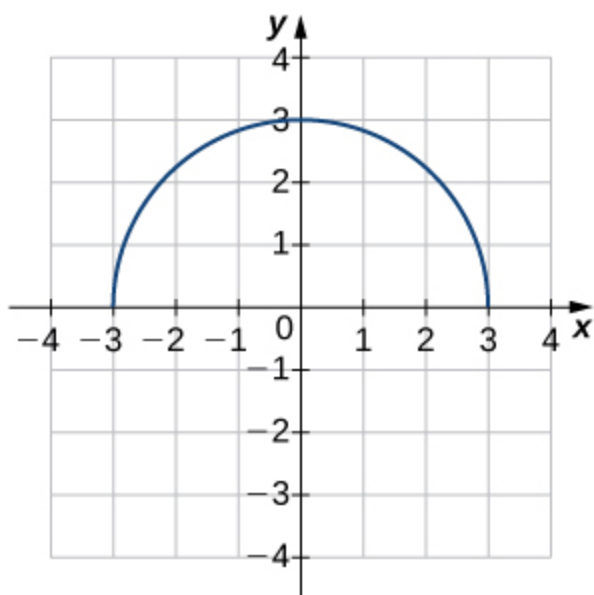
Exercise:

Problem:



Exercise:

Problem:

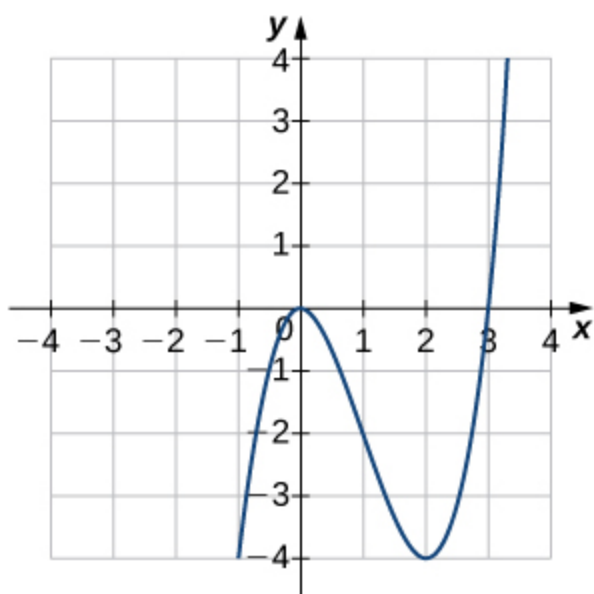


Solution:

Not one-to-one

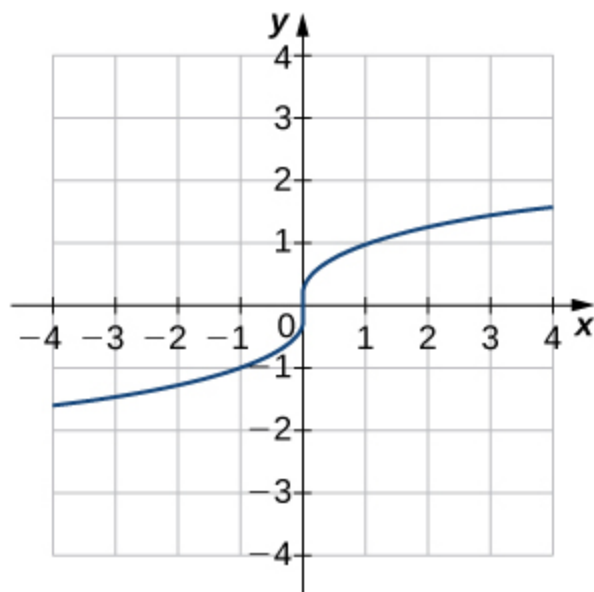
Exercise:

Problem:



Exercise:

Problem:

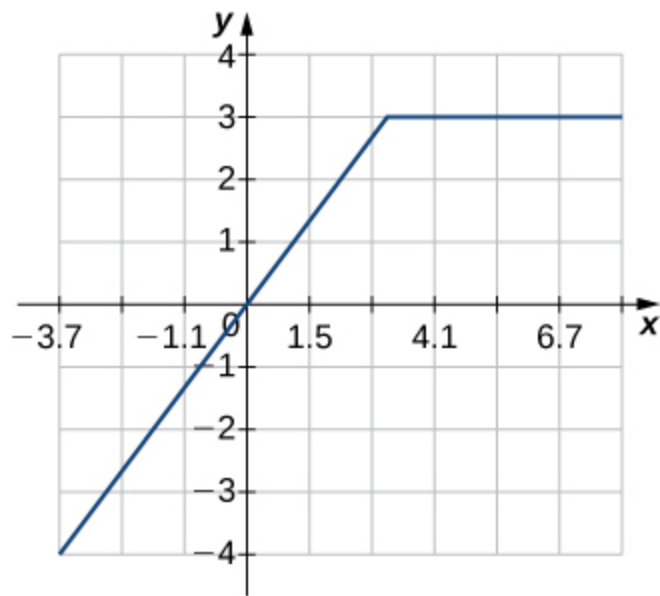


Solution:

One-to-one

Exercise:

Problem:



For the following exercises, a. find the inverse function, and b. find the domain and range of the inverse function.

Exercise:

Problem: $f(x) = x^2 - 4, x \geq 0$

Solution:

a. $f^{-1}(x) = \sqrt{x + 4}$ b. Domain : $x \geq -4$, range: $y \geq 0$

Exercise:

Problem: $f(x) = \sqrt[3]{x - 4}$

Exercise:

Problem: $f(x) = x^3 + 1$

Solution:

a. $f^{-1}(x) = \sqrt[3]{x-1}$ b. Domain: all real numbers, range: all real numbers

Exercise:

Problem: $f(x) = (x-1)^2, x \leq 1$

Exercise:

Problem: $f(x) = \sqrt{x-1}$

Solution:

a. $f^{-1}(x) = x^2 + 1$, b. Domain: $x \geq 0$, range: $y \geq 1$

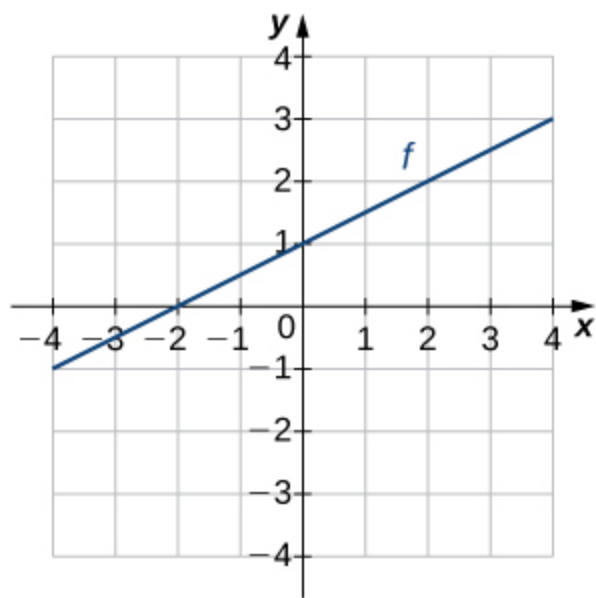
Exercise:

Problem: $f(x) = \frac{1}{x+2}$

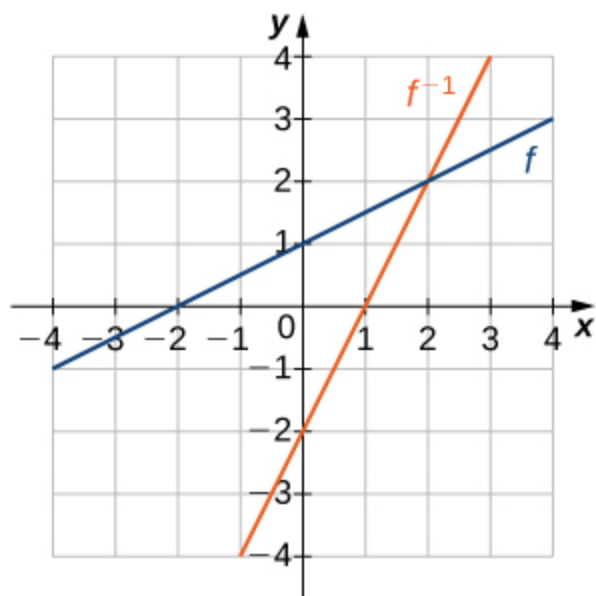
For the following exercises, use the graph of f to sketch the graph of its inverse function.

Exercise:

Problem:

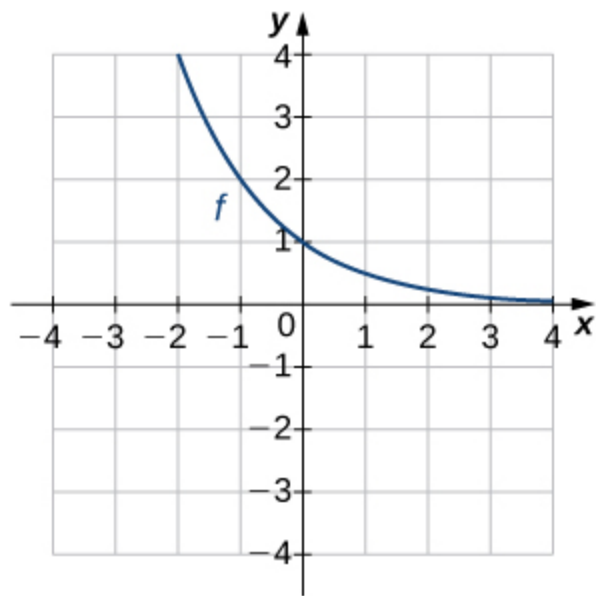


Solution:



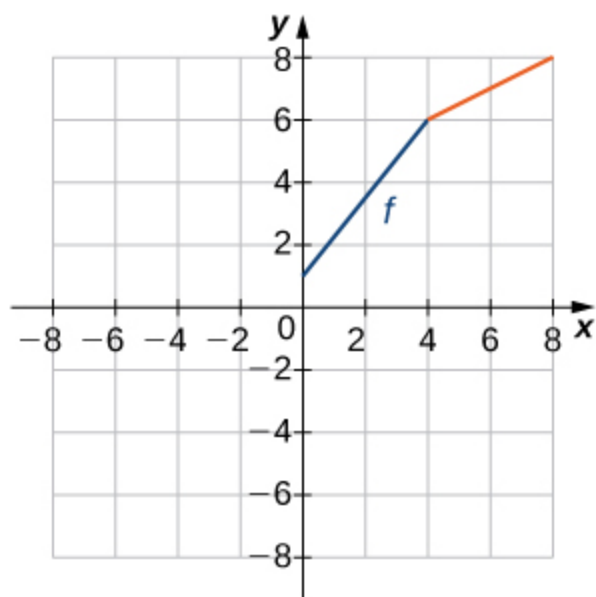
Exercise:

Problem:

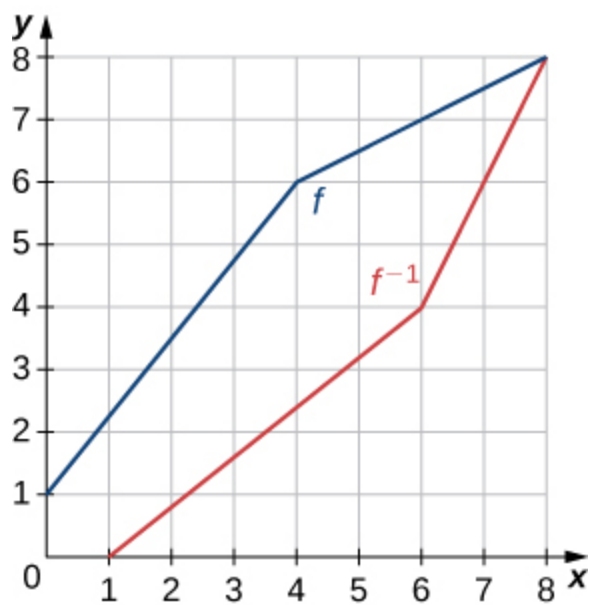


Exercise:

Problem:

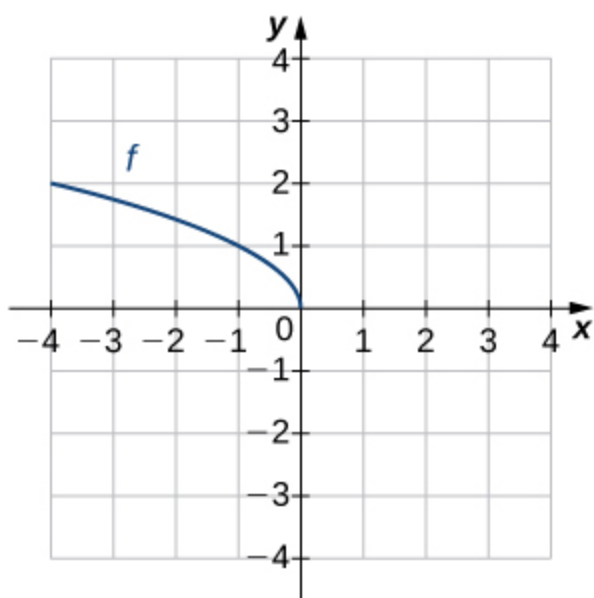


Solution:



Exercise:

Problem:



For the following exercises, use composition to determine which pairs of functions are inverses.

Exercise:

Problem: $f(x) = 8x, g(x) = \frac{x}{8}$

Solution:

These are inverses.

Exercise:

Problem: $f(x) = 8x + 3, g(x) = \frac{x-3}{8}$

Exercise:

Problem: $f(x) = 5x - 7, g(x) = \frac{x+5}{7}$

Solution:

These are not inverses.

Exercise:

Problem: $f(x) = \frac{2}{3}x + 2, g(x) = \frac{3}{2}x + 3$

Exercise:

Problem: $f(x) = \frac{1}{x-1}, x \neq 1, g(x) = \frac{1}{x} + 1, x \neq 0$

Solution:

These are inverses.

Exercise:

Problem: $f(x) = x^3 + 1, g(x) = (x - 1)^{1/3}$

Exercise:

Problem: $f(x) = x^2 + 2x + 1, x \geq -1, \quad g(x) = -1 + \sqrt{x}, x \geq 0$

Solution:

These are inverses.

Exercise:

Problem:

$$f(x) = \sqrt{4 - x^2}, 0 \leq x \leq 2, g(x) = \sqrt{4 - x^2}, 0 \leq x \leq 2$$

For the following exercises, evaluate the functions. Give the exact value.

Exercise:

Problem: $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

Solution:

$$\frac{\pi}{6}$$

Exercise:

Problem: $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Exercise:

Problem: $\cot^{-1}(1)$

Solution:

$$\frac{\pi}{4}$$

Exercise:

Problem: $\sin^{-1}(-1)$

Exercise:

Problem: $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution:

$$\frac{\pi}{6}$$

Exercise:

Problem: $\cos\left(\tan^{-1}\left(\sqrt{3}\right)\right)$

Exercise:

Problem: $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

Solution:

$$\frac{\sqrt{2}}{2}$$

Exercise:

Problem: $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

Exercise:

Problem: $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

Solution:

$$-\frac{\pi}{6}$$

Exercise:

Problem:

The function $C = T(F) = (5/9)(F - 32)$ converts degrees Fahrenheit to degrees Celsius.

- Find the inverse function $F = T^{-1}(C)$
- What is the inverse function used for?

Exercise:**Problem:**

[T] The velocity V (in centimeters per second) of blood in an artery at a distance x cm from the center of the artery can be modeled by the function $V = f(x) = 500(0.04 - x^2)$ for $0 \leq x \leq 0.2$.

- Find $x = f^{-1}(V)$.
- Interpret what the inverse function is used for.
- Find the distance from the center of an artery with a velocity of 15 cm/sec, 10 cm/sec, and 5 cm/sec.

Solution:

- a. $x = f^{-1}(V) = \sqrt{0.04 - \frac{V}{500}}$ b. The inverse function determines the distance from the center of the artery at which blood is flowing with velocity V . c. 0.1 cm; 0.14 cm; 0.17 cm

Exercise:**Problem:**

A function that converts dress sizes in the United States to those in Europe is given by $D(x) = 2x + 24$.

- Find the European dress sizes that correspond to sizes 6, 8, 10, and 12 in the United States.

- b. Find the function that converts European dress sizes to U.S. dress sizes.
- c. Use part b. to find the dress sizes in the United States that correspond to 46, 52, 62, and 70.

Exercise:

Problem:

[T] The cost to remove a toxin from a lake is modeled by the function

$C(p) = 75p/(85 - p)$, where C is the cost (in thousands of dollars) and p is the amount of toxin in a small lake (measured in parts per billion [ppb]). This model is valid only when the amount of toxin is less than 85 ppb.

- a. Find the cost to remove 25 ppb, 40 ppb, and 50 ppb of the toxin from the lake.
- b. Find the inverse function. c. Use part b. to determine how much of the toxin is removed for \$50,000.

Solution:

- a. \$31,250, \$66,667, \$107,143 b. $(p = \frac{85C}{C+75})$ c. 34 ppb

Exercise:

Problem: [T] A race car is accelerating at a velocity given by

$$v(t) = \frac{25}{4}t + 54,$$

where v is the velocity (in feet per second) at time t .

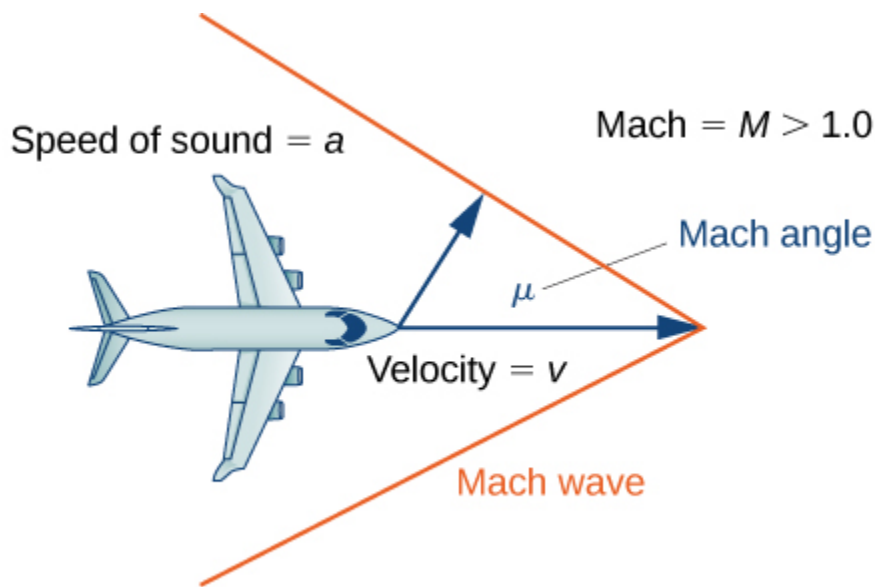
- a. Find the velocity of the car at 10 sec.
- b. Find the inverse function.
- c. Use part b. to determine how long it takes for the car to reach a speed of 150 ft/sec.

Exercise:

Problem:

[T] An airplane's Mach number M is the ratio of its speed to the speed of sound. When a plane is flying at a constant altitude, then its Mach angle is given by $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$.

Find the Mach angle (to the nearest degree) for the following Mach numbers.



- a. $\mu = 1.4$
- b. $\mu = 2.8$
- c. $\mu = 4.3$

Solution:

- a. $\sim 92^\circ$ b. $\sim 42^\circ$ c. $\sim 27^\circ$

Exercise:

Problem:

[T] Using $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$, find the Mach number M for the following angles.

- a. $\mu = \frac{\pi}{6}$
- b. $\mu = \frac{2\pi}{7}$
- c. $\mu = \frac{3\pi}{8}$

Exercise:**Problem:**

[T] The temperature (in degrees Celsius) of a city in the northern United States can be modeled by the function

$$T(x) = 5 + 18\sin\left[\frac{\pi}{6}(x - 4.6)\right],$$

where x is time in months and $x = 1.00$ corresponds to January 1. Determine the month and day when the temperature is 21°C .

Solution:

$x \approx 6.69, 8.51$; so, the temperature occurs on June 21 and August 15

Exercise:**Problem:**

[T] The depth (in feet) of water at a dock changes with the rise and fall of tides. It is modeled by the function

$$D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8,$$

where t is the number of hours after midnight. Determine the first time after midnight when the depth is 11.75 ft.

Exercise:

Problem:

[T] An object moving in simple harmonic motion is modeled by the function

$$s(t) = -6 \cos\left(\frac{\pi t}{2}\right),$$

where s is measured in inches and t is measured in seconds. Determine the first time when the distance moved is 4.5 in.

Solution:

~1.5 sec

Exercise:**Problem:**

[T] A local art gallery has a portrait 3 ft in height that is hung 2.5 ft above the eye level of an average person. The viewing angle θ can be modeled by the function

$$\theta = \tan^{-1} \frac{5.5}{x} - \tan^{-1} \frac{2.5}{x},$$

where x is the distance (in feet) from the portrait. Find the viewing angle when a person is 4 ft from the portrait.

Exercise:**Problem:**

[T] Use a calculator to evaluate $\tan^{-1}(\tan(2.1))$ and $\cos^{-1}(\cos(2.1))$. Explain the results of each.

Solution:

$\tan^{-1}(\tan(2.1)) \approx -1.0416$; the expression does not equal 2.1 since $2.1 > 1.57 = \frac{\pi}{2}$ —in other words, it is not in the restricted domain of

$\tan x \cdot \cos^{-1}(\cos(2.1)) = 2.1$, since 2.1 is in the restricted domain of $\cos x$.

Exercise:

Problem:

[T] Use a calculator to evaluate $\sin(\sin^{-1}(-2))$ and $\tan(\tan^{-1}(-2))$. Explain the results of each.

Glossary

horizontal line test

a function f is one-to-one if and only if every horizontal line intersects the graph of f , at most, once

inverse function

for a function f , the inverse function f^{-1} satisfies $f^{-1}(y) = x$ if $f(x) = y$

inverse trigonometric functions

the inverses of the trigonometric functions are defined on restricted domains where they are one-to-one functions

one-to-one function

a function f is one-to-one if $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

restricted domain

a subset of the domain of a function f

Exponential and Logarithmic Functions

- Identify the form of an exponential function.
- Explain the difference between the graphs of x^b and b^x .
- Recognize the significance of the number e .
- Identify the form of a logarithmic function.
- Explain the relationship between exponential and logarithmic functions.
- Describe how to calculate a logarithm to a different base.
- Identify the hyperbolic functions, their graphs, and basic identities.

In this section we examine exponential and logarithmic functions. We use the properties of these functions to solve equations involving exponential or logarithmic terms, and we study the meaning and importance of the number e . We also define hyperbolic and inverse hyperbolic functions, which involve combinations of exponential and logarithmic functions. (Note that we present alternative definitions of exponential and logarithmic functions in the chapter [Applications of Integrations](#), and prove that the functions have the same properties with either definition.)

Exponential Functions

Exponential functions arise in many applications. One common example is population growth.

For example, if a population starts with P_0 individuals and then grows at an annual rate of 2%, its population after 1 year is

Equation:

$$P(1) = P_0 + 0.02P_0 = P_0(1 + 0.02) = P_0(1.02).$$

Its population after 2 years is

Equation:

$$P(2) = P(1) + 0.02P(1) = P(1)(1.02) = P_0(1.02)^2.$$

In general, its population after t years is

Equation:

$$P(t) = P_0(1.02)^t,$$

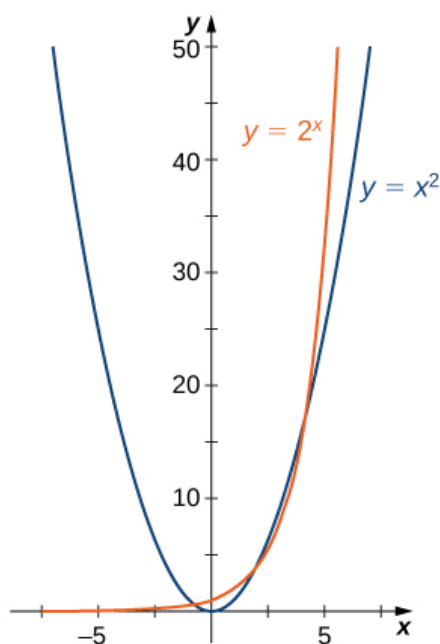
which is an exponential function. More generally, any function of the form $f(x) = b^x$, where $b > 0$, $b \neq 1$, is an exponential function with **base** b and **exponent** x . Exponential functions have constant bases and variable exponents. Note that a function of the form $f(x) = x^b$ for some constant b is not an exponential function but a power function.

To see the difference between an exponential function and a power function, we compare the functions $y = x^2$ and $y = 2^x$. In [\[link\]](#), we see that both 2^x and x^2 approach infinity as $x \rightarrow \infty$. Eventually, however, 2^x becomes larger than x^2 and grows more rapidly as $x \rightarrow \infty$. In the opposite direction, as $x \rightarrow -\infty$, $x^2 \rightarrow \infty$, whereas $2^x \rightarrow 0$. The line $y = 0$ is a horizontal asymptote for $y = 2^x$.

x	-3	-2	-1	0	1	2	3	4	5	6
x^2	9	4	1	0	1	4	9	16	25	36
2^x	1/8	1/4	1/2	1	2	4	8	16	32	64

Values of x^2 and 2^x

In [\[link\]](#), we graph both $y = x^2$ and $y = 2^x$ to show how the graphs differ.



Both 2^x and x^2 approach infinity as $x \rightarrow \infty$, but 2^x grows more rapidly than x^2 . As $x \rightarrow -\infty$, $x^2 \rightarrow \infty$, whereas $2^x \rightarrow 0$.

Evaluating Exponential Functions

Recall the properties of exponents: If x is a positive integer, then we define $b^x = b \cdot b \cdots b$ (with x factors of b). If x is a negative integer, then $x = -y$ for some positive integer y , and we define $b^x = b^{-y} = 1/b^y$. Also, b^0 is defined to be 1. If x is a rational number, then $x = p/q$, where p and q are integers and $b^x = b^{p/q} = \sqrt[q]{b^p}$. For example, $9^{3/2} = \sqrt{9^3} = 27$. However, how is b^x defined if x is an irrational number? For example, what do we mean by $2^{\sqrt{2}}$? This is too complex a question for

us to answer fully right now; however, we can make an approximation. In [\[link\]](#), we list some rational numbers approaching $\sqrt{2}$, and the values of 2^x for each rational number x are presented as well. We claim that if we choose rational numbers x getting closer and closer to $\sqrt{2}$, the values of 2^x get closer and closer to some number L . We define that number L to be $2^{\sqrt{2}}$.

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
2^x	2.639	2.65737	2.66475	2.665119	2.665138	2.665143

Values of 2^x for a List of Rational Numbers Approximating $\sqrt{2}$

Example:

Exercise:

Problem:

Bacterial Growth

Suppose a particular population of bacteria is known to double in size every 4 hours. If a culture starts with 1000 bacteria, the number of bacteria after 4 hours is $n(4) = 1000 \cdot 2$. The number of bacteria after 8 hours is $n(8) = n(4) \cdot 2 = 1000 \cdot 2^2$. In general, the number of bacteria after $4m$ hours is $n(4m) = 1000 \cdot 2^m$. Letting $t = 4m$, we see that the number of bacteria after t hours is $n(t) = 1000 \cdot 2^{t/4}$. Find the number of bacteria after 6 hours, 10 hours, and 24 hours.

Solution:

The number of bacteria after 6 hours is given by $n(6) = 1000 \cdot 2^{6/4} \approx 2828$ bacteria. The number of bacteria after 10 hours is given by $n(10) = 1000 \cdot 2^{10/4} \approx 5657$ bacteria. The number of bacteria after 24 hours is given by $n(24) = 1000 \cdot 2^6 = 64,000$ bacteria.

Note:

Exercise:

Problem: Given the exponential function $f(x) = 100 \cdot 3^{x/2}$, evaluate $f(4)$ and $f(10)$.

Solution:

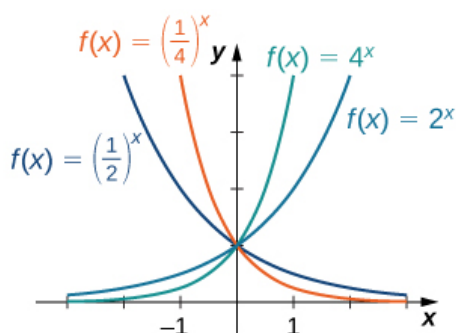
$f(4) = 900$; $f(10) = 24,300$.

Note:

Go to [World Population Balance](#) for another example of exponential population growth.

Graphing Exponential Functions

For any base $b > 0, b \neq 1$, the exponential function $f(x) = b^x$ is defined for all real numbers x and $b^x > 0$. Therefore, the domain of $f(x) = b^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. To graph b^x , we note that for $b > 1$, b^x is increasing on $(-\infty, \infty)$ and $b^x \rightarrow \infty$ as $x \rightarrow \infty$, whereas $b^x \rightarrow 0$ as $x \rightarrow -\infty$. On the other hand, if $0 < b < 1$, $f(x) = b^x$ is decreasing on $(-\infty, \infty)$ and $b^x \rightarrow 0$ as $x \rightarrow \infty$ whereas $b^x \rightarrow \infty$ as $x \rightarrow -\infty$ ([link](#)).



If $b > 1$, then b^x is increasing on $(-\infty, \infty)$. If $0 < b < 1$, then b^x is decreasing on $(-\infty, \infty)$.

Note:

Visit this [site](#) for more exploration of the graphs of exponential functions.

Note that exponential functions satisfy the general laws of exponents. To remind you of these laws, we state them as rules.

Note:**Rule: Laws of Exponents**

For any constants $a > 0, b > 0$, and for all x and y ,

1. $b^x \cdot b^y = b^{x+y}$
2. $\frac{b^x}{b^y} = b^{x-y}$

3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

Example:

Exercise:

Problem:

Using the Laws of Exponents

Use the laws of exponents to simplify each of the following expressions.

a. $\frac{(2x^{2/3})^3}{(4x^{-1/3})^2}$

b. $\frac{(x^3y^{-1})^2}{(xy^2)^{-2}}$

Solution:

a. We can simplify as follows:

Equation:

$$\frac{(2x^{2/3})^3}{(4x^{-1/3})^2} = \frac{2^3(x^{2/3})^3}{4^2(x^{-1/3})^2} = \frac{8x^2}{16x^{-2/3}} = \frac{x^2x^{2/3}}{2} = \frac{x^{8/3}}{2}.$$

b. We can simplify as follows:

Equation:

$$\frac{(x^3y^{-1})^2}{(xy^2)^{-2}} = \frac{(x^3)^2(y^{-1})^2}{x^{-2}(y^2)^{-2}} = \frac{x^6y^{-2}}{x^{-2}y^{-4}} = x^6x^2y^{-2}y^4 = x^8y^2.$$

Note:

Exercise:

Problem: Use the laws of exponents to simplify $(6x^{-3}y^2) / (12x^{-4}y^5)$.

Solution:

$$x / (2y^3)$$

Hint

$$x^a/x^b = x^{a-b}$$

The Number e

A special type of exponential function appears frequently in real-world applications. To describe it, consider the following example of exponential growth, which arises from compounding interest in a savings account. Suppose a person invests P dollars in a savings account with an annual interest rate r , compounded annually. The amount of money after 1 year is

Equation:

$$A(1) = P + rP = P(1 + r).$$

The amount of money after 2 years is

Equation:

$$A(2) = A(1) + rA(1) = P(1 + r) + rP(1 + r) = P(1 + r)^2.$$

More generally, the amount after t years is

Equation:

$$A(t) = P(1 + r)^t.$$

If the money is compounded 2 times per year, the amount of money after half a year is

Equation:

$$A\left(\frac{1}{2}\right) = P + \left(\frac{r}{2}\right)P = P\left(1 + \left(\frac{r}{2}\right)\right).$$

The amount of money after 1 year is

Equation:

$$A(1) = A\left(\frac{1}{2}\right) + \left(\frac{r}{2}\right)A\left(\frac{1}{2}\right) = P\left(1 + \frac{r}{2}\right) + \frac{r}{2}\left(P\left(1 + \frac{r}{2}\right)\right) = P\left(1 + \frac{r}{2}\right)^2.$$

After t years, the amount of money in the account is

Equation:

$$A(t) = P\left(1 + \frac{r}{2}\right)^{2t}.$$

More generally, if the money is compounded n times per year, the amount of money in the account after t years is given by the function

Equation:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

What happens as $n \rightarrow \infty$? To answer this question, we let $m = n/r$ and write

Equation:

$$\left(1 + \frac{r}{n}\right)^{nt} = \left(1 + \frac{1}{m}\right)^{mrt},$$

and examine the behavior of $\left(1 + 1/m\right)^m$ as $m \rightarrow \infty$, using a table of values ([link](#)).

m	10	100	1000	10,000	100,000	1,000,000
$\left(1 + \frac{1}{m}\right)^m$	2.5937	2.7048	2.71692	2.71815	2.718268	2.718280

Values of $\left(1 + \frac{1}{m}\right)^m$ as $m \rightarrow \infty$

Looking at this table, it appears that $\left(1 + 1/m\right)^m$ is approaching a number between 2.7 and 2.8 as $m \rightarrow \infty$. In fact, $\left(1 + 1/m\right)^m$ does approach some number as $m \rightarrow \infty$. We call this **number** e . To six decimal places of accuracy,

Equation:

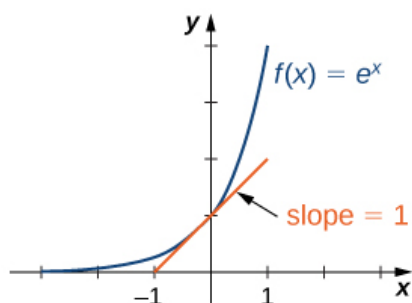
$$e \approx 2.718282.$$

The letter e was first used to represent this number by the Swiss mathematician Leonhard Euler during the 1720s. Although Euler did not discover the number, he showed many important connections between e and logarithmic functions. We still use the notation e today to honor Euler's work because it appears in many areas of mathematics and because we can use it in many practical applications.

Returning to our savings account example, we can conclude that if a person puts P dollars in an account at an annual interest rate r , compounded continuously, then $A(t) = Pe^{rt}$. This function may be familiar. Since functions involving base e arise often in applications, we call the function $f(x) = e^x$ the **natural exponential function**. Not only is this function interesting because of the definition of the number e , but also, as discussed next, its graph has an important property.

Since $e > 1$, we know e^x is increasing on $(-\infty, \infty)$. In [link](#), we show a graph of $f(x) = e^x$ along with a *tangent line* to the graph of at $x = 0$. We give a precise definition of tangent line in the next chapter; but, informally, we say a tangent line to a graph of f at $x = a$ is a line that passes through the point $(a, f(a))$ and has the same "slope" as f at that point. The function $f(x) = e^x$ is the only exponential function b^x with tangent line at $x = 0$ that has a slope of 1. As we see later in the text,

having this property makes the natural exponential function the most simple exponential function to use in many instances.



The graph of $f(x) = e^x$ has a tangent line with slope 1 at $x = 0$.

Example:

Exercise:

Problem:

Compounding Interest

Suppose \$500 is invested in an account at an annual interest rate of $r = 5.5\%$, compounded continuously.

- Let t denote the number of years after the initial investment and $A(t)$ denote the amount of money in the account at time t . Find a formula for $A(t)$.
- Find the amount of money in the account after 10 years and after 20 years.

Solution:

- If P dollars are invested in an account at an annual interest rate r , compounded continuously, then $A(t) = Pe^{rt}$. Here $P = \$500$ and $r = 0.055$. Therefore, $A(t) = 500e^{0.055t}$.
- After 10 years, the amount of money in the account is

Equation:

$$A(10) = 500e^{0.055 \cdot 10} = 500e^{0.55} \approx \$866.63.$$

After 20 years, the amount of money in the account is

Equation:

$$A(20) = 500e^{0.055 \cdot 20} = 500e^{1.1} \approx \$1,502.08.$$

Note:

Exercise:

Problem:

If \$750 is invested in an account at an annual interest rate of 4%, compounded continuously, find a formula for the amount of money in the account after t years. Find the amount of money after 30 years.

Solution:

$A(t) = 750e^{0.04t}$. After 30 years, there will be approximately \$2,490.09.

Hint

$$A(t) = Pe^{rt}.$$

Logarithmic Functions

Using our understanding of exponential functions, we can discuss their inverses, which are the logarithmic functions. These come in handy when we need to consider any phenomenon that varies over a wide range of values, such as pH in chemistry or decibels in sound levels.

The exponential function $f(x) = b^x$ is one-to-one, with domain $(-\infty, \infty)$ and range $(0, \infty)$. Therefore, it has an inverse function, called the *logarithmic function with base b* . For any $b > 0, b \neq 1$, the logarithmic function with base b , denoted \log_b , has domain $(0, \infty)$ and range $(-\infty, \infty)$, and satisfies

Equation:

$$\log_b(x) = y \text{ if and only if } b^y = x.$$

For example,

Equation:

$$\begin{array}{lll} \log_2(8) = 3 & \text{since} & 2^3 = 8, \\ \log_{10}\left(\frac{1}{100}\right) = -2 & \text{since} & 10^{-2} = \frac{1}{10^2} = \frac{1}{100}, \\ \log_b(1) = 0 & \text{since} & b^0 = 1 \text{ for any base } b > 0. \end{array}$$

Furthermore, since $y = \log_b(x)$ and $y = b^x$ are inverse functions,

Equation:

$$\log_b(b^x) = x \text{ and } b^{\log_b(x)} = x.$$

The most commonly used logarithmic function is the function \log_e . Since this function uses natural e as its base, it is called the **natural logarithm**. Here we use the notation $\ln(x)$ or $\ln x$ to mean $\log_e(x)$. For example,

Equation:

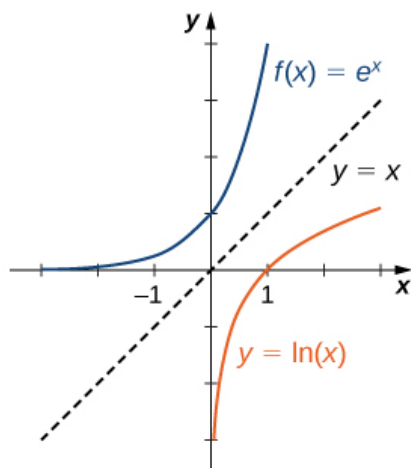
$$\ln(e) = \log_e(e) = 1, \ln(e^3) = \log_e(e^3) = 3, \ln(1) = \log_e(1) = 0.$$

Since the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other,

Equation:

$$\ln(e^x) = x \text{ and } e^{\ln x} = x,$$

and their graphs are symmetric about the line $y = x$ ([link](#)).

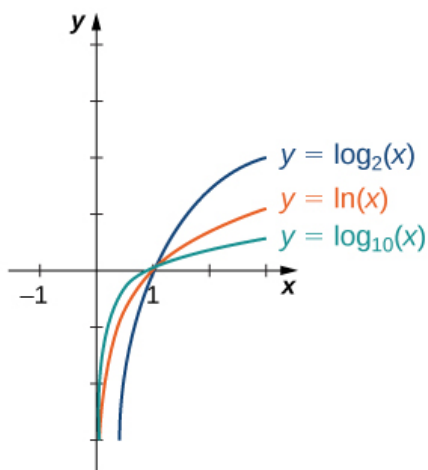


The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$.

Note:

At this [site](#) you can see an example of a base-10 logarithmic scale.

In general, for any base $b > 0, b \neq 1$, the function $g(x) = \log_b(x)$ is symmetric about the line $y = x$ with the function $f(x) = b^x$. Using this fact and the graphs of the exponential functions, we graph functions \log_b for several values of $b > 1$ ([link](#)).



Graphs of $y = \log_b(x)$ are depicted for $b = 2, e, 10$.

Before solving some equations involving exponential and logarithmic functions, let's review the basic properties of logarithms.

Note:

Rule: Properties of Logarithms

If $a, b, c > 0$, $b \neq 1$, and r is any real number, then

Equation:

1. $\log_b(ac) = \log_b(a) + \log_b(c)$ (Product property)
2. $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$ (Quotient property)
3. $\log_b(a^r) = r\log_b(a)$ (Power property)

Example:

Exercise:

Problem:

Solving Equations Involving Exponential Functions

Solve each of the following equations for x .

- a. $5^x = 2$
- b. $e^x + 6e^{-x} = 5$

Solution:

- a. Applying the natural logarithm function to both sides of the equation, we have

Equation:

$$\ln 5^x = \ln 2.$$

Using the power property of logarithms,

Equation:

$$x \ln 5 = \ln 2.$$

Therefore, $x = \ln 2 / \ln 5$.

- b. Multiplying both sides of the equation by e^x , we arrive at the equation

Equation:

$$e^{2x} + 6 = 5e^x.$$

Rewriting this equation as

Equation:

$$e^{2x} - 5e^x + 6 = 0,$$

we can then rewrite it as a quadratic equation in e^x :

Equation:

$$(e^x)^2 - 5(e^x) + 6 = 0.$$

Now we can solve the quadratic equation. Factoring this equation, we obtain

Equation:

$$(e^x - 3)(e^x - 2) = 0.$$

Therefore, the solutions satisfy $e^x = 3$ and $e^x = 2$. Taking the natural logarithm of both sides gives us the solutions $x = \ln 3, \ln 2$.

Note:

Exercise:

Problem: Solve $e^{2x}/(3 + e^{2x}) = 1/2$.

Solution:

$$x = \frac{\ln 3}{2}$$

Hint

First solve the equation for e^{2x} .

Example:**Exercise:****Problem:****Solving Equations Involving Logarithmic Functions**

Solve each of the following equations for x .

- a. $\ln\left(\frac{1}{x}\right) = 4$
- b. $\log_{10}\sqrt{x} + \log_{10}x = 2$
- c. $\ln(2x) - 3\ln(x^2) = 0$

Solution:

- a. By the definition of the natural logarithm function,

Equation:

$$\ln\left(\frac{1}{x}\right) = 4 \text{ if and only if } e^4 = \frac{1}{x}.$$

Therefore, the solution is $x = 1/e^4$.

- b. Using the product and power properties of logarithmic functions, rewrite the left-hand side of the equation as

Equation:

$$\log_{10}\sqrt{x} + \log_{10}x = \log_{10}x\sqrt{x} = \log_{10}x^{3/2} = \frac{3}{2}\log_{10}x.$$

Therefore, the equation can be rewritten as

Equation:

$$\frac{3}{2}\log_{10}x = 2 \text{ or } \log_{10}x = \frac{4}{3}.$$

The solution is $x = 10^{4/3} = 10\sqrt[3]{10}$.

- c. Using the power property of logarithmic functions, we can rewrite the equation as $\ln(2x) - \ln(x^6) = 0$.

Using the quotient property, this becomes

Equation:

$$\ln \left(\frac{2}{x^5} \right) = 0.$$

Therefore, $2/x^5 = 1$, which implies $x = \sqrt[5]{2}$. We should then check for any extraneous solutions.

Note:

Exercise:

Problem: Solve $\ln(x^3) - 4\ln(x) = 1$.

Solution:

$$x = \frac{1}{e}$$

Hint

First use the power property, then use the product property of logarithms.

When evaluating a logarithmic function with a calculator, you may have noticed that the only options are \log_{10} or \log , called the *common logarithm*, or \ln , which is the natural logarithm. However, exponential functions and logarithm functions can be expressed in terms of any desired base b . If you need to use a calculator to evaluate an expression with a different base, you can apply the change-of-base formulas first. Using this change of base, we typically write a given exponential or logarithmic function in terms of the natural exponential and natural logarithmic functions.

Note:

Rule: Change-of-Base Formulas

Let $a > 0$, $b > 0$, and $a \neq 1$, $b \neq 1$.

1. $a^x = b^{x \log_b a}$ for any real number x .
If $b = e$, this equation reduces to $a^x = e^{x \log_e a} = e^{x \ln a}$.
2. $\log_a x = \frac{\log_b x}{\log_b a}$ for any real number $x > 0$.
If $b = e$, this equation reduces to $\log_a x = \frac{\ln x}{\ln a}$.

Proof

For the first change-of-base formula, we begin by making use of the power property of logarithmic functions. We know that for any base $b > 0, b \neq 1, \log_b(a^x) = x \log_b a$. Therefore,

Equation:

$$b^{\log_b(a^x)} = b^{x \log_b a}.$$

In addition, we know that b^x and $\log_b(x)$ are inverse functions. Therefore,

Equation:

$$b^{\log_b(a^x)} = a^x.$$

Combining these last two equalities, we conclude that $a^x = b^{x \log_b a}$.

To prove the second property, we show that

Equation:

$$(\log_b a) \cdot (\log_a x) = \log_b x.$$

Let $u = \log_b a, v = \log_a x$, and $w = \log_b x$. We will show that $u \cdot v = w$. By the definition of logarithmic functions, we know that $b^u = a, a^v = x$, and $b^w = x$. From the previous equations, we see that

Equation:

$$b^{uv} = (b^u)^v = a^v = x = b^w.$$

Therefore, $b^{uv} = b^w$. Since exponential functions are one-to-one, we can conclude that $u \cdot v = w$.

□

Example:

Exercise:

Problem:

Changing Bases

Use a calculating utility to evaluate $\log_3 7$ with the change-of-base formula presented earlier.

Solution:

Use the second equation with $a = 3$ and $e = 3$:

$$\log_3 7 = \frac{\ln 7}{\ln 3} \approx 1.77124.$$

Note:

Exercise:

Problem: Use the change-of-base formula and a calculating utility to evaluate $\log_4 6$.

Solution:

1.29248

Hint

Use the change of base to rewrite this expression in terms of expressions involving the natural logarithm function.

Example:**Exercise:**

Problem:

Chapter Opener: The Richter Scale for Earthquakes

(credit: modification of work by
Robb Hannawacker, NPS)

In 1935, Charles Richter developed a scale (now known as the *Richter scale*) to measure the magnitude of an earthquake. The scale is a base-10 logarithmic scale, and it can be described as follows: Consider one earthquake with magnitude R_1 on the Richter scale and a second earthquake with magnitude R_2 on the Richter scale. Suppose $R_1 > R_2$, which means the earthquake of magnitude R_1 is stronger, but how much stronger is it than the other earthquake? A way of measuring the intensity of an earthquake is by using a seismograph to measure the amplitude of the earthquake waves. If A_1 is the amplitude measured for the first earthquake and A_2 is the amplitude measured for the second earthquake, then the amplitudes and magnitudes of the two earthquakes satisfy the following equation:

Equation:

$$R_1 - R_2 = \log_{10} \left(\frac{A_1}{A_2} \right).$$

Consider an earthquake that measures 8 on the Richter scale and an earthquake that measures 7 on the Richter scale. Then,

Equation:

$$8 - 7 = \log_{10} \left(\frac{A_1}{A_2} \right).$$

Therefore,

Equation:

$$\log_{10} \left(\frac{A_1}{A_2} \right) = 1,$$

which implies $A_1/A_2 = 10$ or $A_1 = 10A_2$. Since A_1 is 10 times the size of A_2 , we say that the first earthquake is 10 times as intense as the second earthquake. On the other hand, if one earthquake measures 8 on the Richter scale and another measures 6, then the relative intensity of the two earthquakes satisfies the equation

Equation:

$$\log_{10} \left(\frac{A_1}{A_2} \right) = 8 - 6 = 2.$$

Therefore, $A_1 = 100A_2$. That is, the first earthquake is 100 times more intense than the second earthquake.

How can we use logarithmic functions to compare the relative severity of the magnitude 9 earthquake in Japan in 2011 with the magnitude 7.3 earthquake in Haiti in 2010?

Solution:

To compare the Japan and Haiti earthquakes, we can use an equation presented earlier:

$$9 - 7.3 = \log_{10} \left(\frac{A_1}{A_2} \right).$$

Therefore, $A_1/A_2 = 10^{1.7}$, and we conclude that the earthquake in Japan was approximately 50 times more intense than the earthquake in Haiti.

Note:

Exercise:

Problem:

Compare the relative severity of a magnitude 8.4 earthquake with a magnitude 7.4 earthquake.

Solution:

The magnitude 8.4 earthquake is roughly 10 times as severe as the magnitude 7.4 earthquake.

Hint

$$R_1 - R_2 = \log_{10} (A_1/A_2).$$

Hyperbolic Functions

The hyperbolic functions are defined in terms of certain combinations of e^x and e^{-x} . These functions arise naturally in various engineering and physics applications, including the study of water waves and vibrations of elastic membranes. Another common use for a hyperbolic function is the representation of a hanging chain or cable, also known as a catenary ([link](#)). If we introduce a coordinate system so that the low point of the chain lies along the y -axis, we can describe the height of the chain in terms of a hyperbolic function. First, we define the **hyperbolic functions**.



The shape of a strand of silk in a spider's web can be described in terms of a hyperbolic function. The same shape applies to a chain or cable hanging from two supports with only its own weight. (credit: "Mtpaley", Wikimedia Commons)

Note:

Definition

Hyperbolic cosine

Equation:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic sine**Equation:**

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic tangent**Equation:**

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cosecant**Equation:**

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Hyperbolic secant**Equation:**

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cotangent**Equation:**

$$\operatorname{coth} x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The name *cosh* rhymes with “gosh,” whereas the name *sinh* is pronounced “cinch.” *Tanh*, *sech*, *csch*, and *coth* are pronounced “tanch,” “seech,” “coseech,” and “cotanch,” respectively.

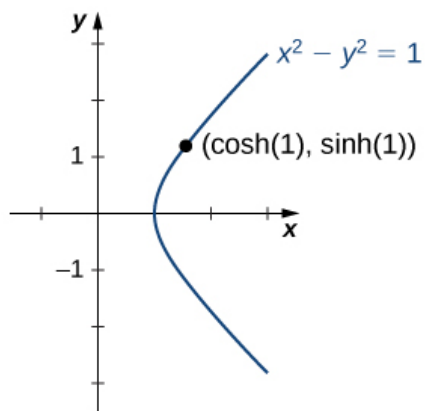
Using the definition of $\cosh(x)$ and principles of physics, it can be shown that the height of a hanging chain, such as the one in [\[link\]](#), can be described by the function $h(x) = a \cosh(x/a) + c$ for certain constants a and c .

But why are these functions called *hyperbolic functions*? To answer this question, consider the quantity $\cosh^2 t - \sinh^2 t$. Using the definition of \cosh and \sinh , we see that

Equation:

$$\cosh^2 t - \sinh^2 t = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = 1.$$

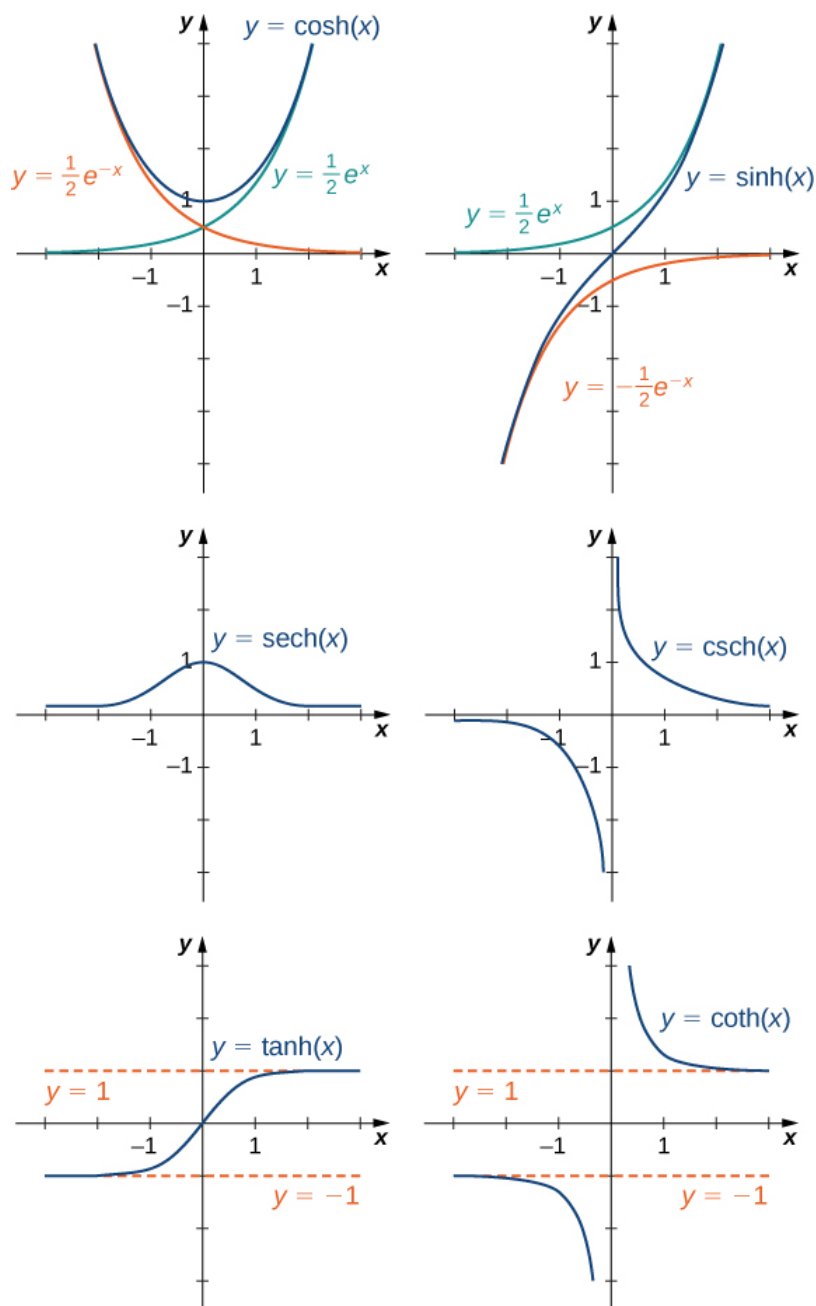
This identity is the analog of the trigonometric identity $\cos^2 t + \sin^2 t = 1$. Here, given a value t , the point $(x, y) = (\cosh t, \sinh t)$ lies on the unit hyperbola $x^2 - y^2 = 1$ ([\[link\]](#)).



The unit hyperbola
 $\cosh^2 t - \sinh^2 t = 1$.

Graphs of Hyperbolic Functions

To graph $\cosh x$ and $\sinh x$, we make use of the fact that both functions approach $(1/2)e^x$ as $x \rightarrow \infty$, since $e^{-x} \rightarrow 0$ as $x \rightarrow \infty$. As $x \rightarrow -\infty$, $\cosh x$ approaches $1/2e^{-x}$, whereas $\sinh x$ approaches $-1/2e^{-x}$. Therefore, using the graphs of $1/2e^x$, $1/2e^{-x}$, and $-1/2e^{-x}$ as guides, we graph $\cosh x$ and $\sinh x$. To graph $\tanh x$, we use the fact that $\tanh(0) = 0$, $-1 < \tanh(x) < 1$ for all x , $\tanh x \rightarrow 1$ as $x \rightarrow \infty$, and $\tanh x \rightarrow -1$ as $x \rightarrow -\infty$. The graphs of the other three hyperbolic functions can be sketched using the graphs of $\cosh x$, $\sinh x$, and $\tanh x$ ([link](#)).



The hyperbolic functions involve combinations of e^x and e^{-x} .

Identities Involving Hyperbolic Functions

The identity $\cosh^2 t - \sinh^2 t$, shown in [\[link\]](#), is one of several identities involving the hyperbolic functions, some of which are listed next. The first four properties follow easily from the definitions of

hyperbolic sine and hyperbolic cosine. Except for some differences in signs, most of these properties are analogous to identities for trigonometric functions.

Note:

Rule: Identities Involving Hyperbolic Functions

1. $\cosh(-x) = \cosh x$
2. $\sinh(-x) = -\sinh x$
3. $\cosh x + \sinh x = e^x$
4. $\cosh x - \sinh x = e^{-x}$
5. $\cosh^2 x - \sinh^2 x = 1$
6. $1 - \tanh^2 x = \operatorname{sech}^2 x$
7. $\coth^2 x - 1 = \operatorname{csch}^2 x$
8. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
9. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

Example:

Exercise:

Problem:

Evaluating Hyperbolic Functions

- a. Simplify $\sinh(5 \ln x)$.
- b. If $\sinh x = 3/4$, find the values of the remaining five hyperbolic functions.

Solution:

- a. Using the definition of the \sinh function, we write

Equation:

$$\sinh(5 \ln x) = \frac{e^{5 \ln x} - e^{-5 \ln x}}{2} = \frac{e^{\ln(x^5)} - e^{\ln(x^{-5})}}{2} = \frac{x^5 - x^{-5}}{2}.$$

- b. Using the identity $\cosh^2 x - \sinh^2 x = 1$, we see that

Equation:

$$\cosh^2 x = 1 + \left(\frac{3}{4}\right)^2 = \frac{25}{16}.$$

Since $\cosh x \geq 1$ for all x , we must have $\cosh x = 5/4$. Then, using the definitions for the other hyperbolic functions, we conclude that $\tanh x = 3/5$, $\operatorname{csch} x = 4/3$, $\operatorname{sech} x = 4/5$, and $\coth x = 5/3$.

Note:

Exercise:

Problem: Simplify $\cosh(2\ln x)$.

Solution:

$$(x^2 + x^{-2})/2$$

Hint

Use the definition of the cosh function and the power property of logarithm functions.

Inverse Hyperbolic Functions

From the graphs of the hyperbolic functions, we see that all of them are one-to-one except $\cosh x$ and $\operatorname{sech} x$. If we restrict the domains of these two functions to the interval $[0, \infty)$, then all the hyperbolic functions are one-to-one, and we can define the **inverse hyperbolic functions**. Since the hyperbolic functions themselves involve exponential functions, the inverse hyperbolic functions involve logarithmic functions.

Note:

Definition

Inverse Hyperbolic Functions

Equation:

$$\begin{aligned}\sinh^{-1}x &= \operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1}) & \cosh^{-1}x &= \operatorname{arccosh} x = \ln(x + \sqrt{x^2 - 1}) \\ \tanh^{-1}x &= \operatorname{arctanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) & \coth^{-1}x &= \operatorname{arccot} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right) \\ \operatorname{sech}^{-1}x &= \operatorname{arcsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) & \operatorname{csch}^{-1}x &= \operatorname{arccsch} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)\end{aligned}$$

Let's look at how to derive the first equation. The others follow similarly. Suppose $y = \sinh^{-1}x$. Then, $x = \sinh y$ and, by the definition of the hyperbolic sine function, $x = \frac{e^y - e^{-y}}{2}$. Therefore,

Equation:

$$e^y - 2x - e^{-y} = 0.$$

Multiplying this equation by e^y , we obtain

Equation:

$$e^{2y} - 2xe^y - 1 = 0.$$

This can be solved like a quadratic equation, with the solution

Equation:

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Since $e^y > 0$, the only solution is the one with the positive sign. Applying the natural logarithm to both sides of the equation, we conclude that

Equation:

$$y = \ln \left(x + \sqrt{x^2 + 1} \right).$$

Example:

Exercise:

Problem:

Evaluating Inverse Hyperbolic Functions

Evaluate each of the following expressions.

Equation:

$$\sinh^{-1}(2)$$

Equation:

$$\tanh^{-1}(1/4)$$

Solution:

$$\sinh^{-1}(2) = \ln \left(2 + \sqrt{2^2 + 1} \right) = \ln \left(2 + \sqrt{5} \right) \approx 1.4436$$

$$\tanh^{-1}(1/4) = \frac{1}{2} \ln \left(\frac{1+1/4}{1-1/4} \right) = \frac{1}{2} \ln \left(\frac{5/4}{3/4} \right) = \frac{1}{2} \ln \left(\frac{5}{3} \right) \approx 0.2554$$

Note:

Exercise:

Problem: Evaluate $\tanh^{-1}(1/2)$.

Solution:

$$\frac{1}{2} \ln(3) \approx 0.5493.$$

Hint

Use the definition of $\tanh^{-1}x$ and simplify.

Key Concepts

- The exponential function $y = b^x$ is increasing if $b > 1$ and decreasing if $0 < b < 1$. Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- The logarithmic function $y = \log_b(x)$ is the inverse of $y = b^x$. Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.
- The natural exponential function is $y = e^x$ and the natural logarithmic function is $y = \ln x = \log_e x$.
- Given an exponential function or logarithmic function in base a , we can make a change of base to convert this function to any base $b > 0, b \neq 1$. We typically convert to base e .
- The hyperbolic functions involve combinations of the exponential functions e^x and e^{-x} . As a result, the inverse hyperbolic functions involve the natural logarithm.

For the following exercises, evaluate the given exponential functions as indicated, accurate to two significant digits after the decimal.

Exercise:

Problem: $f(x) = 5^x$ a. $x = 3$ b. $x = \frac{1}{2}$ c. $x = \sqrt{2}$

Solution:

a. 125 b. 2.24 c. 9.74

Exercise:

Problem: $f(x) = (0.3)^x$ a. $x = -1$ b. $x = 4$ c. $x = -1.5$

Exercise:

Problem: $f(x) = 10^x$ a. $x = -2$ b. $x = 4$ c. $x = \frac{5}{3}$

Solution:

a. 0.01 b. 10,000 c. 46.42

Exercise:

Problem: $f(x) = e^x$ a. $x = 2$ b. $x = -3.2$ c. $x = \pi$

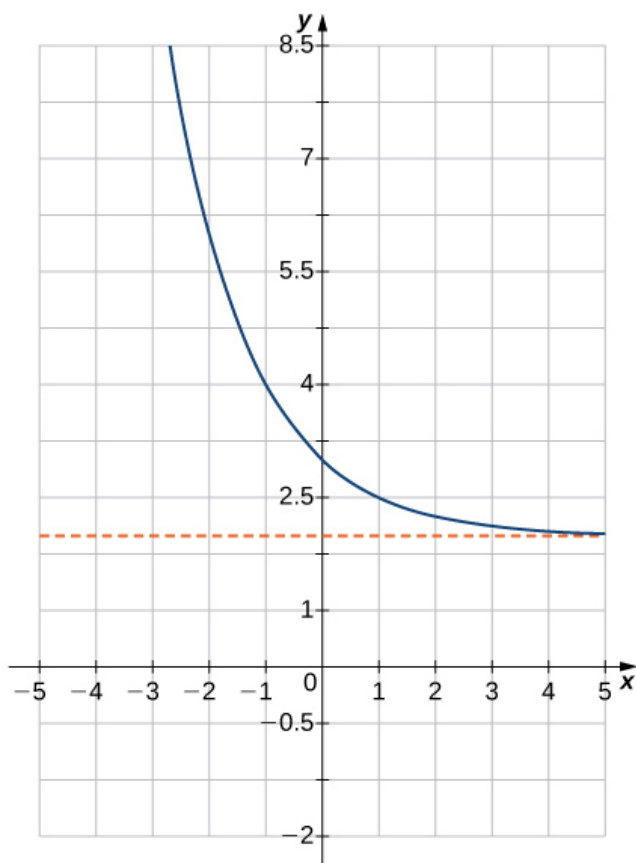
For the following exercises, match the exponential equation to the correct graph.

- a. $y = 4^{-x}$
- b. $y = 3^{x-1}$

- c. $y = 2^{x+1}$
- d. $y = \left(\frac{1}{2}\right)^x + 2$
- e. $y = -3^{-x}$
- f. $y = 1 - 5^x$

Exercise:

Problem:

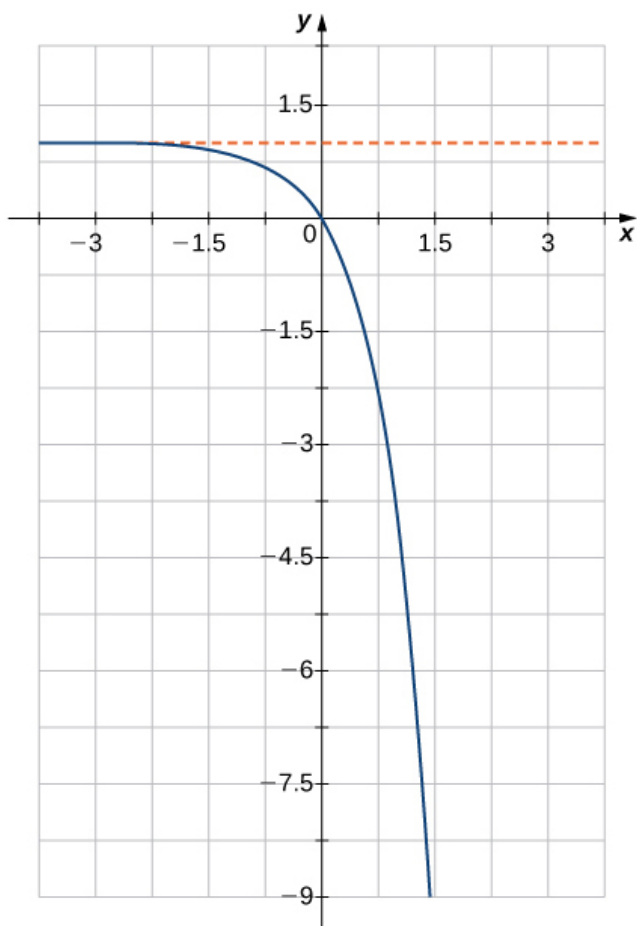


Solution:

d

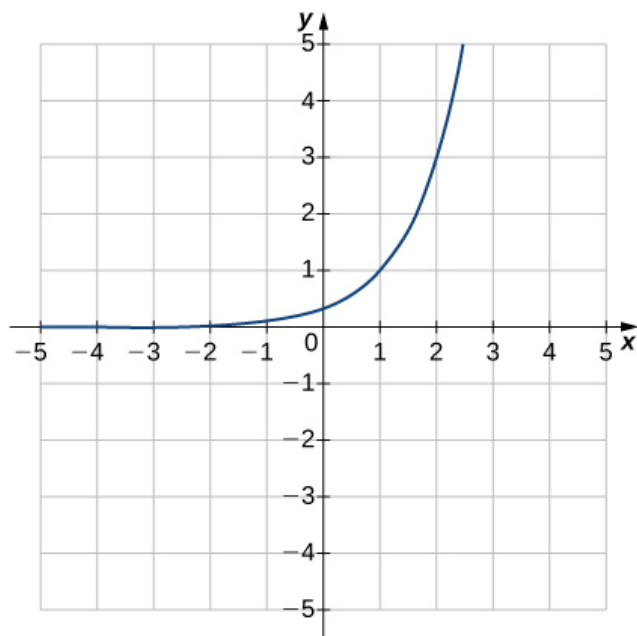
Exercise:

Problem:



Exercise:

Problem:

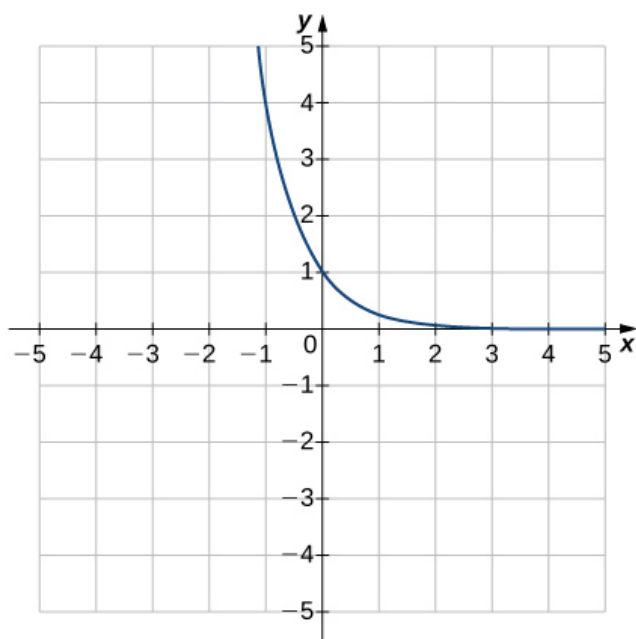


Solution:

b

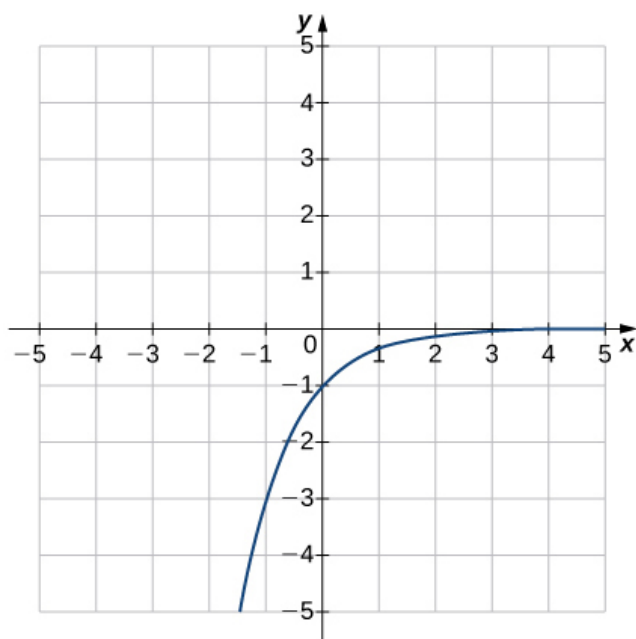
Exercise:

Problem:



Exercise:

Problem:

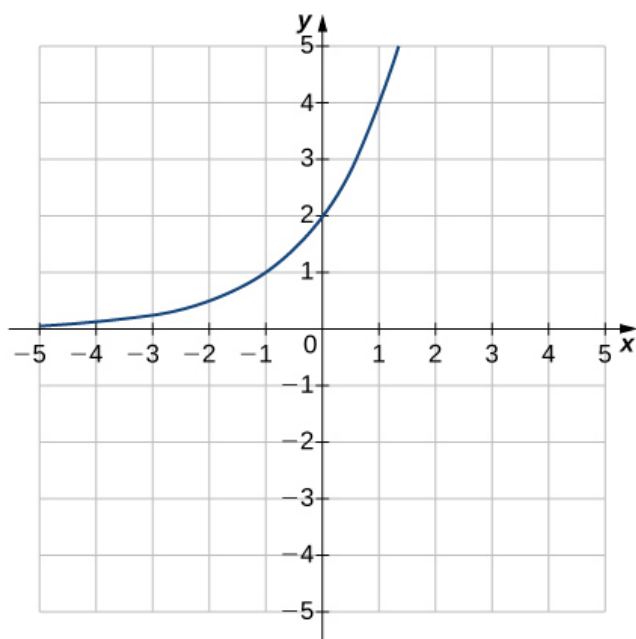


Solution:

e

Exercise:

Problem:



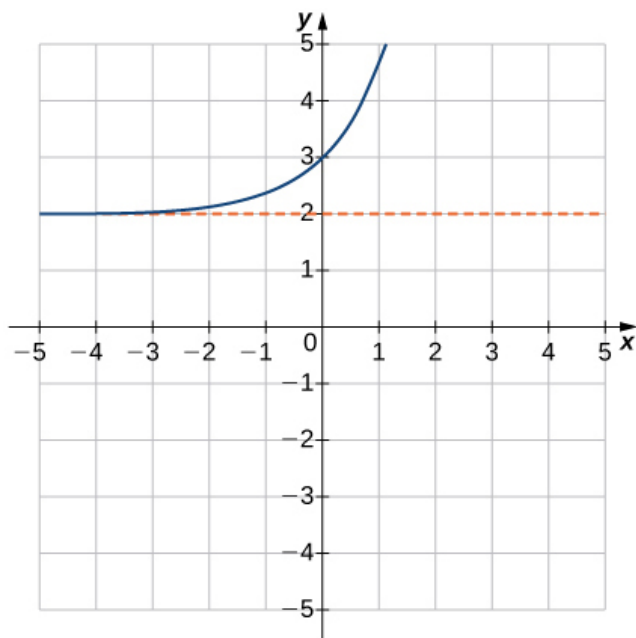
For the following exercises, sketch the graph of the exponential function. Determine the domain, range, and horizontal asymptote.

Exercise:

Problem: $f(x) = e^x + 2$

Solution:

Domain: all real numbers, range: $(2, \infty)$, $y = 2$



Exercise:

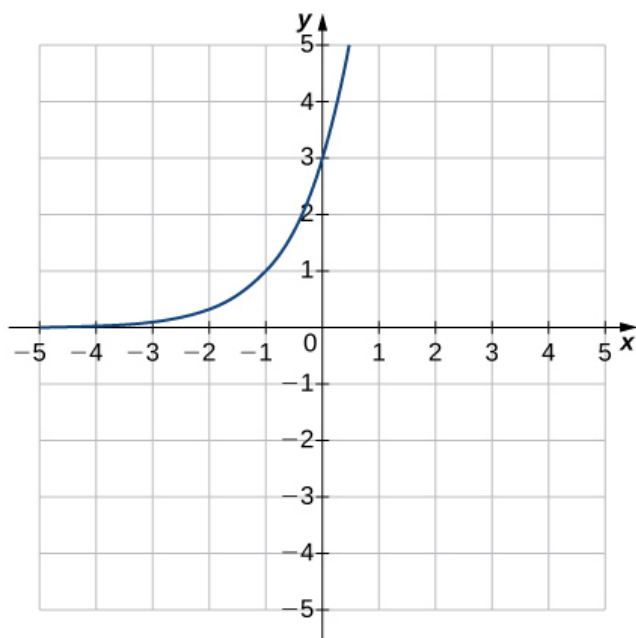
Problem: $f(x) = -2^x$

Exercise:

Problem: $f(x) = 3^{x+1}$

Solution:

Domain: all real numbers, range: $(0, \infty)$, $y = 0$



Exercise:

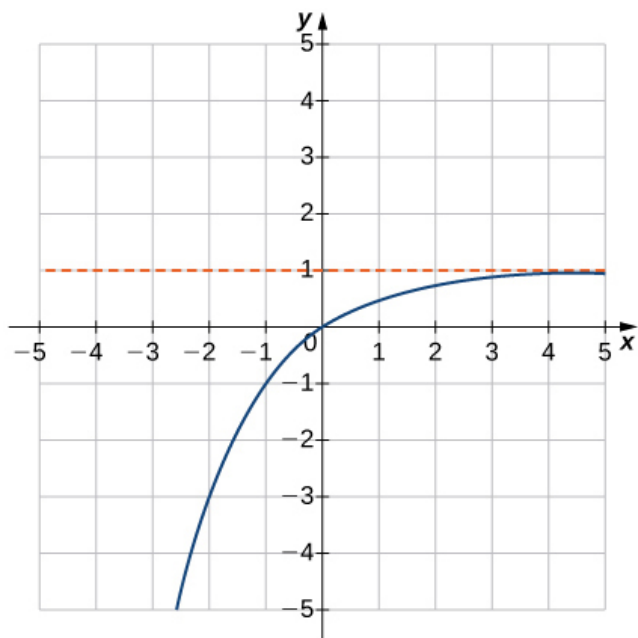
Problem: $f(x) = 4^x - 1$

Exercise:

Problem: $f(x) = 1 - 2^{-x}$

Solution:

Domain: all real numbers, range: $(-\infty, 1)$, $y = 1$



Exercise:

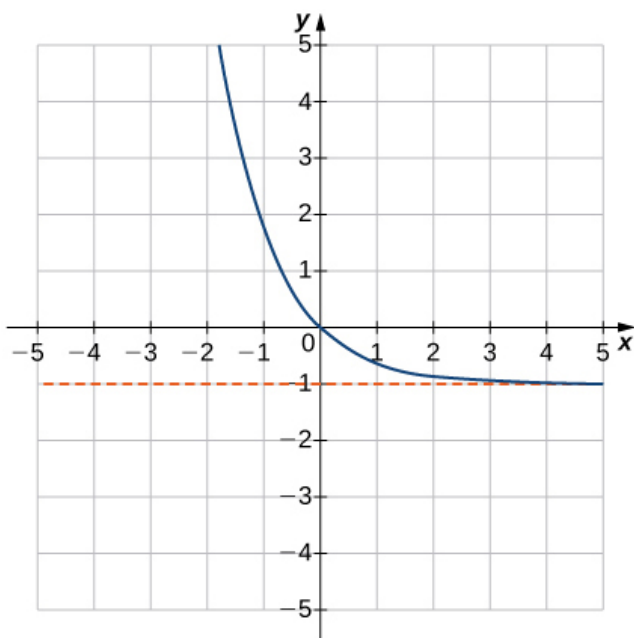
Problem: $f(x) = 5^{x+1} + 2$

Exercise:

Problem: $f(x) = e^{-x} - 1$

Solution:

Domain: all real numbers, range: $(-1, \infty)$, $y = -1$



For the following exercises, write the equation in equivalent exponential form.

Exercise:

Problem: $\log_3 81 = 4$

Exercise:

Problem: $\log_8 2 = \frac{1}{3}$

Solution:

$$8^{1/3} = 2$$

Exercise:

Problem: $\log_5 1 = 0$

Exercise:

Problem: $\log_5 25 = 2$

Solution:

$$5^2 = 25$$

Exercise:

Problem: $\log 0.1 = -1$

Exercise:

Problem: $\ln\left(\frac{1}{e^3}\right) = -3$

Solution:

$$e^{-3} = \frac{1}{e^3}$$

Exercise:

Problem: $\log_9 3 = 0.5$

Exercise:

Problem: $\ln 1 = 0$

Solution:

$$e^0 = 1$$

For the following exercises, write the equation in equivalent logarithmic form.

Exercise:

Problem: $2^3 = 8$

Exercise:

Problem: $4^{-2} = \frac{1}{16}$

Solution:

$$\log_4\left(\frac{1}{16}\right) = -2$$

Exercise:

Problem: $10^2 = 100$

Exercise:

Problem: $9^0 = 1$

Solution:

$$\log_9 1 = 0$$

Exercise:

Problem: $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

Exercise:

Problem: $\sqrt[3]{64} = 4$

Solution:

$$\log_6 4 = \frac{1}{3}$$

Exercise:

Problem: $e^x = y$

Exercise:

Problem: $9^y = 150$

Solution:

$$\log_9 150 = y$$

Exercise:

Problem: $b^3 = 45$

Exercise:

Problem: $4^{-3/2} = 0.125$

Solution:

$$\log_4 0.125 = -\frac{3}{2}$$

For the following exercises, sketch the graph of the logarithmic function. Determine the domain, range, and vertical asymptote.

Exercise:

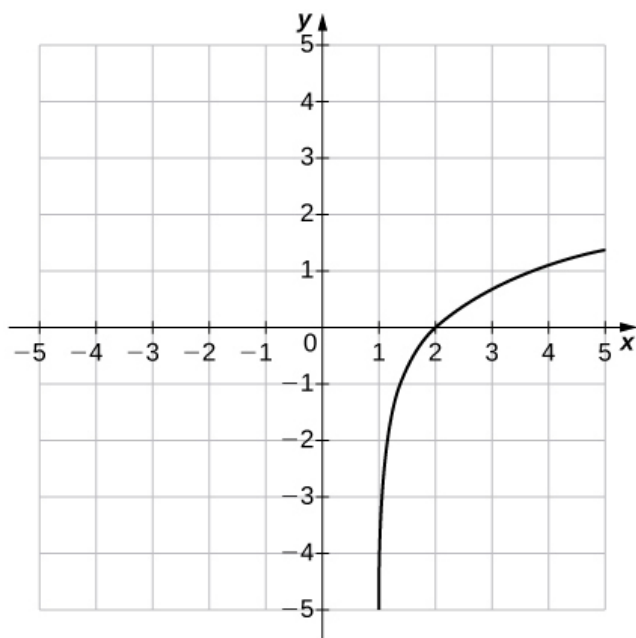
Problem: $f(x) = 3 + \ln x$

Exercise:

Problem: $f(x) = \ln(x - 1)$

Solution:

Domain: $(1, \infty)$, range: $(-\infty, \infty)$, $x = 1$



Exercise:

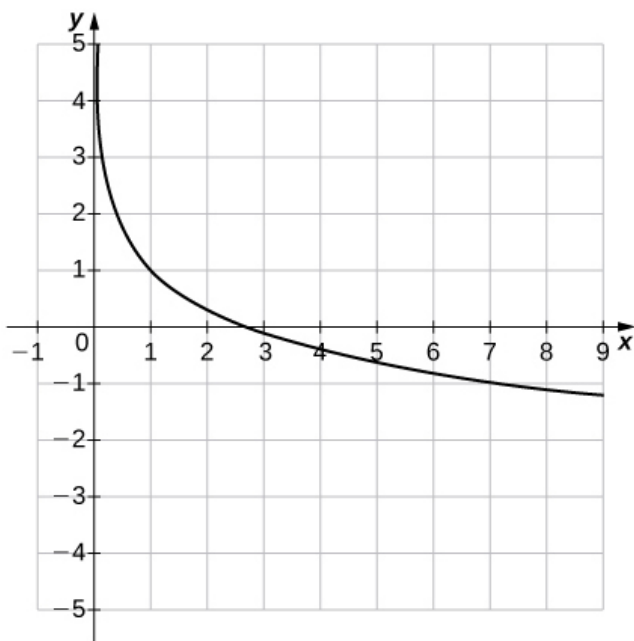
Problem: $f(x) = \ln(-x)$

Exercise:

Problem: $f(x) = 1 - \ln x$

Solution:

Domain: $(0, \infty)$, range: $(-\infty, \infty)$, $x = 0$



Exercise:

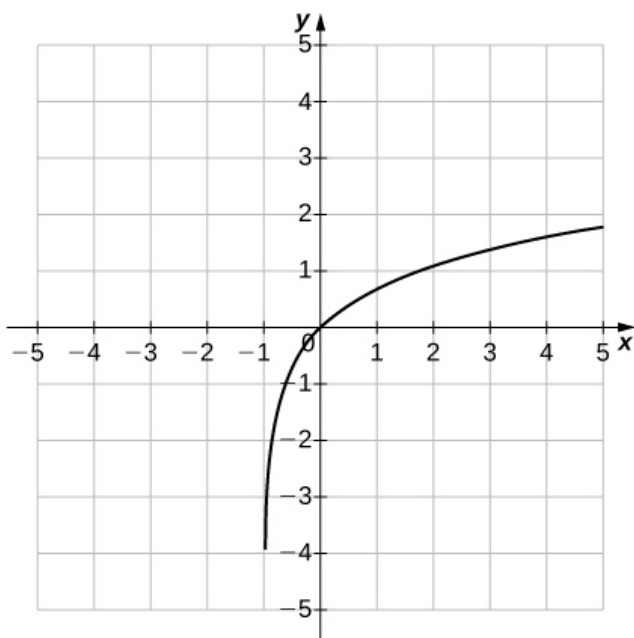
Problem: $f(x) = \log x - 1$

Exercise:

Problem: $f(x) = \ln(x + 1)$

Solution:

Domain: $(-1, \infty)$, range: $(-\infty, \infty)$, $x = -1$



For the following exercises, use properties of logarithms to write the expressions as a sum, difference, and/or product of logarithms.

Exercise:

Problem: $\log x^4 y$

Exercise:

Problem: $\log_3 \frac{9a^3}{b}$

Solution:

$$2 + 3\log_3 a - \log_3 b$$

Exercise:

Problem: $\ln a \sqrt[3]{b}$

Exercise:

Problem: $\log_5 \sqrt{125xy^3}$

Solution:

$$\frac{3}{2} + \frac{1}{2}\log_5 x + \frac{3}{2}\log_5 y$$

Exercise:

Problem: $\log_4 \frac{\sqrt[3]{xy}}{64}$

Exercise:

Problem: $\ln \left(\frac{6}{\sqrt{e^3}} \right)$

Solution:

$$-\frac{3}{2} + \ln 6$$

For the following exercises, solve the exponential equation exactly.

Exercise:

Problem: $5^x = 125$

Exercise:

Problem: $e^{3x} - 15 = 0$

Solution:

$$\frac{\ln 15}{3}$$

Exercise:

Problem: $8^x = 4$

Exercise:

Problem: $4^{x+1} - 32 = 0$

Solution:

$$\frac{3}{2}$$

Exercise:

Problem: $3^{x/14} = \frac{1}{10}$

Exercise:

Problem: $10^x = 7.21$

Solution:

$$\log 7.21$$

Exercise:

Problem: $4 \cdot 2^{3x} - 20 = 0$

Exercise:

Problem: $7^{3x-2} = 11$

Solution:

$$\frac{2}{3} + \frac{\log 11}{3 \log 7}$$

For the following exercises, solve the logarithmic equation exactly, if possible.

Exercise:

Problem: $\log_3 x = 0$

Exercise:

Problem: $\log_5 x = -2$

Solution:

$$x = \frac{1}{25}$$

Exercise:

Problem: $\log_4 (x + 5) = 0$

Exercise:

Problem: $\log (2x - 7) = 0$

Solution:

$$x = 4$$

Exercise:

Problem: $\ln \sqrt{x + 3} = 2$

Exercise:

Problem: $\log_6 (x + 9) + \log_6 x = 2$

Solution:

$$x = 3$$

Exercise:

Problem: $\log_4(x+2) - \log_4(x-1) = 0$

Exercise:

Problem: $\ln x + \ln(x-2) = \ln 4$

Solution:

$$1 + \sqrt{5}$$

For the following exercises, use the change-of-base formula and either base 10 or base e to evaluate the given expressions. Answer in exact form and in approximate form, rounding to four decimal places.

Exercise:

Problem: $\log_5 47$

Exercise:

Problem: $\log_7 82$

Solution:

$$\left(\frac{\log 82}{\log 7} \approx 2.2646 \right)$$

Exercise:

Problem: $\log_6 103$

Exercise:

Problem: $\log_{0.5} 211$

Solution:

$$\left(\frac{\log 211}{\log 0.5} \approx -7.7211 \right)$$

Exercise:

Problem: $\log_2 \pi$

Exercise:

Problem: $\log_{0.2} 0.452$

Solution:

$$\left(\frac{\log 0.452}{\log 0.2} \approx 0.4934 \right)$$

Exercise:

Problem: Rewrite the following expressions in terms of exponentials and simplify.

- a. $2 \cosh(\ln x)$ b. $\cosh 4x + \sinh 4x$ c. $\cosh 2x - \sinh 2x$ d.
 $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

Exercise:

Problem:

[T] The number of bacteria N in a culture after t days can be modeled by the function $N(t) = 1300 \cdot (2)^{t/4}$. Find the number of bacteria present after 15 days.

Solution:

~17,491

Exercise:

Problem:

[T] The demand D (in millions of barrels) for oil in an oil-rich country is given by the function $D(p) = 150 \cdot (2.7)^{-0.25p}$, where p is the price (in dollars) of a barrel of oil. Find the amount of oil demanded (to the nearest million barrels) when the price is between \$15 and \$20.

Exercise:

Problem:

[T] The amount A of a \$100,000 investment paying continuously and compounded for t years is given by $A(t) = 100,000 \cdot e^{0.055t}$. Find the amount A accumulated in 5 years.

Solution:

Approximately \$131,653 is accumulated in 5 years.

Exercise:

Problem:

[T] An investment is compounded monthly, quarterly, or yearly and is given by the function $A = P \left(1 + \frac{j}{n} \right)^{nt}$, where A is the value of the investment at time t , P is the initial principle that was invested, j is the annual interest rate, and n is the number of time the interest is compounded per year. Given a yearly interest rate of 3.5% and an initial principle of \$100,000, find the amount A accumulated in 5 years for interest that is compounded a. daily, b., monthly, c. quarterly, and d. yearly.

Exercise:

Problem:

[T] The concentration of hydrogen ions in a substance is denoted by $[H^+]$, measured in moles per liter. The pH of a substance is defined by the logarithmic function $pH = -\log [H^+]$. This function is used to measure the acidity of a substance. The pH of water is 7. A substance with a pH less than 7 is an acid, whereas one that has a pH of more than 7 is a base.

- a. Find the pH of the following substances. Round answers to one digit.
- b. Determine whether the substance is an acid or a base.

- i. Eggs: $[H^+] = 1.6 \times 10^{-8} \text{ mol/L}$
- ii. Beer: $[H^+] = 3.16 \times 10^{-3} \text{ mol/L}$
- iii. Tomato Juice: $[H^+] = 7.94 \times 10^{-5} \text{ mol/L}$

Solution:

- i. a. pH = 8 b. Base ii. a. pH = 3 b. Acid iii. a. pH = 4 b. Acid

Exercise:**Problem:**

[T] Iodine-131 is a radioactive substance that decays according to the function $Q(t) = Q_0 \cdot e^{-0.08664t}$, where Q_0 is the initial quantity of a sample of the substance and t is in days. Determine how long it takes (to the nearest day) for 95% of a quantity to decay.

Exercise:**Problem:**

[T] According to the World Bank, at the end of 2013 ($t = 0$) the U.S. population was 316 million and was increasing according to the following model:

$$P(t) = 316e^{0.0074t},$$

where P is measured in millions of people and t is measured in years after 2013.

- a. Based on this model, what will be the population of the United States in 2020?
- b. Determine when the U.S. population will be twice what it is in 2013.

Solution:

- a. ~333 million b. 94 years from 2013, or in 2107

Exercise:**Problem:**

[T] The amount A accumulated after 1000 dollars is invested for t years at an interest rate of 4% is modeled by the function $A(t) = 1000(1.04)^t$.

- a. Find the amount accumulated after 5 years and 10 years.
- b. Determine how long it takes for the original investment to triple.

Exercise:

Problem:

[T] A bacterial colony grown in a lab is known to double in number in 12 hours. Suppose, initially, there are 1000 bacteria present.

- a. Use the exponential function $Q = Q_0 e^{kt}$ to determine the value k , which is the growth rate of the bacteria. Round to four decimal places.
- b. Determine approximately how long it takes for 200,000 bacteria to grow.

Solution:

- a. $k \approx 0.0578$ b. ≈ 92 hours

Exercise:

Problem:

[T] The rabbit population on a game reserve doubles every 6 months. Suppose there were 120 rabbits initially.

- a. Use the exponential function $P = P_0 a^t$ to determine the growth rate constant a . Round to four decimal places.
- b. Use the function in part a. to determine approximately how long it takes for the rabbit population to reach 3500.

Exercise:

Problem:

[T] The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time, in Japan, an earthquake with magnitude 4.9 caused only minor damage. Approximately how much more energy was released by the San Francisco earthquake than by the Japanese earthquake?

Solution:

The San Francisco earthquake had $10^{3.4}$ or ~ 2512 times more energy than the Japan earthquake.

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

Exercise:

Problem: A function is always one-to-one.

Exercise:

Problem: $f \circ g = g \circ f$, assuming f and g are functions.

Solution:

False

Exercise:

Problem: A relation that passes the horizontal and vertical line tests is a one-to-one function.

Exercise:

Problem: A relation passing the horizontal line test is a function.

Solution:

False

For the following problems, state the domain and range of the given functions:

$$f = x^2 + 2x - 3, \quad g = \ln(x - 5), \quad h = \frac{1}{x+4}$$

Exercise:

Problem: h

Exercise:

Problem: g

Solution:

Domain: $x > 5$, range: all real numbers

Exercise:

Problem: $h \circ f$

Exercise:

Problem: $g \circ f$

Solution:

Domain: $x > 2$ and $x < -4$, range: all real numbers

Find the degree, y-intercept, and zeros for the following polynomial functions.

Exercise:

Problem: $f(x) = 2x^2 + 9x - 5$

Exercise:

Problem: $f(x) = x^3 + 2x^2 - 2x$

Solution:

Degree of 3, y -intercept: 0, zeros: 0, $\sqrt{3} - 1$, $-1 - \sqrt{3}$

Simplify the following trigonometric expressions.

Exercise:

Problem: $\frac{\tan^2 x}{\sec^2 x} + \cos^2 x$

Exercise:

Problem: $\cos(2x) = \sin^2 x$

Solution:

$\cos(2x)$ or $\frac{1}{2}(\cos(2x) + 1)$

Solve the following trigonometric equations on the interval $\theta = [-2\pi, 2\pi]$ exactly.

Exercise:

Problem: $6\cos^2 x - 3 = 0$

Exercise:

Problem: $\sec^2 x - 2\sec x + 1 = 0$

Solution:

$0, \pm 2\pi$

Solve the following logarithmic equations.

Exercise:

Problem: $5^x = 16$

Exercise:

Problem: $\log_2(x + 4) = 3$

Solution:

Are the following functions one-to-one over their domain of existence? Does the function have an inverse? If so, find the inverse $f^{-1}(x)$ of the function. Justify your answer.

Exercise:

Problem: $f(x) = x^2 + 2x + 1$

Exercise:

Problem: $f(x) = \frac{1}{x}$

Solution:

One-to-one; yes, the function has an inverse; inverse: $f^{-1}(x) = \frac{1}{y}$

For the following problems, determine the largest domain on which the function is one-to-one and find the inverse on that domain.

Exercise:

Problem: $f(x) = \sqrt{9 - x}$

Exercise:

Problem: $f(x) = x^2 + 3x + 4$

Solution:

$$x \geq -\frac{3}{2}, f^{-1}(x) = -\frac{3}{2} + \frac{1}{2}\sqrt{4y - 7}$$

Exercise:

Problem:

A car is racing along a circular track with diameter of 1 mi. A trainer standing in the center of the circle marks his progress every 5 sec. After 5 sec, the trainer has to turn 55° to keep up with the car. How fast is the car traveling?

For the following problems, consider a restaurant owner who wants to sell T-shirts advertising his brand. He recalls that there is a fixed cost and variable cost, although he does not remember the values. He does know that the T-shirt printing company charges \$440 for 20 shirts and \$1000 for 100 shirts.

Exercise:

Problem:

- Find the equation $C = f(x)$ that describes the total cost as a function of number of shirts and
- determine how many shirts he must sell to break even if he sells the shirts for \$10 each.

Solution:

a. $C(x) = 300 + 7x$ b. 100 shirts

Exercise:**Problem:**

a. Find the inverse function $x = f^{-1}(C)$ and describe the meaning of this function. b. Determine how many shirts the owner can buy if he has \$8000 to spend.

For the following problems, consider the population of Ocean City, New Jersey, which is cyclical by season.

Exercise:**Problem:**

The population can be modeled by $P(t) = 82.5 - 67.5 \cos[(\pi/6)t]$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). During a year, in what intervals is the population less than 20,000? During what intervals is the population more than 140,000?

Solution:

The population is less than 20,000 from December 8 through January 23 and more than 140,000 from May 29 through August 2

Exercise:**Problem:**

In reality, the overall population is most likely increasing or decreasing throughout each year. Let's reformulate the model as $P(t) = 82.5 - 67.5 \cos[(\pi/6)t] + t$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). When is the first time the population reaches 200,000?

For the following problems, consider radioactive dating. A human skeleton is found in an archeological dig. Carbon dating is implemented to determine how old the skeleton is by using the equation $y = e^{rt}$, where y is the percentage of radiocarbon still present in the material, t is the number of years passed, and $r = -0.0001210$ is the decay rate of radiocarbon.

Exercise:**Problem:**

If the skeleton is expected to be 2000 years old, what percentage of radiocarbon should be present?

Solution:

78.51%

Exercise:

Problem:

Find the inverse of the carbon-dating equation. What does it mean? If there is 25% radiocarbon, how old is the skeleton?

Glossary

base

the number b in the exponential function $f(x) = b^x$ and the logarithmic function $f(x) = \log_b x$

exponent

the value x in the expression b^x

hyperbolic functions

the functions denoted \sinh , \cosh , \tanh , csch , sech , and coth , which involve certain combinations of e^x and e^{-x}

inverse hyperbolic functions

the inverses of the hyperbolic functions where \cosh and sech are restricted to the domain $[0, \infty)$; each of these functions can be expressed in terms of a composition of the natural logarithm function and an algebraic function

natural exponential function

the function $f(x) = e^x$

natural logarithm

the function $\ln x = \log_e x$

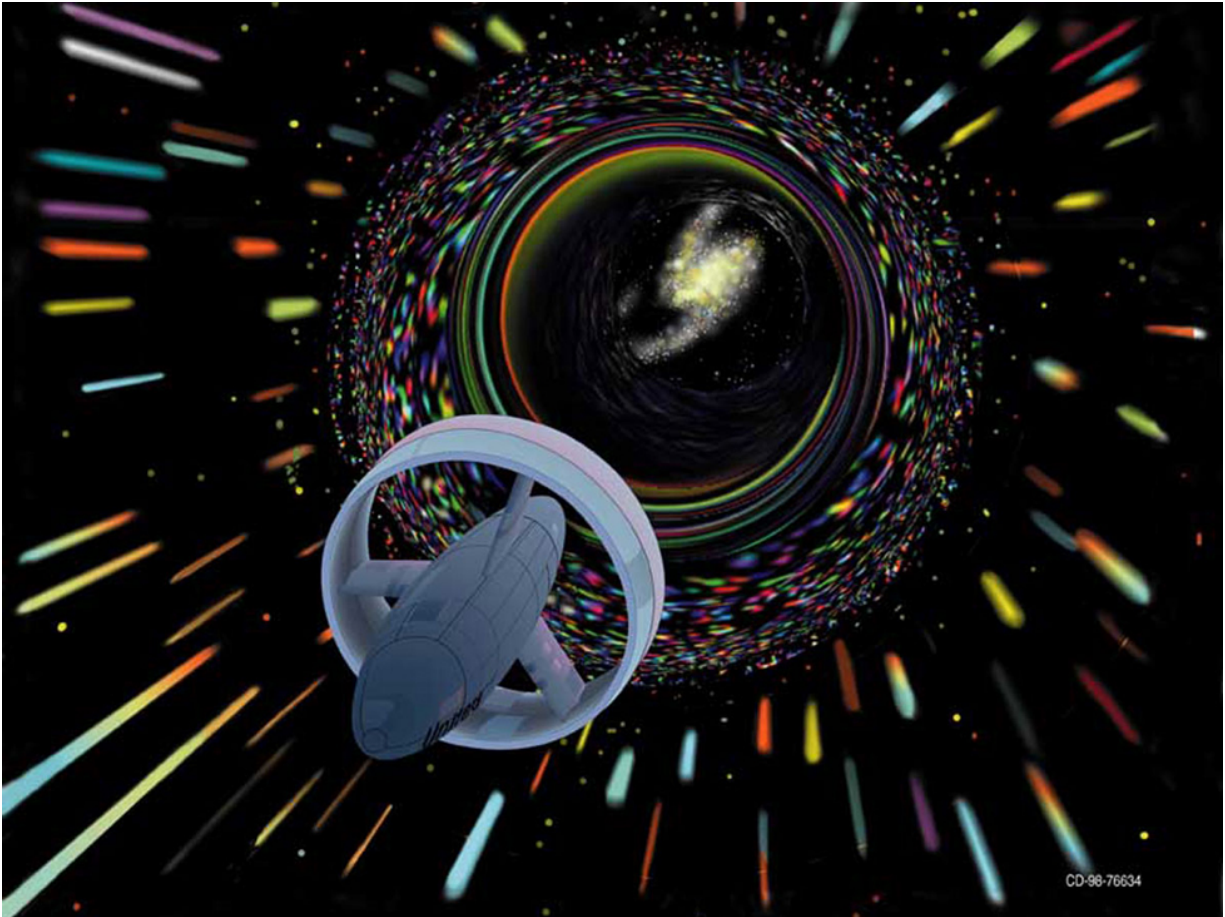
number e

as m gets larger, the quantity $(1 + (1/m))^m$ gets closer to some real number; we define that real number to be e ; the value of e is approximately 2.718282

Introduction

class="introduction"

The vision of human exploration by the National Aeronautics and Space Administration (NASA) to distant parts of the universe illustrates the idea of space travel at high speeds. But, is there a limit to how fast a spacecraft can go? (credit: NASA)



Science fiction writers often imagine spaceships that can travel to far-off planets in distant galaxies. However, back in 1905, Albert Einstein showed that a limit exists to how fast any object can travel. The problem is that the faster an object moves, the more mass it attains (in the form of energy), according to the equation

Equation:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. What is this speed limit? (We explore this problem further in [\[link\]](#).)

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to

find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

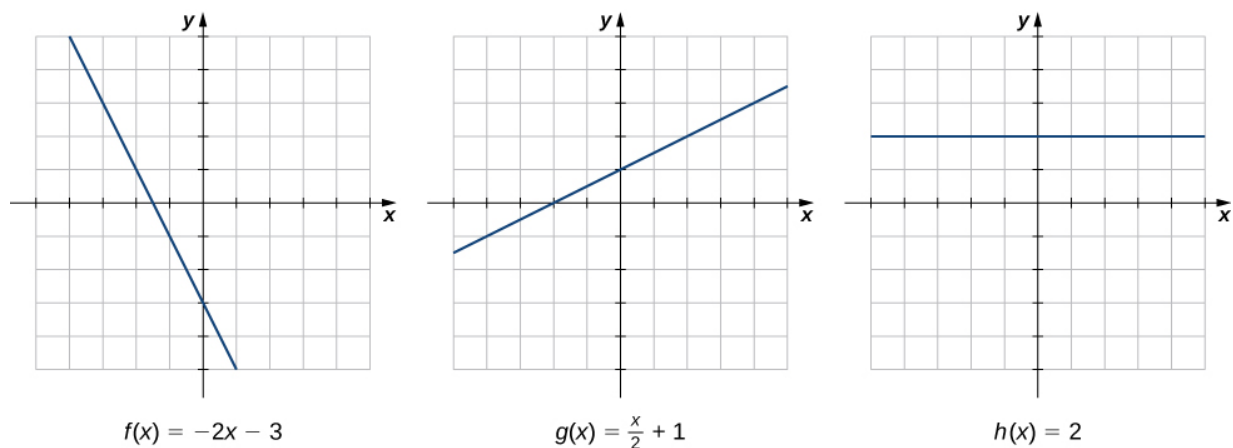
A Preview of Calculus

- Describe the tangent problem and how it led to the idea of a derivative.
- Explain how the idea of a limit is involved in solving the tangent problem.
- Recognize a tangent to a curve at a point as the limit of secant lines.
- Identify instantaneous velocity as the limit of average velocity over a small time interval.
- Describe the area problem and how it was solved by the integral.
- Explain how the idea of a limit is involved in solving the area problem.
- Recognize how the ideas of limit, derivative, and integral led to the studies of infinite series and multivariable calculus.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics—like the space travel problem posed in the chapter opener. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve.

The Tangent Problem and Differential Calculus

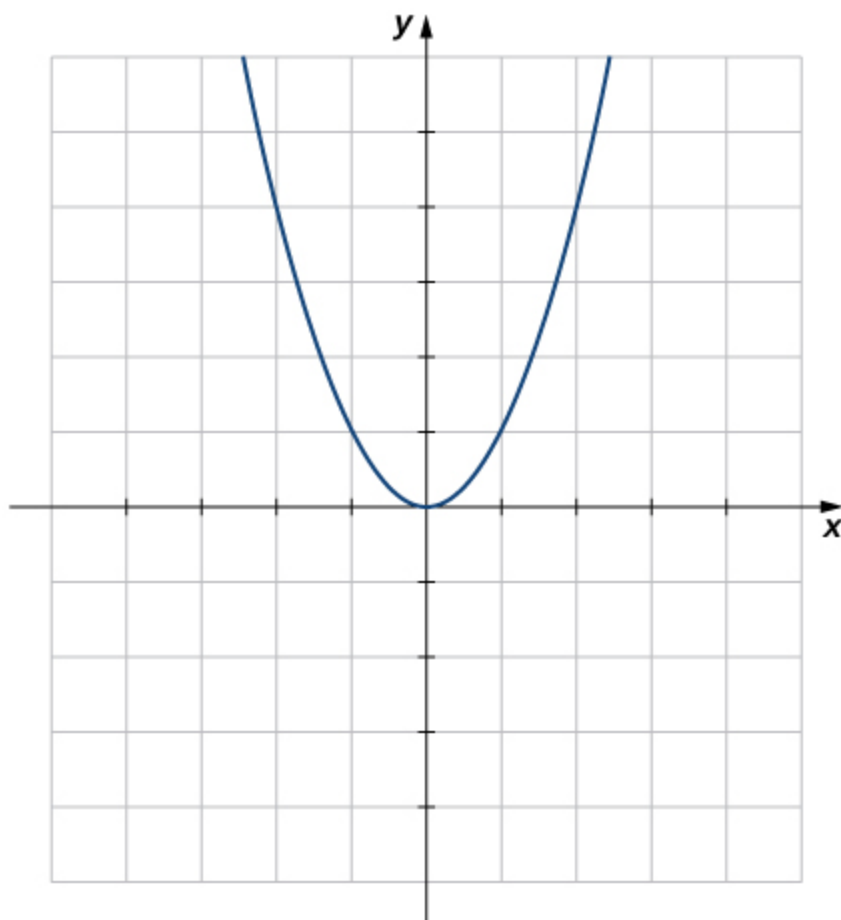
Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x) = -2x - 3$, $g(x) = \frac{1}{2}x + 1$, and $h(x) = 2$, shown in [\[link\]](#).



The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of $f(x) = -2x - 3$, we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the x -axis, the y -coordinate decreases by 2 units. This rate of change is determined by the slope (-2) of the line. Similarly, the slope of $1/2$ in the function $g(x)$ tells us that for every change in x of 1 unit there is a corresponding change in y of $1/2$ unit. The function $h(x) = 2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

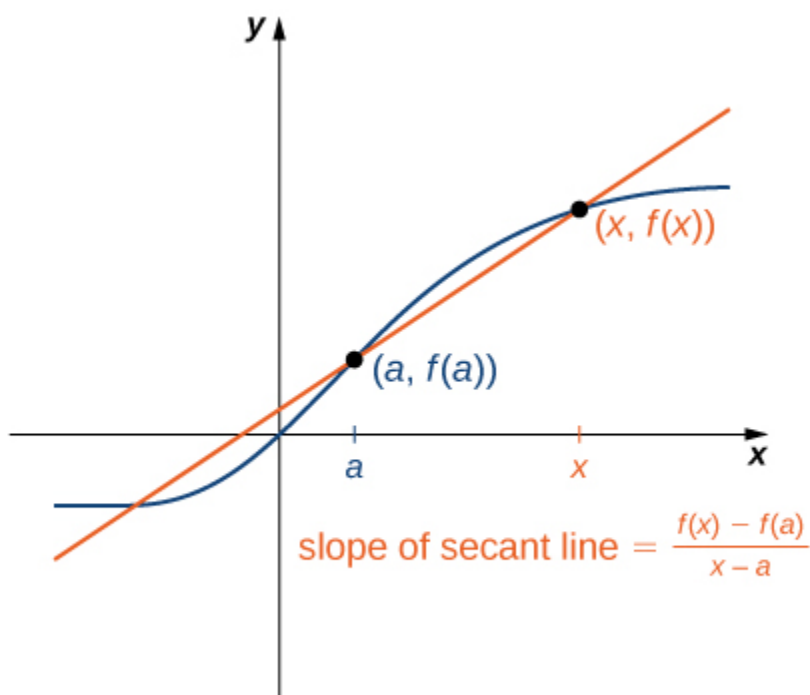
Compare the graphs of these three functions with the graph of $k(x) = x^2$ ([link](#)). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase—slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?



$$k(x) = x^2$$

The function $k(x) = x^2$ does not have a constant rate of change.

We can approximate the rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a **secant** line. [\[link\]](#) shows a secant line to a function $f(x)$ at a point $(a, f(a))$.



The slope of a secant line through a point $(a, f(a))$ estimates the rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

Note:

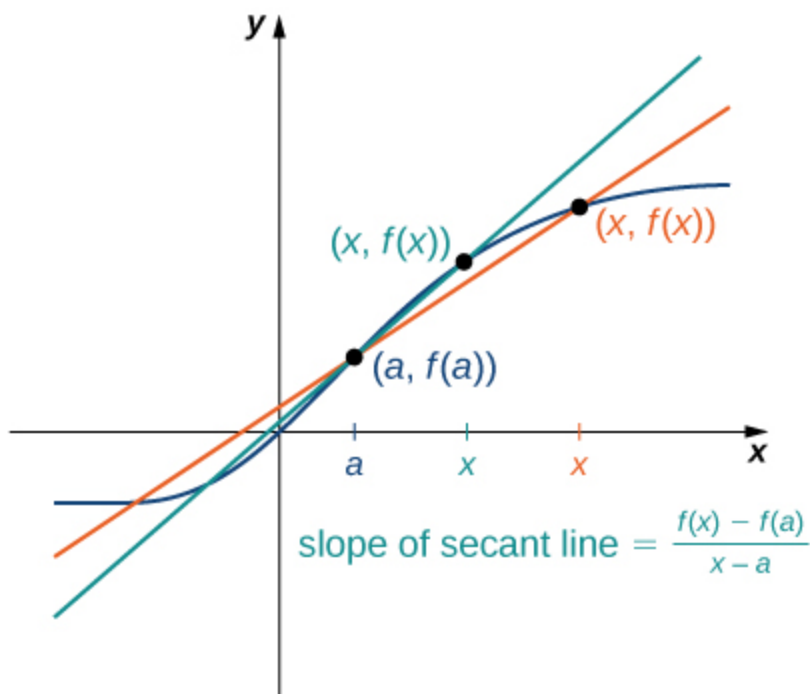
Definition

The **secant** to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

Equation:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

The accuracy of approximating the rate of change of the function with a secant line depends on how close x is to a . As we see in [\[link\]](#), if x is closer to a , the slope of the secant line is a better measure of the rate of change of $f(x)$ at a .

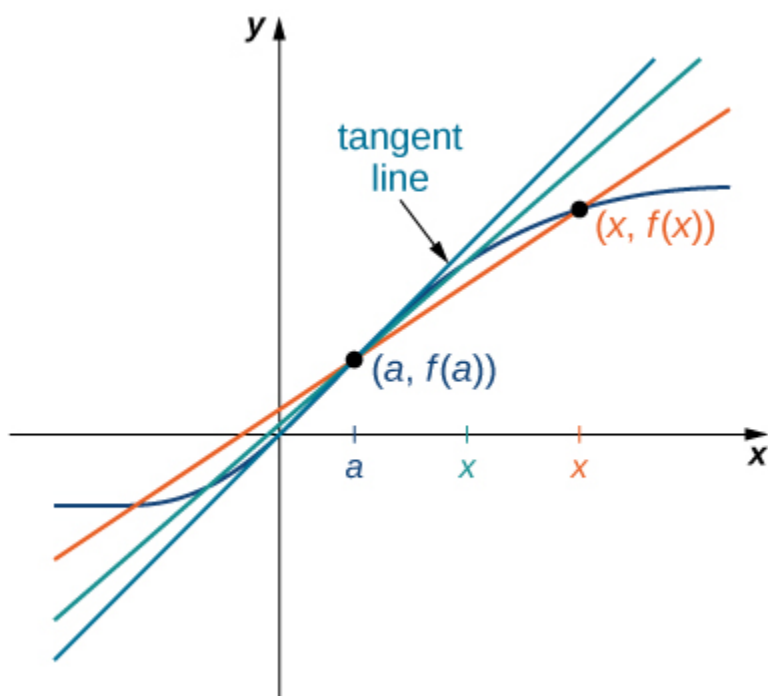


As x gets closer to a , the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at a .

The secant lines themselves approach a line that is called the **tangent** to the function $f(x)$ at a ([\[link\]](#)). The slope of the tangent line to the graph at a measures the rate of change of the function at a . This value also represents the derivative of the function $f(x)$ at a , or the rate of change of the function at a . This derivative is denoted by $f'(a)$. **Differential calculus** is the field of calculus concerned with the study of derivatives and their applications.

Note:

For an interactive demonstration of the slope of a secant line that you can manipulate yourself, visit this applet (*Note:* this site requires a Java browser plugin): [Math Insight](#).



Solving the Tangent Problem: As x approaches a , the secant lines approach the tangent line.

[\[link\]](#) illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the rate of change of the function at the point at which the slopes are calculated.

Example:**Exercise:****Problem:**

Finding Slopes of Secant Lines

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and each of the following points on the graph of $f(x) = x^2$.

- a. $(2, 4)$
- b. $(\frac{3}{2}, \frac{9}{4})$

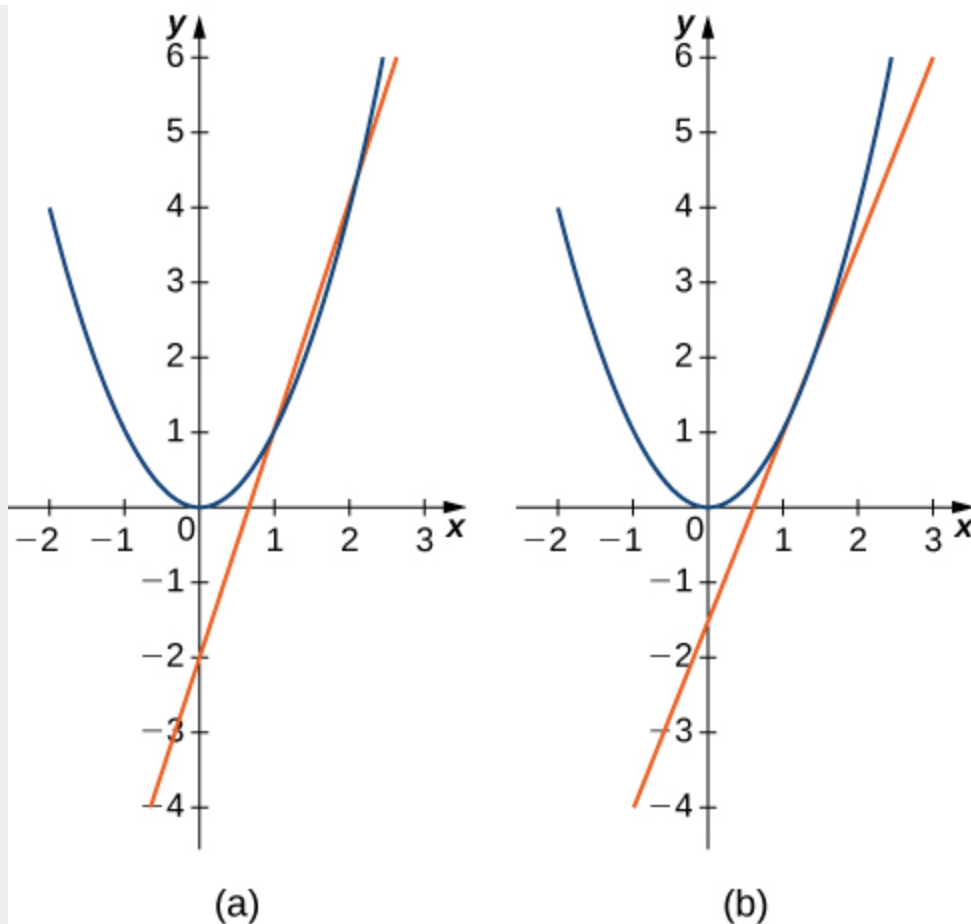
Solution:

Use the formula for the slope of a secant line from the definition.

$$\text{a. } m_{\text{sec}} = \frac{4-1}{2-1} = 3$$

$$\text{b. } m_{\text{sec}} = \frac{\frac{9}{4}-1}{\frac{3}{2}-1} = \frac{5}{2} = 2.5$$

The point in part b. is closer to the point $(1, 1)$, so the slope of 2.5 is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of 2 to 2.5 ([\[link\]](#)).



The secant lines to $f(x) = x^2$ at $(1, 1)$ through (a) $(2, 4)$ and (b) $(\frac{3}{2}, \frac{9}{4})$ provide successively closer approximations to the tangent line to $f(x) = x^2$ at $(1, 1)$.

Note:

Exercise:

Problem:

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and the point $(\frac{5}{4}, \frac{25}{16})$ on the graph of $f(x) = x^2$.

Solution:

2.25

Hint

Use [\[link\]](#) and [\[link\]](#) as a solving guide.

We continue our investigation by exploring a related question. Keeping in mind that velocity may be thought of as the rate of change of position, suppose that we have a function, $s(t)$, that gives the position of an object along a coordinate axis at any given time t . Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t = a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the **average velocity** of an object over a time period to be the change in its position divided by the length of the time period.

Note:**Definition**

Let $s(t)$ be the position of an object moving along a coordinate axis at time t . The **average velocity** of the object over a time interval $[a, t]$ where $a < t$ (or $[t, a]$ if $t < a$) is

Equation:

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}.$$

As t is chosen closer to a , the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point a , we let the x -values approach a in the slope of the secant line. Similarly, to find the instantaneous velocity at time a , we let the t -values approach a in the average velocity. This process of letting x or t approach a in an expression is called taking a **limit**. Thus, we may define the **instantaneous velocity** as follows.

Note:

Definition

For a position function $s(t)$, the **instantaneous velocity** at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of t become closer to a , provided such a value exists.

[\[link\]](#) illustrates this concept of limits and average velocity.

Example:

Exercise:

Problem:

Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground t seconds later (for $0 \leq t \leq 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

- a. $[0.49, 0.5]$
- b. $[0.5, 0.51]$

Solution:

Substitute the data into the formula for the definition of average velocity.

$$\begin{aligned}\text{a. } v_{\text{ave}} &= \frac{s(0.49) - s(0.5)}{0.49 - 0.5} = -15.84 \\ \text{b. } v_{\text{ave}} &= \frac{s(0.51) - s(0.5)}{0.51 - 0.5} = -16.016\end{aligned}$$

The instantaneous velocity is somewhere between -15.84 and -16.16 ft/sec. A good guess might be -16 ft/sec.

Note:

Exercise:

Problem:

An object moves along a coordinate axis so that its position at time t is given by $s(t) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

Solution:

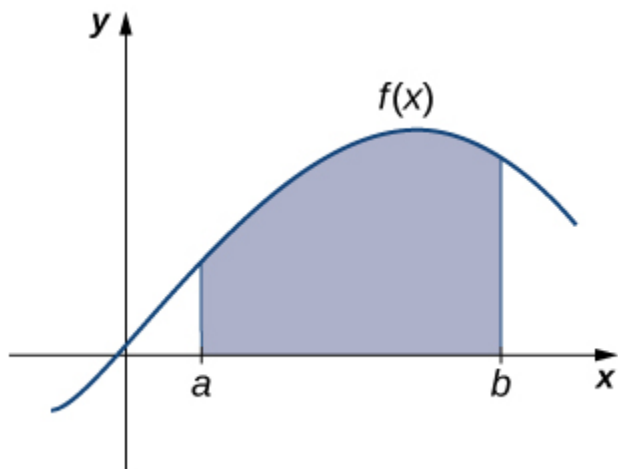
12.006001

Hint

$$\text{Use } v_{\text{ave}} = \frac{s(2.001) - s(2)}{2.001 - 2}.$$

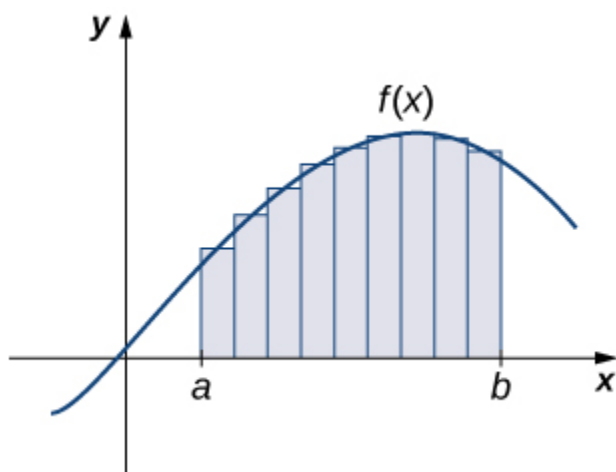
The Area Problem and Integral Calculus

We now turn our attention to a classic question from calculus. Many quantities in physics—for example, quantities of work—may be interpreted as the area under a curve. This leads us to ask the question: How can we find the area between the graph of a function and the x -axis over an interval ([link](#))?



The Area Problem: How do we find the area of the shaded region?

As in the answer to our previous questions on velocity, we first try to approximate the solution. We approximate the area by dividing up the interval $[a, b]$ into smaller intervals in the shape of rectangles. The approximation of the area comes from adding up the areas of these rectangles ([link](#)).



The area of the region under the curve is approximated by summing the areas of thin rectangles.

As the widths of the rectangles become smaller (approach zero), the sums of the areas of the rectangles approach the area between the graph of $f(x)$ and the x -axis over the interval $[a, b]$. Once again, we find ourselves taking a limit. Limits of this type serve as a basis for the definition of the definite integral. **Integral calculus** is the study of integrals and their applications.

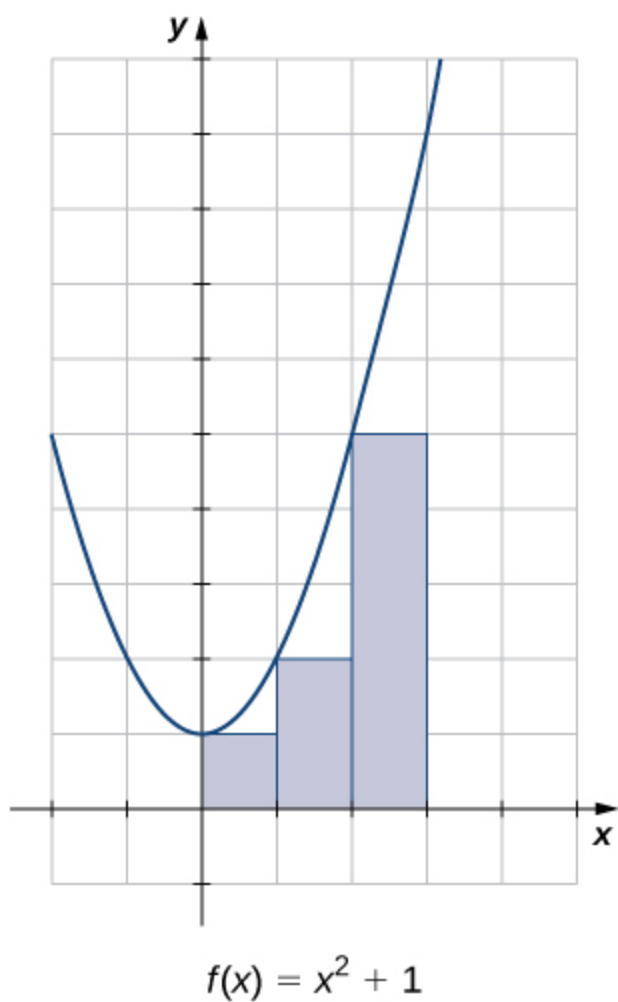
Example:

Exercise:

Problem:

Estimation Using Rectangles

Estimate the area between the x -axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in [\[link\]](#).



The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

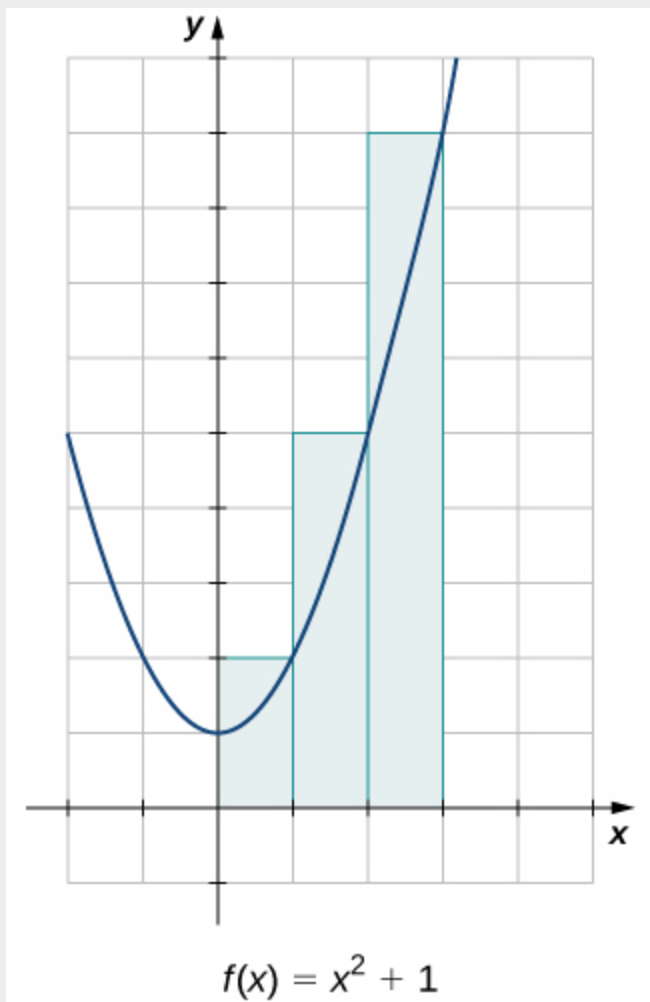
Solution:

The areas of the three rectangles are 1 unit², 2 unit², and 5 unit². Using these rectangles, our area estimate is 8 unit².

Note:

Exercise:**Problem:**

Estimate the area between the x -axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown here:

**Solution:**

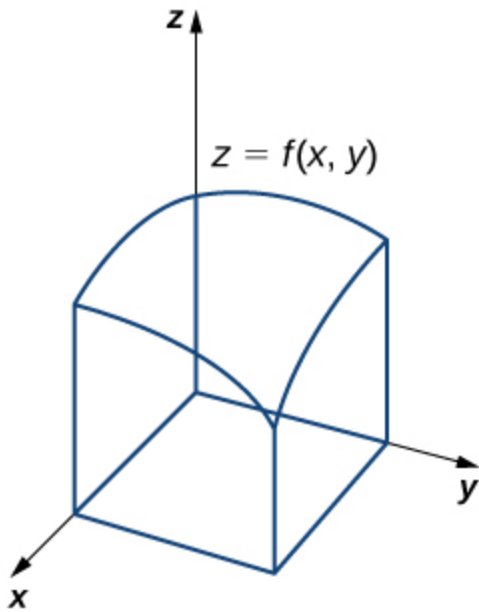
16 unit²

Hint

Use [\[link\]](#) as a guide.

Other Aspects of Calculus

So far, we have studied functions of one variable only. Such functions can be represented visually using graphs in two dimensions; however, there is no good reason to restrict our investigation to two dimensions. Suppose, for example, that instead of determining the velocity of an object moving along a coordinate axis, we want to determine the velocity of a rock fired from a catapult at a given time, or of an airplane moving in three dimensions. We might want to graph real-value functions of two variables or determine volumes of solids of the type shown in [\[link\]](#). These are only a few of the types of questions that can be asked and answered using **multivariable calculus**. Informally, multivariable calculus can be characterized as the study of the calculus of functions of two or more variables. However, before exploring these and other ideas, we must first lay a foundation for the study of calculus in one variable by exploring the concept of a limit.



We can use multivariable calculus to find the volume between a surface defined by a function of two variables and a plane.

Key Concepts

- Differential calculus arose from trying to solve the problem of determining the slope of a line tangent to a curve at a point. The slope of the tangent line indicates the rate of change of the function, also called the *derivative*. Calculating a derivative requires finding a limit.
- Integral calculus arose from trying to solve the problem of finding the area of a region between the graph of a function and the x -axis. We can approximate the area by dividing it into thin rectangles and summing the areas of these rectangles. This summation leads to the value of a function called the *integral*. The integral is also calculated by finding a limit and, in fact, is related to the derivative of a function.
- Multivariable calculus enables us to solve problems in three-dimensional space, including determining motion in space and finding volumes of solids.

Key Equations

- **Slope of a Secant Line**

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}$$

- **Average Velocity over Interval $[a, t]$**

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}$$

For the following exercises, points $P(1, 2)$ and $Q(x, y)$ are on the graph of the function $f(x) = x^2 + 1$.

Exercise:

Problem:

[T] Complete the following table with the appropriate values: y -coordinate of Q , the point $Q(x, y)$, and the slope of the secant line passing through points P and Q . Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

Solution:

a. 2.2100000; b. 2.0201000; c. 2.0020010; d. 2.0002000; e. (1.1000000, 2.2100000); f. (1.0100000, 2.0201000); g. (1.0010000, 2.0020010); h. (1.0001000, 2.0002000); i. 2.1000000; j. 2.0100000; k. 2.0010000; l. 2.0001000

Exercise:

Problem:

Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x = 1$.

Exercise:

Problem:

Use the value in the preceding exercise to find the equation of the tangent line at point P . Graph $f(x)$ and the tangent line.

Solution:

$$y = 2x$$

For the following exercises, points $P(1, 1)$ and $Q(x, y)$ are on the graph of the function $f(x) = x^3$.

Exercise:

Problem:

[T] Complete the following table with the appropriate values: y -coordinate of Q , the point $Q(x, y)$, and the slope of the secant line passing through points P and Q . Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

Exercise:

Problem:

Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x = 1$.

Solution:

Exercise:**Problem:**

Use the value in the preceding exercise to find the equation of the tangent line at point P . Graph $f(x)$ and the tangent line.

For the following exercises, points $P(4, 2)$ and $Q(x, y)$ are on the graph of the function $f(x) = \sqrt{x}$.

Exercise:**Problem:**

[T] Complete the following table with the appropriate values: y -coordinate of Q , the point $Q(x, y)$, and the slope of the secant line passing through points P and Q . Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
4.1	a.	e.	i.
4.01	b.	f.	j.
4.001	c.	g.	k.
4.0001	d.	h.	l.

Solution:

a. 2.0248457; b. 2.0024984; c. 2.0002500; d. 2.0000250; e. (4.1000000, 2.0248457); f. (4.0100000, 2.0024984); g.

(4.0010000,2.0002500); h. (4.00010000,2.0000250); i. 0.24845673; j. 0.24984395; k. 0.24998438; l. 0.24999844

Exercise:

Problem:

Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x = 4$.

Exercise:

Problem:

Use the value in the preceding exercise to find the equation of the tangent line at point P .

Solution:

$$y = \frac{x}{4} + 1$$

For the following exercises, points $P(1.5, 0)$ and $Q(\phi, y)$ are on the graph of the function $f(\phi) = \cos(\pi\phi)$.

Exercise:

Problem:

[T] Complete the following table with the appropriate values: y -coordinate of Q , the point $Q(x, y)$, and the slope of the secant line passing through points P and Q . Round your answer to eight significant digits.

x	y	$Q(\phi, y)$	m_{sec}
-----	-----	--------------	------------------

x	y	$Q(\phi, y)$	m_{sec}
1.4	a.	e.	i.
1.49	b.	f.	j.
1.499	c.	g.	k.
1.4999	d.	h.	l.

Exercise:

Problem:

Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x = 4$.

Solution:

π

Exercise:

Problem:

Use the value in the preceding exercise to find the equation of the tangent line at point P .

For the following exercises, points $P(-1, -1)$ and $Q(x, y)$ are on the graph of the function $f(x) = \frac{1}{x}$.

Exercise:

Problem:

[T] Complete the following table with the appropriate values: y -coordinate of Q , the point $Q(x, y)$, and the slope of the secant line passing through points P and Q . Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
-1.05	a.	e.	i.
-1.01	b.	f.	j.
-1.005	c.	g.	k.
-1.001	d.	h.	l.

Solution:

a. -0.95238095; b. -0.99009901; c. -0.99502488; d. -0.99900100; e. (-1;.0500000,-0;.95238095); f. (-1;.0100000,-0;.9909901); g. (-1;.0050000,-0;.99502488); h. (-1.0010000,-0;.99900100); i. -0.95238095; j. -0.99009901; k. -0.99502488; l. -0.99900100

Exercise:**Problem:**

Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x = -1$.

Exercise:**Problem:**

Use the value in the preceding exercise to find the equation of the tangent line at point P .

Solution:

$$y = -x - 2$$

For the following exercises, the position function of a ball dropped from the top of a 200-meter tall building is given by $s(t) = 200 - 4.9t^2$, where position s is measured in meters and time t is measured in seconds. Round your answer to eight significant digits.

Exercise:**Problem:**

[T] Compute the average velocity of the ball over the given time intervals.

- a. $[4.99, 5]$
- b. $[5, 5.01]$
- c. $[4.999, 5]$
- d. $[5, 5.001]$

Exercise:**Problem:**

Use the preceding exercise to guess the instantaneous velocity of the ball at $t = 5$ sec.

Solution:

-49 m/sec (velocity of the ball is 49 m/sec downward)

For the following exercises, consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time t seconds is $h(t) = 15t - 4.9t^2$.

Exercise:

Problem:

[T] Compute the average velocity of the stone over the given time intervals.

- a. $[1, 1.05]$
- b. $[1, 1.01]$
- c. $[1, 1.005]$
- d. $[1, 1.001]$

Exercise:

Problem:

Use the preceding exercise to guess the instantaneous velocity of the stone at $t = 1$ sec.

Solution:

5.2 m/sec

For the following exercises, consider a rocket shot into the air that then returns to Earth. The height of the rocket in meters is given by $h(t) = 600 + 78.4t - 4.9t^2$, where t is measured in seconds.

Exercise:

Problem:

[T] Compute the average velocity of the rocket over the given time intervals.

- a. $[9, 9.01]$
- b. $[8.99, 9]$

- c. $[9, 9.001]$
- d. $[8.999, 9]$

Exercise:

Problem:

Use the preceding exercise to guess the instantaneous velocity of the rocket at $t = 9$ sec.

Solution:

-9.8 m/sec

For the following exercises, consider an athlete running a 40-m dash. The position of the athlete is given by $d(t) = \frac{t^3}{6} + 4t$, where d is the position in meters and t is the time elapsed, measured in seconds.

Exercise:

Problem:

[T] Compute the average velocity of the runner over the given time intervals.

- a. $[1.95, 2.05]$
- b. $[1.995, 2.005]$
- c. $[1.9995, 2.0005]$
- d. $[2, 2.00001]$

Exercise:

Problem:

Use the preceding exercise to guess the instantaneous velocity of the runner at $t = 2$ sec.

Solution:

6 m/sec

For the following exercises, consider the function $f(x) = |x|$.

Exercise:

Problem:

Sketch the graph of f over the interval $[-1, 2]$ and shade the region above the x -axis.

Exercise:

Problem:

Use the preceding exercise to find the exact value of the area between the x -axis and the graph of f over the interval $[-1, 2]$ using rectangles. For the rectangles, use the square units, and approximate both above and below the lines. Use geometry to find the exact answer.

Solution:

Under, 1 unit²; over: 4 unit². The exact area of the two triangles is $\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = 2.5 \text{ units}^2$.

For the following exercises, consider the function $f(x) = \sqrt{1 - x^2}$. (*Hint: This is the upper half of a circle of radius 1 positioned at (0, 0).*)

Exercise:

Problem: Sketch the graph of f over the interval $[-1, 1]$.

Exercise:

Problem:

Use the preceding exercise to find the exact area between the x -axis and the graph of f over the interval $[-1, 1]$ using rectangles. For the rectangles, use squares 0.4 by 0.4 units, and approximate both above and below the lines. Use geometry to find the exact answer.

Solution:

Under, 0.96 unit^2 ; over, 1.92 unit^2 . The exact area of the semicircle with radius 1 is $\frac{\pi(1)^2}{2} = \frac{\pi}{2} \text{ unit}^2$.

For the following exercises, consider the function $f(x) = -x^2 + 1$.

Exercise:

Problem: Sketch the graph of f over the interval $[-1, 1]$.

Exercise:**Problem:**

Approximate the area of the region between the x -axis and the graph of f over the interval $[-1, 1]$.

Solution:

Approximately 1.3333333 unit^2

Glossary

average velocity

the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval $[t, a]$ (if $t < a$ or $[a, t]$ if $t > a$), with a position given by $s(t)$, that is

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}$$

differential calculus

the field of calculus concerned with the study of derivatives and their applications

instantaneous velocity

The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the

form $[t, a]$ and $[a, t]$ approach as the values of t move closer to a , provided such a value exists

integral calculus

the study of integrals and their applications

limit

the process of letting x or t approach a in an expression; the limit of a function $f(x)$ as x approaches a is the value that $f(x)$ approaches as x approaches a

multivariable calculus

the study of the calculus of functions of two or more variables

secant

A secant line to a function $f(x)$ at a is a line through the point $(a, f(a))$ and another point on the function; the slope of the secant line is given by $m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}$

tangent

A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with x -values that approach a ; the slope of the tangent line to a graph at a measures the rate of change of the function at a

The Limit of a Function

- Using correct notation, describe the limit of a function.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- Use a graph to estimate the limit of a function or to identify when the limit does not exist.
- Define one-sided limits and provide examples.
- Explain the relationship between one-sided and two-sided limits.
- Using correct notation, describe an infinite limit.
- Define a vertical asymptote.

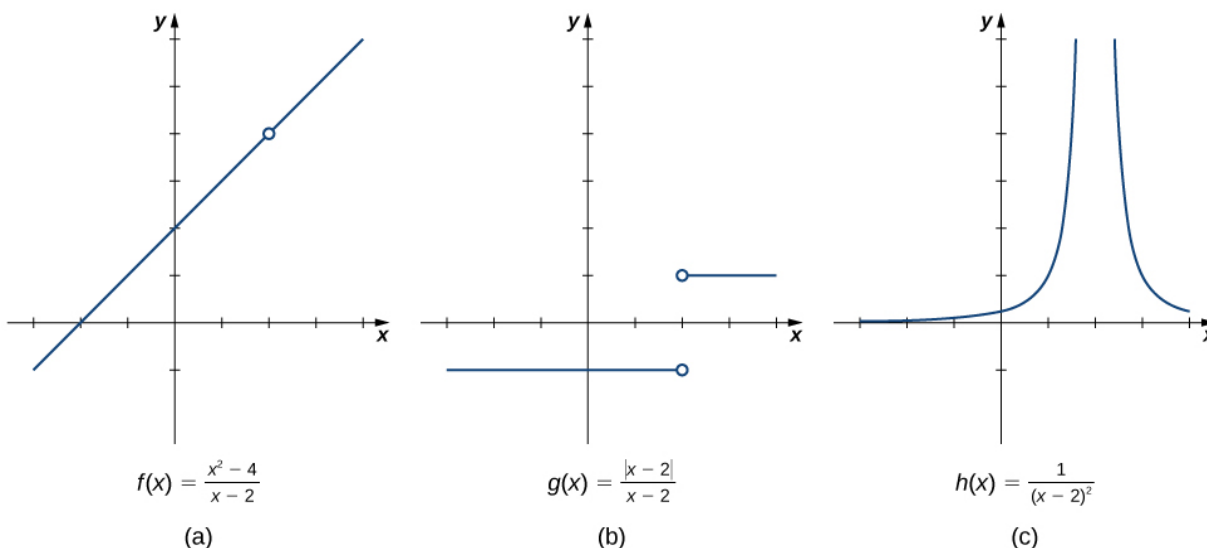
The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions

Equation:

$$f(x) = \frac{x^2 - 4}{x - 2}, \quad g(x) = \frac{|x - 2|}{x - 2}, \quad \text{and} \quad h(x) = \frac{1}{(x - 2)^2},$$

which are shown in [\[link\]](#). In particular, let's focus our attention on the behavior of each graph at and around $x = 2$.



These graphs show the behavior of three different functions around $x = 2$.

Each of the three functions is undefined at $x = 2$, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2$. To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

Intuitive Definition of a Limit

Let's first take a closer look at how the function $f(x) = (x^2 - 4)/(x - 2)$ behaves around $x = 2$ in [\[link\]](#). As the values of x approach 2 from either side of 2, the values of $y = f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as x approaches 2 is 4. Symbolically, we express this limit as

Equation:

$$\lim_{x \rightarrow 2} f(x) = 4.$$

From this very brief informal look at one limit, let's start to develop an **intuitive definition of the limit**. We can think of the limit of a function at a number a as being the one real number L that the functional values approach as the x -values approach a , provided such a real number L exists. Stated more carefully, we have the following definition:

Note:**Definition**

Let $f(x)$ be a function defined at all values in an open interval containing a , with the possible exception of a itself, and let L be a real number. If *all* values of the function $f(x)$ approach the real number L as the values of x ($\neq a$) approach the number a , then we say that the limit of $f(x)$ as x approaches a is L . (More succinct, as x gets closer to a , $f(x)$ gets closer and stays close to L .) Symbolically, we express this idea as

Equation:

$$\lim_{x \rightarrow a} f(x) = L.$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

Note:**Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values**

1. To evaluate $\lim_{x \rightarrow a} f(x)$, we begin by completing a table of functional values. We should choose two sets of x -values—one set of values approaching a and less than a , and another set of values approaching a and greater than a . [\[link\]](#) demonstrates what your tables might look like.

x	$f(x)$		x	$f(x)$
$a - 0.1$	$f(a - 0.1)$		$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$		$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$		$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$		$a + 0.0001$	$f(a + 0.0001)$

x	$f(x)$		x	$f(x)$
Use additional values as necessary.			Use additional values as necessary.	

Table of Functional Values for $\lim_{x \rightarrow a} f(x)$

- Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a - 0.1), f(a - 0.01), f(a - 0.001), f(a - 0.0001)$, and so on, and $f(a + 0.1), f(a + 0.01), f(a + 0.001), f(a + 0.0001)$, and so on. (Note: Although we have chosen the x -values $a \pm 0.1, a \pm 0.01, a \pm 0.001, a \pm 0.0001$, and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)
- If both columns approach a common y -value L , we state $\lim_{x \rightarrow a} f(x) = L$. We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.
- Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for x -values near a are in our window. We can use the trace feature to move along the graph of the function and watch the y -value readout as the x -values approach a . If the y -values approach L as our x -values approach a from both directions, then $\lim_{x \rightarrow a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in [\[link\]](#).

Example:

Exercise:

Problem:

Evaluating a Limit Using a Table of Functional Values 1

Evaluate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ using a table of functional values.

Solution:

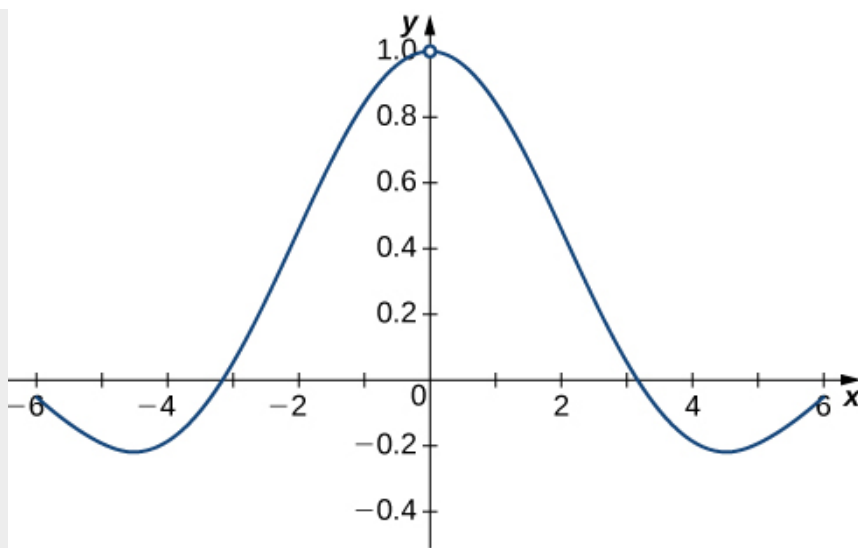
We have calculated the values of $f(x) = (\sin x)/x$ for the values of x listed in [\[link\]](#).

x	$\frac{\sin x}{x}$		x	$\frac{\sin x}{x}$
-0.1	0.998334166468		0.1	0.998334166468
-0.01	0.999983333417		0.01	0.999983333417
-0.001	0.999998333333		0.001	0.999998333333
-0.0001	0.999999983333		0.0001	0.999999983333

Table of Functional Values for $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output.

As we read down each $\frac{(\sin x)}{x}$ column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. A calculator-or computer-generated graph of $f(x) = \frac{(\sin x)}{x}$ would be similar to that shown in [\[link\]](#), and it confirms our estimate.



The graph of $f(x) = (\sin x)/x$ confirms the estimate from [\[link\]](#).

Example:

Exercise:

Problem:

Evaluating a Limit Using a Table of Functional Values 2

Evaluate $\lim_{x \rightarrow 4} \frac{\sqrt{x}-2}{x-4}$ using a table of functional values.

Solution:

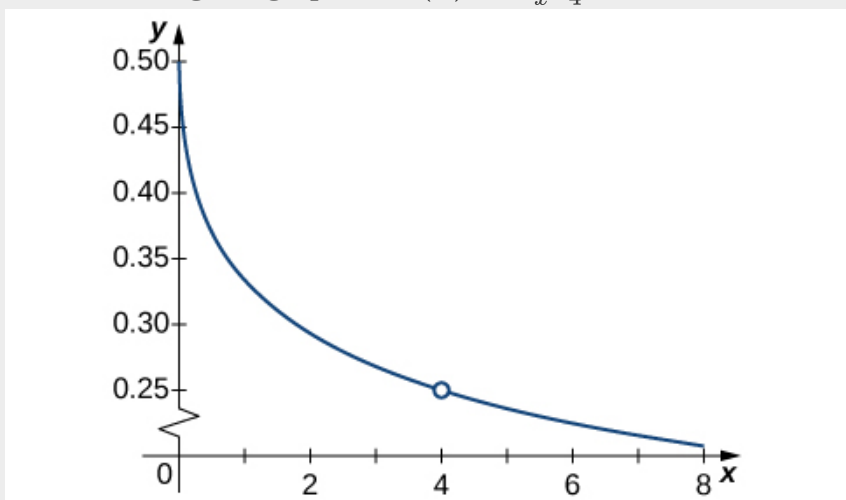
As before, we use a table—in this case, [\[link\]](#)—to list the values of the function for the given values of x .

x	$\frac{\sqrt{x}-2}{x-4}$		x	$\frac{\sqrt{x}-2}{x-4}$
-----	--------------------------	--	-----	--------------------------

x	$\frac{\sqrt{x}-2}{x-4}$		x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869		4.1	0.248456731317
3.99	0.25015644562		4.01	0.24984394501
3.999	0.250015627		4.001	0.249984377
3.9999	0.250001563		4.0001	0.249998438
3.99999	0.25000016		4.00001	0.24999984

Table of Functional Values for $\lim_{x \rightarrow 4} \frac{\sqrt{x}-2}{x-4}$

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that $\lim_{x \rightarrow 4} \frac{\sqrt{x}-2}{x-4} = 0.25$. We confirm this estimate using the graph of $f(x) = \frac{\sqrt{x}-2}{x-4}$ shown in [\[link\]](#).



The graph of $f(x) = \frac{\sqrt{x}-2}{x-4}$ confirms the estimate from [\[link\]](#).

Note:**Exercise:****Problem:**

Estimate $\lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x - 1}$ using a table of functional values. Use a graph to confirm your estimate.

Solution:

$$\lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x - 1} = -1$$

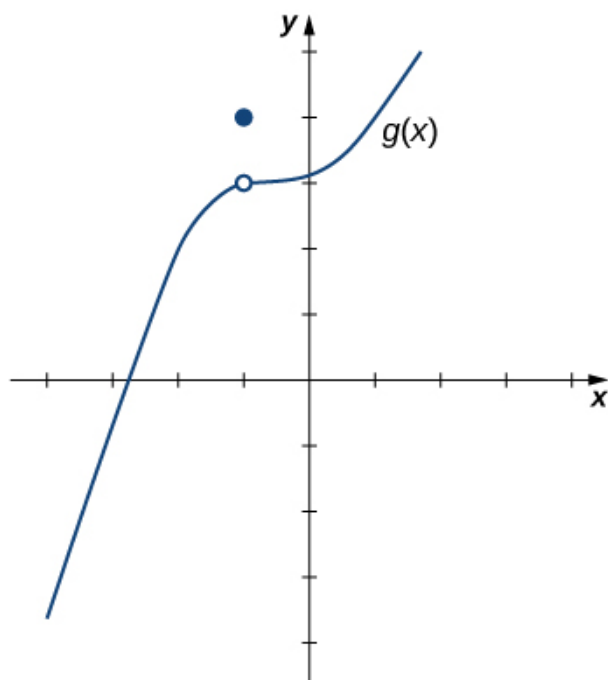
Hint

Use 0.9, 0.99, 0.999, 0.9999, 0.99999 and 1.1, 1.01, 1.001, 1.0001, 1.00001 as your table values.

At this point, we see from [\[link\]](#) and [\[link\]](#) that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In [\[link\]](#), we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

Example:**Exercise:****Problem:****Evaluating a Limit Using a Graph**

For $g(x)$ shown in [\[link\]](#), evaluate $\lim_{x \rightarrow -1} g(x)$.



The graph of $g(x)$ includes one value not on a smooth curve.

Solution:

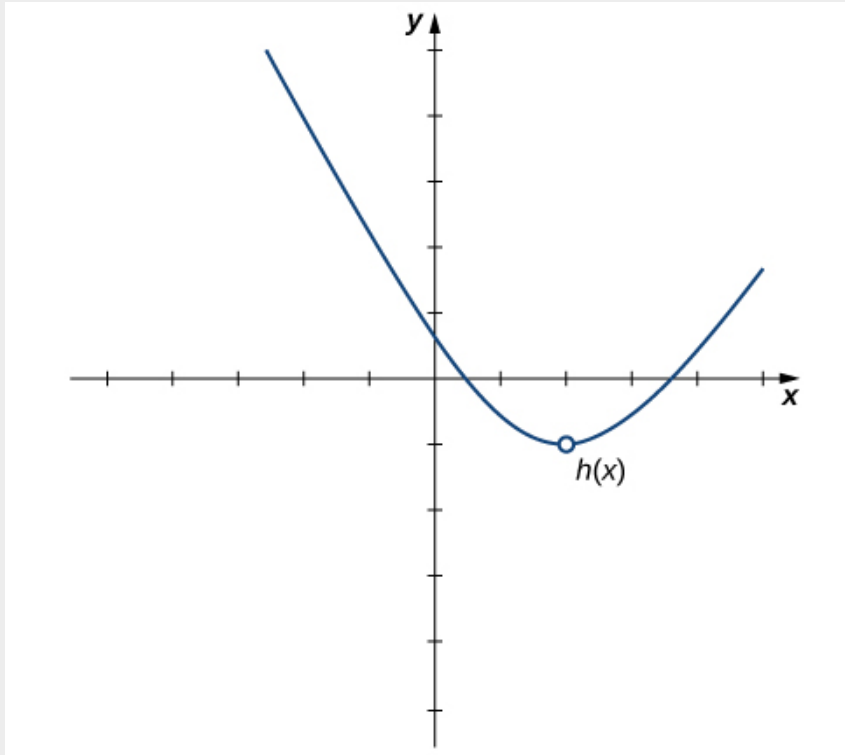
Despite the fact that $g(-1) = 4$, as the x -values approach -1 from either side, the $g(x)$ values approach 3 . Therefore, $\lim_{x \rightarrow -1} g(x) = 3$. Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on [\[link\]](#), we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

Note:

Exercise:

Problem: Use the graph of $h(x)$ in [\[link\]](#) to evaluate $\lim_{x \rightarrow 2} h(x)$, if possible.



Solution:

$$\lim_{x \rightarrow 2} h(x) = -1.$$

Hint

What y-value does the function approach as the x-values approach 2?

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

Note:

Two Important Limits

Let a be a real number and c be a constant.

i. **Equation:**

$$\lim_{x \rightarrow a} x = a$$

ii. **Equation:**

$$\lim_{x \rightarrow a} c = c$$

We can make the following observations about these two limits.

- i. For the first limit, observe that as x approaches a , so does $f(x)$, because $f(x) = x$. Consequently, $\lim_{x \rightarrow a} x = a$.
- ii. For the second limit, consider [\[link\]](#).

x	$f(x) = c$		x	$f(x) = c$
$a - 0.1$	c		$a + 0.1$	c
$a - 0.01$	c		$a + 0.01$	c
$a - 0.001$	c		$a + 0.001$	c
$a - 0.0001$	c		$a + 0.0001$	c

Table of Functional Values for $\lim_{x \rightarrow a} c = c$

Observe that for all values of x (regardless of whether they are approaching a), the values $f(x)$ remain constant at c . We have no choice but to conclude $\lim_{x \rightarrow a} c = c$.

The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

Example:
Exercise:

Problem:
Evaluating a Limit That Fails to Exist

 Evaluate $\lim_{x \rightarrow 0} \sin \left(\frac{1}{x} \right)$ using a table of values.

Solution:

[\[link\]](#) lists values for the function $\sin(1/x)$ for the given values of x .

x	$\sin \left(\frac{1}{x} \right)$		x	$\sin \left(\frac{1}{x} \right)$
-0.1	0.544021110889		0.1	-0.544021110889
-0.01	0.50636564111		0.01	-0.50636564111
-0.001	-0.8268795405312		0.001	0.826879540532
-0.0001	0.305614388888		0.0001	-0.305614388888
-0.00001	-0.035748797987		0.00001	0.035748797987
-0.000001	0.349993504187		0.000001	-0.349993504187

Table of Functional Values for $\lim_{x \rightarrow 0} \sin \left(\frac{1}{x} \right)$

 After examining the table of functional values, we can see that the y-values do not seem to approach any one single value. It appears the limit does not exist.

Before drawing this conclusion, let's take a more systematic approach. Take the following sequence of x -values approaching 0:

Equation:

$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \dots$$

The corresponding y -values are

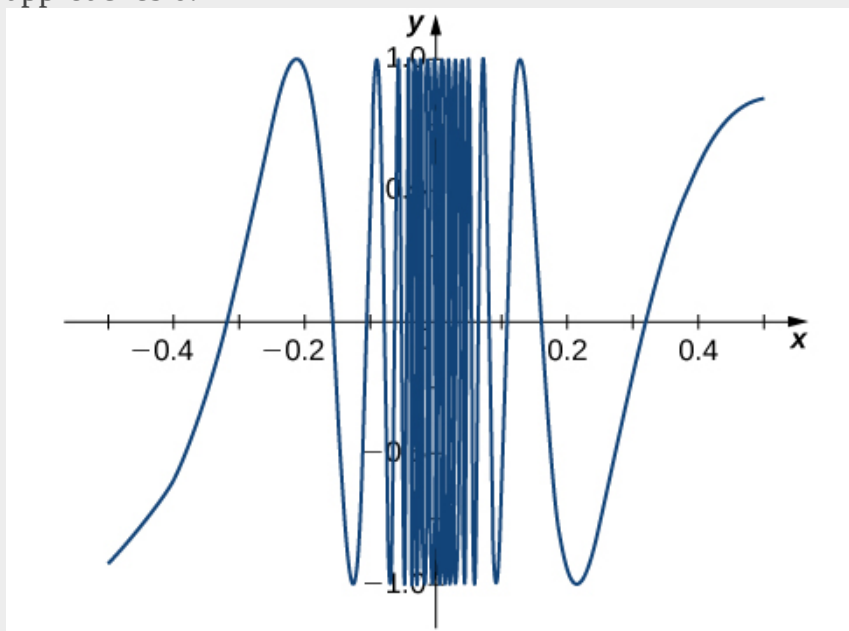
Equation:

$$1, -1, 1, -1, 1, -1, \dots$$

At this point we can indeed conclude that $\lim_{x \rightarrow 0} \sin(1/x)$ does not exist.

(Mathematicians frequently abbreviate “does not exist” as DNE. Thus, we would write $\lim_{x \rightarrow 0} \sin(1/x)$ DNE.) The graph of $f(x) = \sin(1/x)$ is shown in [\[link\]](#)

and it gives a clearer picture of the behavior of $\sin(1/x)$ as x approaches 0. You can see that $\sin(1/x)$ oscillates ever more wildly between -1 and 1 as x approaches 0.



The graph of $f(x) = \sin(1/x)$ oscillates rapidly between -1 and 1 as x approaches 0.

Note:

Exercise:

Problem: Use a table of functional values to evaluate $\lim_{x \rightarrow 2} \frac{|x^2 - 4|}{x - 2}$, if possible.

Solution:

$\lim_{x \rightarrow 2} \frac{|x^2 - 4|}{x - 2}$ does not exist.

Hint

Use x -values 1.9, 1.99, 1.999, 1.9999, 1.9999 and 2.1, 2.01, 2.001, 2.0001, 2.00001 in your table.

One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g(x) = |x - 2| / (x - 2)$ introduced at the beginning of the section (see [\[link\]](#)(b)). As we pick values of x close to 2, $g(x)$ does not approach a single value, so the limit as x approaches 2 does not exist—that is, $\lim_{x \rightarrow 2} g(x)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the x -value 2. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of 2 (or *the negative side of 2*), $g(x) = -1$. Thus, as x approaches 2 from the left, $g(x)$ approaches -1 . Mathematically, we say that the limit as x approaches 2 from the left is -1 . Symbolically, we express this idea as

Equation:

$$\lim_{x \rightarrow 2^-} g(x) = -1.$$

Similarly, as x approaches 2 from the right (or *from the positive side*), $g(x)$ approaches 1. Symbolically, we express this idea as

Equation:

$$\lim_{x \rightarrow 2^+} g(x) = 1.$$

We can now present an informal definition of one-sided limits.

Note:

Definition

We define two types of **one-sided limits**.

Limit from the left: Let $f(x)$ be a function defined at all values in an open interval of the form z , and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach the number a , then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

Equation:

$$\lim_{x \rightarrow a^-} f(x) = L.$$

Limit from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c) , and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a , then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

Equation:

$$\lim_{x \rightarrow a^+} f(x) = L.$$

Example:

Exercise:

Problem:

Evaluating One-Sided Limits

For the function $f(x) = \begin{cases} x + 1 & \text{if } x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim_{x \rightarrow 2^-} f(x)$

b. $\lim_{x \rightarrow 2^+} f(x)$

Solution:

We can use tables of functional values again [\[link\]](#). Observe that for values of x less than 2, we use $f(x) = x + 1$ and for values of x greater than 2, we use $f(x) = x^2 - 4$.

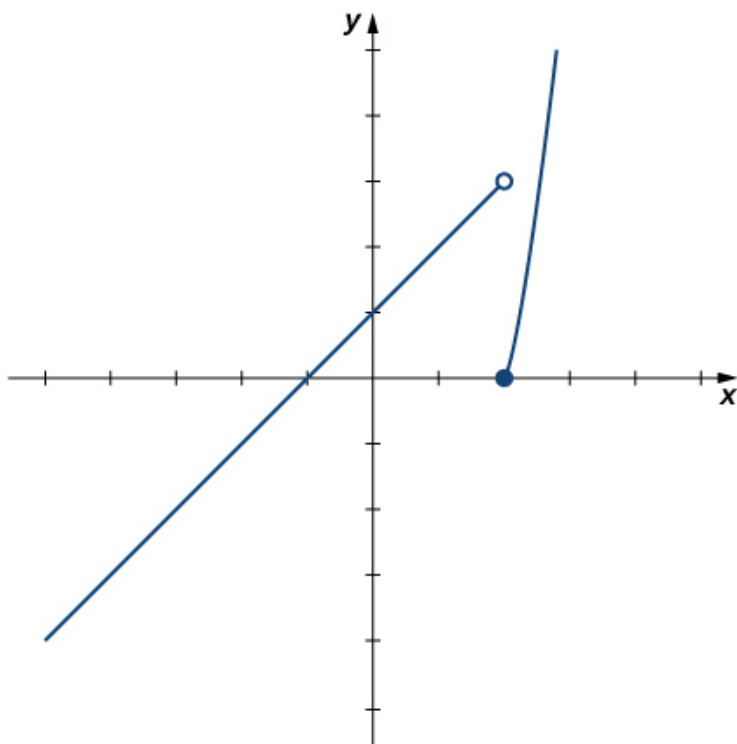
x	$f(x) = x + 1$		x	$f(x) = x^2 - 4$
1.9	2.9		2.1	0.41
1.99	2.99		2.01	0.0401
1.999	2.999		2.001	0.004001
1.9999	2.9999		2.0001	0.00040001
1.99999	2.99999		2.00001	0.0000400001

Table of Functional Values for $f(x) = \begin{cases} x + 1 & \text{if } x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$

Based on this table, we can conclude that a. $\lim_{x \rightarrow 2^-} f(x) = 3$ and b.

$\lim_{x \rightarrow 2^+} f(x) = 0$. Therefore, the (two-sided) limit of $f(x)$ does not exist at $x = 2$.

[\[link\]](#) shows a graph of $f(x)$ and reinforces our conclusion about these limits.



The graph of $f(x) = \begin{cases} x + 1 & \text{if } x < 2 \\ x^2 - 4 & \text{if } x \geq 2 \end{cases}$ has a break at $x = 2$.

Note:

Exercise:

Problem:

Use a table of functional values to estimate the following limits, if possible.

a. $\lim_{x \rightarrow 2^-} \frac{|x^2 - 4|}{x - 2}$

b. $\lim_{x \rightarrow 2^+} \frac{|x^2 - 4|}{x - 2}$

Solution:

$$\text{a. } \lim_{x \rightarrow 2^-} \frac{|x^2 - 4|}{x - 2} = -4; \text{ b. } \lim_{x \rightarrow 2^+} \frac{|x^2 - 4|}{x - 2} = 4$$

Hint

- a. Use x -values 1.9, 1.99, 1.999, 1.9999, 1.9999 to estimate $\lim_{x \rightarrow 2^-} \frac{|x^2 - 4|}{x - 2}$.
- b. Use x -values 2.1, 2.01, 2.001, 2.0001, 2.00001 to estimate $\lim_{x \rightarrow 2^+} \frac{|x^2 - 4|}{x - 2}$.
- (These tables are available from a previous Checkpoint problem.)

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in [\[link\]](#).

Note:

Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a , with the possible exception of a itself, and let L be a real number. Then,

Equation:

$$\lim_{x \rightarrow a} f(x) = L \text{ if and only if } \lim_{x \rightarrow a^-} f(x) = L \text{ and } \lim_{x \rightarrow a^+} f(x) = L.$$

Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x - 2)^2$, the third and final function introduced at the beginning of this section (see [\[link\]](#)(c)). From its graph we see that as the values of x approach 2, the values of $h(x) = 1/(x - 2)^2$ become larger and larger

and, in fact, become infinite. Mathematically, we say that the limit of $h(x)$ as x approaches 2 is positive infinity. Symbolically, we express this idea as

Equation:

$$\lim_{x \rightarrow 2} h(x) = +\infty.$$

More generally, we define **infinite limits** as follows:

Note:

Definition

We define three types of **infinite limits**.

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a) .

- i. If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a , then we say that the limit as x approaches a from the left is positive infinity and we write

Equation:

$$\lim_{x \rightarrow a^-} f(x) = +\infty.$$

- ii. If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a , then we say that the limit as x approaches a from the left is negative infinity and we write

Equation:

$$\lim_{x \rightarrow a^-} f(x) = -\infty.$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c) .

- i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a , then we say that the limit as x approaches a from the left is positive infinity and we write

Equation:

$$\lim_{x \rightarrow a^+} f(x) = +\infty.$$

- ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a , then we say that the limit as x approaches a from the left is negative infinity and we write

Equation:

$$\lim_{x \rightarrow a^+} f(x) = -\infty.$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a .

- i. If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a , then we say that the limit as x approaches a is positive infinity and we write

Equation:

$$\lim_{x \rightarrow a} f(x) = +\infty.$$

- ii. If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a , then we say that the limit as x approaches a is negative infinity and we write

Equation:

$$\lim_{x \rightarrow a} f(x) = -\infty.$$

It is important to understand that when we write statements such as $\lim_{x \rightarrow a} f(x) = +\infty$ or $\lim_{x \rightarrow a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at a , it must approach a real number L as x approaches a . That said, if, for example, $\lim_{x \rightarrow a} f(x) = +\infty$, we always write $\lim_{x \rightarrow a} f(x) = +\infty$ rather than $\lim_{x \rightarrow a} f(x)$ DNE.

Example:

Exercise:

Problem:

Recognizing an Infinite Limit

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

- a. $\lim_{x \rightarrow 0^-} \frac{1}{x}$
- b. $\lim_{x \rightarrow 0^+} \frac{1}{x}$
- c. $\lim_{x \rightarrow 0} \frac{1}{x}$

Solution:

Begin by constructing a table of functional values.

x	$\frac{1}{x}$		x	$\frac{1}{x}$
-0.1	-10		0.1	10
-0.01	-100		0.01	100
-0.001	-1000		0.001	1000
-0.0001	-10,000		0.0001	10,000
-0.00001	-100,000		0.00001	100,000
-0.000001	-1,000,000		0.000001	1,000,000

Table of Functional Values for $f(x) = \frac{1}{x}$

- a. The values of $1/x$ decrease without bound as x approaches 0 from the left.
We conclude that

Equation:

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

- b. The values of $1/x$ increase without bound as x approaches 0 from the right.
We conclude that

Equation:

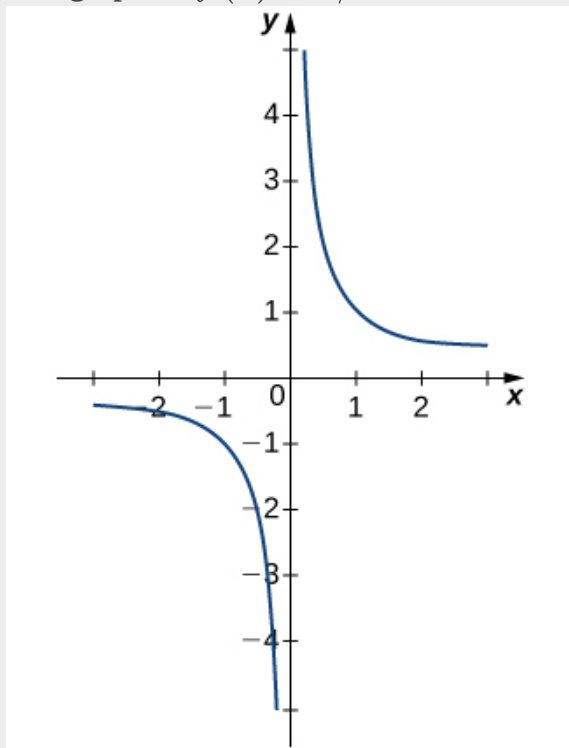
$$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty.$$

- c. Since $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

Equation:

$$\lim_{x \rightarrow 0} \frac{1}{x} \text{ DNE.}$$

The graph of $f(x) = 1/x$ in [\[link\]](#) confirms these conclusions.



The graph of $f(x) = 1/x$
confirms that the limit as x
approaches 0 does not exist.

Note:**Exercise:****Problem:**

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x^2$ to confirm your conclusion.

a. $\lim_{x \rightarrow 0^-} \frac{1}{x^2}$

b. $\lim_{x \rightarrow 0^+} \frac{1}{x^2}$

c. $\lim_{x \rightarrow 0} \frac{1}{x^2}$

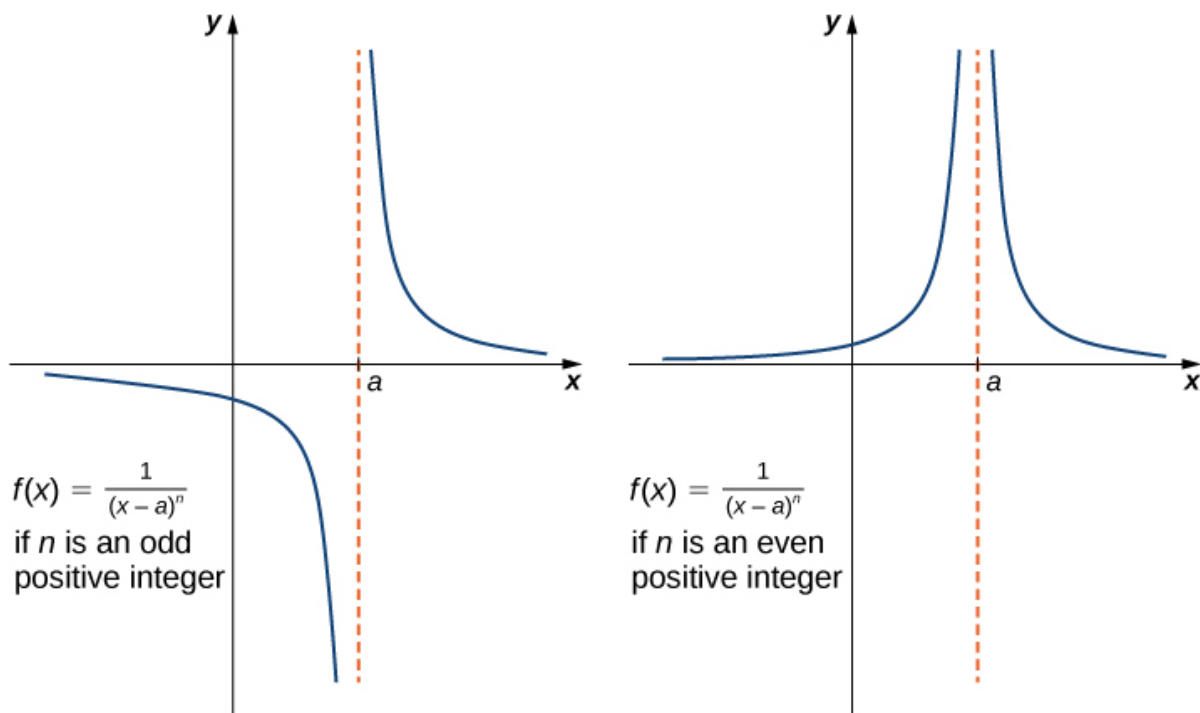
Solution:

a. $\lim_{x \rightarrow 0^-} \frac{1}{x^2} = +\infty$; b. $\lim_{x \rightarrow 0^+} \frac{1}{x^2} = +\infty$; c. $\lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty$

Hint

Follow the procedures from [\[link\]](#).

It is useful to point out that functions of the form $f(x) = 1/(x - a)^n$, where n is a positive integer, have infinite limits as x approaches a from either the left or right ([\[link\]](#)). These limits are summarized in [\[link\]](#).



The function $f(x) = 1/(x-a)^n$ has infinite limits at a .

Note:

Infinite Limits from Positive Integers

If n is a positive even integer, then

Equation:

$$\lim_{x \rightarrow a} \frac{1}{(x-a)^n} = +\infty.$$

If n is a positive odd integer, then

Equation:

$$\lim_{x \rightarrow a^+} \frac{1}{(x-a)^n} = +\infty$$

and

Equation:

$$\lim_{x \rightarrow a^-} \frac{1}{(x - a)^n} = -\infty.$$

We should also point out that in the graphs of $f(x) = 1/(x - a)^n$, points on the graph having x -coordinates very near to a are very close to the vertical line $x = a$. That is, as x approaches a , the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a **vertical asymptote** of the graph. We formally define a vertical asymptote as follows:

Note:

Definition

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a **vertical asymptote** of $f(x)$.

Equation:

$$\begin{aligned} \lim_{x \rightarrow a^-} f(x) &= +\infty \text{ or } -\infty \\ \lim_{x \rightarrow a^+} f(x) &= +\infty \text{ or } -\infty \\ &\text{or} \\ \lim_{x \rightarrow a} f(x) &= +\infty \text{ or } -\infty \end{aligned}$$

Example:

Exercise:

Problem:

Finding a Vertical Asymptote

Evaluate each of the following limits using [\[link\]](#). Identify any vertical asymptotes of the function $f(x) = 1/(x + 3)^4$.

- a. $\lim_{x \rightarrow -3^-} \frac{1}{(x+3)^4}$
- b. $\lim_{x \rightarrow -3^+} \frac{1}{(x+3)^4}$
- c. $\lim_{x \rightarrow -3} \frac{1}{(x+3)^4}$

Solution:

We can use [\[link\]](#) directly.

a. $\lim_{x \rightarrow -3^-} \frac{1}{(x+3)^4} = +\infty$

b. $\lim_{x \rightarrow -3^+} \frac{1}{(x+3)^4} = +\infty$

c. $\lim_{x \rightarrow -3} \frac{1}{(x+3)^4} = +\infty$

The function $f(x) = 1/(x+3)^4$ has a vertical asymptote of $x = -3$.

Note:**Exercise:****Problem:**

Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \frac{1}{(x-2)^3}$.

a. $\lim_{x \rightarrow 2^-} \frac{1}{(x-2)^3}$

b. $\lim_{x \rightarrow 2^+} \frac{1}{(x-2)^3}$

c. $\lim_{x \rightarrow 2} \frac{1}{(x-2)^3}$

Solution:

a. $\lim_{x \rightarrow 2^-} \frac{1}{(x-2)^3} = -\infty$; b. $\lim_{x \rightarrow 2^+} \frac{1}{(x-2)^3} = +\infty$; c. $\lim_{x \rightarrow 2} \frac{1}{(x-2)^3}$ DNE. The line $x = 2$ is the vertical asymptote of $f(x) = 1/(x-2)^3$.

Hint

Use [\[link\]](#).

In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

Example:

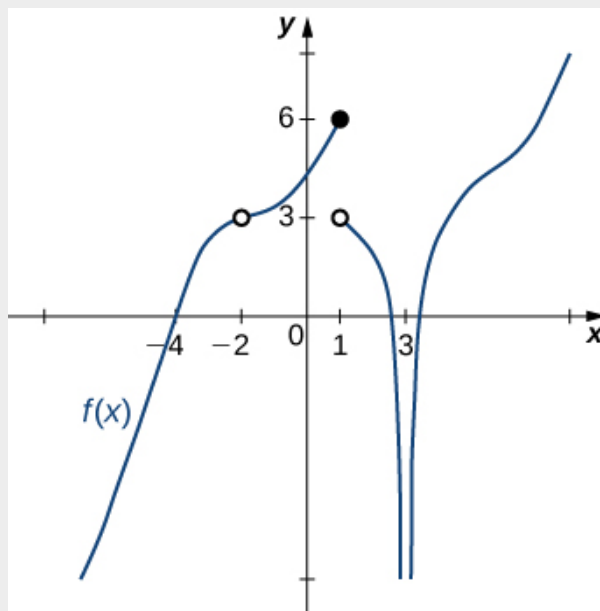
Exercise:

Problem:

Behavior of a Function at Different Points

Use the graph of $f(x)$ in [\[link\]](#) to determine each of the following values:

- a. $\lim_{x \rightarrow -4^-} f(x)$; $\lim_{x \rightarrow -4^+} f(x)$; $\lim_{x \rightarrow -4} f(x)$; $f(-4)$
- b. $\lim_{x \rightarrow -2^-} f(x)$; $\lim_{x \rightarrow -2^+} f(x)$; $\lim_{x \rightarrow -2} f(x)$; $f(-2)$
- c. $\lim_{x \rightarrow 1^-} f(x)$; $\lim_{x \rightarrow 1^+} f(x)$; $\lim_{x \rightarrow 1} f(x)$; $f(1)$
- d. $\lim_{x \rightarrow 3^-} f(x)$; $\lim_{x \rightarrow 3^+} f(x)$; $\lim_{x \rightarrow 3} f(x)$; $f(3)$



The graph shows $f(x)$.

Solution:

Using [\[link\]](#) and the graph for reference, we arrive at the following values:

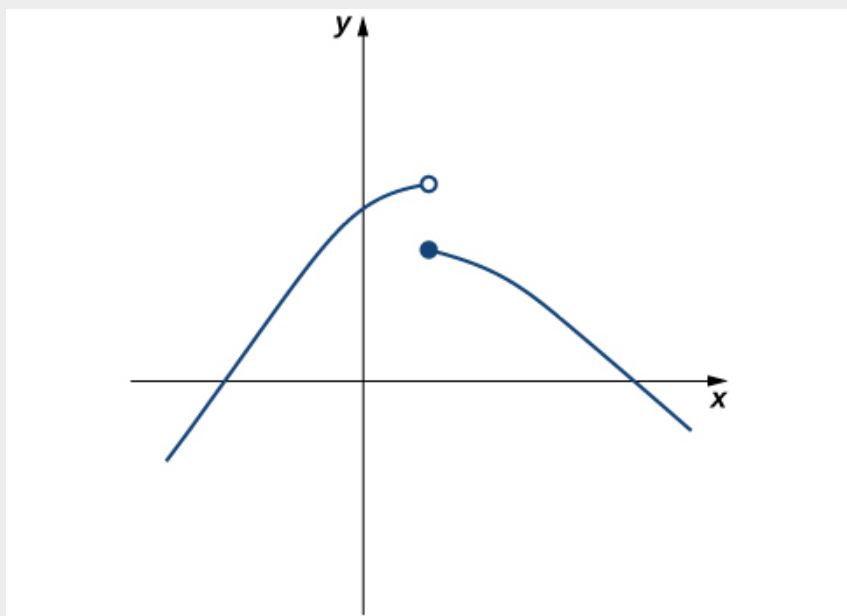
- a. $\lim_{x \rightarrow -4^-} f(x) = 0$; $\lim_{x \rightarrow -4^+} f(x) = 0$; $\lim_{x \rightarrow -4} f(x) = 0$; $f(-4) = 0$
- b. $\lim_{x \rightarrow -2^-} f(x) = 3$; $\lim_{x \rightarrow -2^+} f(x) = 3$; $\lim_{x \rightarrow -2} f(x) = 3$; $f(-2)$ is undefined

- c. $\lim_{x \rightarrow 1^-} f(x) = 6$; $\lim_{x \rightarrow 1^+} f(x) = 3$; $\lim_{x \rightarrow 1} f(x)$ DNE; $f(1) = 6$
- d. $\lim_{x \rightarrow 3^-} f(x) = -\infty$; $\lim_{x \rightarrow 3^+} f(x) = -\infty$; $\lim_{x \rightarrow 3} f(x) = -\infty$; $f(3)$ is undefined

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow 1} f(x)$ for $f(x)$ shown here:



Solution:

Does not exist.

Hint

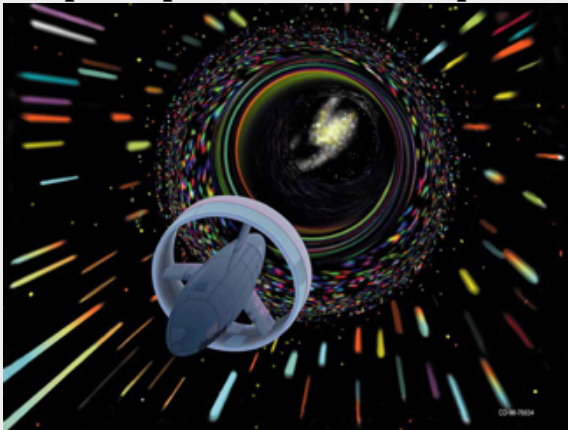
Compare the limit from the right with the limit from the left.

Example:

Exercise:

Problem:

Chapter Opener: Einstein's Equation



(credit: NASA)

In the chapter opener we mentioned briefly how Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein's equation for the mass of a moving object, what is the value of this bound?

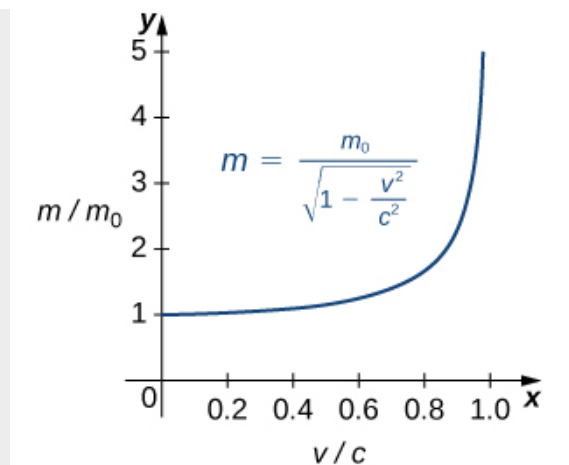
Solution:

Our starting point is Einstein's equation for the mass of a moving object,

Equation:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. To see how the mass changes at high speeds, we can graph the ratio of masses m/m_0 as a function of the ratio of speeds, v/c ([link](#)).



This graph shows the ratio of masses as a function of the ratio of speeds in Einstein's equation for the mass of a moving object.

We can see that as the ratio of speeds approaches 1—that is, as the speed of the object approaches the speed of light—the ratio of masses increases without bound. In other words, the function has a vertical asymptote at $v/c = 1$. We can try a few values of this ratio to test this idea.

$\frac{v}{c}$	$\sqrt{1 - \frac{v^2}{c^2}}$	$\frac{m}{m_0}$
0.99	0.1411	7.089
0.999	0.0447	22.37
0.9999	0.0141	70.71

Ratio of Masses and Speeds for a Moving Object

Thus, according to [\[link\]](#), if an object with mass 100 kg is traveling at $0.9999c$, its mass becomes 7071 kg. Since no object can have an infinite mass, we

conclude that no object can travel at or more than the speed of light.

Key Concepts

- A table of values or graph may be used to estimate a limit.
- If the limit of a function at a point does not exist, it is still possible that the limits from the left and right at that point may exist.
- If the limits of a function from the left and right exist and are equal, then the limit of the function is that common value.
- We may use limits to describe infinite behavior of a function at a point.

Key Equations

- **Intuitive Definition of the Limit**

$$\lim_{x \rightarrow a} f(x) = L$$

- **Two Important Limits**

$$\lim_{x \rightarrow a} x = a \quad \lim_{x \rightarrow a} c = c$$

- **One-Sided Limits**

$$\lim_{x \rightarrow a^-} f(x) = L \quad \lim_{x \rightarrow a^+} f(x) = L$$

- **Infinite Limits from the Left**

$$\lim_{x \rightarrow a^-} f(x) = +\infty \quad \lim_{x \rightarrow a^-} f(x) = -\infty$$

- **Infinite Limits from the Right**

$$\lim_{x \rightarrow a^+} f(x) = +\infty \quad \lim_{x \rightarrow a^+} f(x) = -\infty$$

- **Two-Sided Infinite Limits**

$$\lim_{x \rightarrow a} f(x) = +\infty : \lim_{x \rightarrow a^-} f(x) = +\infty \text{ and } \lim_{x \rightarrow a^+} f(x) = +\infty$$

$$\lim_{x \rightarrow a} f(x) = -\infty : \lim_{x \rightarrow a^-} f(x) = -\infty \text{ and } \lim_{x \rightarrow a^+} f(x) = -\infty$$

For the following exercises, consider the function $f(x) = \frac{x^2-1}{|x-1|}$.

Exercise:

Problem:

[T] Complete the following table for the function. Round your solutions to four decimal places.

x	$f(x)$	x	$f(x)$
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.
0.9999	d.	1.0001	h.

Exercise:

Problem:

What do your results in the preceding exercise indicate about the two-sided limit $\lim_{x \rightarrow 1} f(x)$? Explain your response.

Solution:

$\lim_{x \rightarrow 1} f(x)$ does not exist because $\lim_{x \rightarrow 1^-} f(x) = -2 \neq \lim_{x \rightarrow 1^+} f(x) = 2$.

For the following exercises, consider the function $f(x) = (1 + x)^{1/x}$.

Exercise:

Problem:

[T] Make a table showing the values of f for $x = -0.01, -0.001, -0.0001, -0.00001$ and for $x = 0.01, 0.001, 0.0001, 0.00001$. Round your solutions to five decimal places.

x	$f(x)$	x	$f(x)$
-0.01	a.	0.01	e.
-0.001	b.	0.001	f.

x	$f(x)$	x	$f(x)$
-0.0001	c.	0.0001	g.
-0.00001	d.	0.00001	h.

Exercise:

Problem:

What does the table of values in the preceding exercise indicate about the function $f(x) = (1+x)^{1/x}$?

Solution:

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = 2.7183$$

Exercise:

Problem:

To which mathematical constant does the limit in the preceding exercise appear to be getting closer?

In the following exercises, use the given values to set up a table to evaluate the limits. Round your solutions to eight decimal places.

Exercise:

Problem: [T] $\lim_{x \rightarrow 0} \frac{\sin 2x}{x}; \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

x	$\frac{\sin 2x}{x}$	x	$\frac{\sin 2x}{x}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.

x	$\frac{\sin 2x}{x}$	x	$\frac{\sin 2x}{x}$
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

Solution:

a. 1.98669331; b. 1.99986667; c. 1.99999867; d. 1.99999999; e. 1.98669331; f. 1.99986667; g. 1.99999867; h. 1.99999999; $\lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2$

Exercise:

Problem: [T] $\lim_{x \rightarrow 0} \frac{\sin 3x}{x} \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$

X	$\frac{\sin 3x}{x}$	x	$\frac{\sin 3x}{x}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

Exercise:

Problem:

Use the preceding two exercises to conjecture (guess) the value of the following limit: $\lim_{x \rightarrow 0} \frac{\sin ax}{x}$ for a , a positive real value.

Solution:

$$\lim_{x \rightarrow 0} \frac{\sin ax}{x} = a$$

[T] In the following exercises, set up a table of values to find the indicated limit. Round to eight digits.

Exercise:

Problem: $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 + x - 6}$

x	$\frac{x^2 - 4}{x^2 + x - 6}$	x	$\frac{x^2 - 4}{x^2 + x - 6}$
1.9	a.	2.1	e.
1.99	b.	2.01	f.
1.999	c.	2.001	g.
1.9999	d.	2.0001	h.

Exercise:

Problem: $\lim_{x \rightarrow 1} (1 - 2x)$

x	$1 - 2x$	x	$1 - 2x$
0.9	a.	1.1	e.
0.99	b.	1.01	f.
0.999	c.	1.001	g.

x	$1 - 2x$	x	$1 - 2x$
0.9999	d.	1.0001	h.

Solution:

a. -0.80000000 ; b. -0.98000000 ; c. -0.99800000 ; d. -0.99980000 ; e. -1.20000000 ; f. -1.02000000 ; g. -1.00200000 ; h. -1.00020000 ; $\lim_{x \rightarrow 1} (1 - 2x) = -1$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{5}{1 - e^{1/x}}$

x	$\frac{5}{1 - e^{1/x}}$	x	$\frac{5}{1 - e^{1/x}}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

Exercise:

Problem: $\lim_{z \rightarrow 0} \frac{z-1}{z^2(z+3)}$

z	$\frac{z-1}{z^3(z+3)}$	z	$\frac{z-1}{z^3(z+3)}$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

Solution:

a. -37.931934; b. -3377.9264; c. -333,777.93; d. -33,337,778; e. -29.032258; f. -3289.0365; g. -332,889.04; h. -33,328,889 $\lim_{x \rightarrow 0} \frac{z-1}{z^2(z+3)} = -\infty$

Exercise:

Problem: $\lim_{t \rightarrow 0^+} \frac{\cos t}{t}$

t	$\frac{\cos t}{t}$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

Exercise:

Problem: $\lim_{x \rightarrow 2} \frac{1-\frac{2}{x}}{x^2-4}$

x	$\frac{1-\frac{2}{x}}{x^2-4}$	x	$\frac{1-\frac{2}{x}}{x^2-4}$
1.9	a.	2.1	e.
1.99	b.	2.01	f.
1.999	c.	2.001	g.
1.9999	d.	2.0001	h.

Solution:

a. 0.13495277; b. 0.12594300; c. 0.12509381; d. 0.12500938; e. 0.11614402; f. 0.12406794; g. 0.12490631; h. 0.12499063; $\therefore \lim_{x \rightarrow 2} \frac{1-\frac{2}{x}}{x^2-4} = 0.1250 = \frac{1}{8}$

[T] In the following exercises, set up a table of values and round to eight significant digits. Based on the table of values, make a guess about what the limit is. Then, use a calculator to graph the function and determine the limit. Was the conjecture correct? If not, why does the method of tables fail?

Exercise:

Problem: $\lim_{\theta \rightarrow 0} \sin\left(\frac{\pi}{\theta}\right)$

θ	$\sin\left(\frac{\pi}{\theta}\right)$	θ	$\sin\left(\frac{\pi}{\theta}\right)$
-0.1	a.	0.1	e.
-0.01	b.	0.01	f.
-0.001	c.	0.001	g.
-0.0001	d.	0.0001	h.

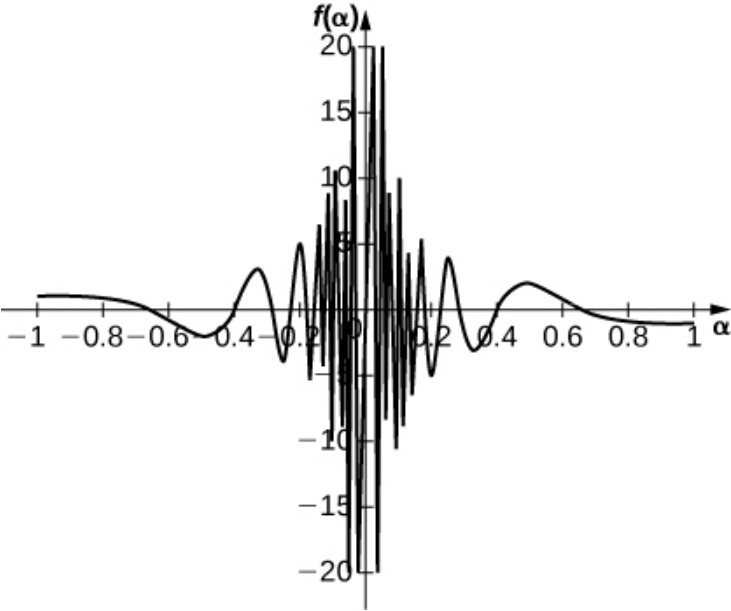
Exercise:

Problem: $\lim_{\alpha \rightarrow 0^+} \frac{1}{\alpha} \cos \left(\frac{\pi}{\alpha} \right)$

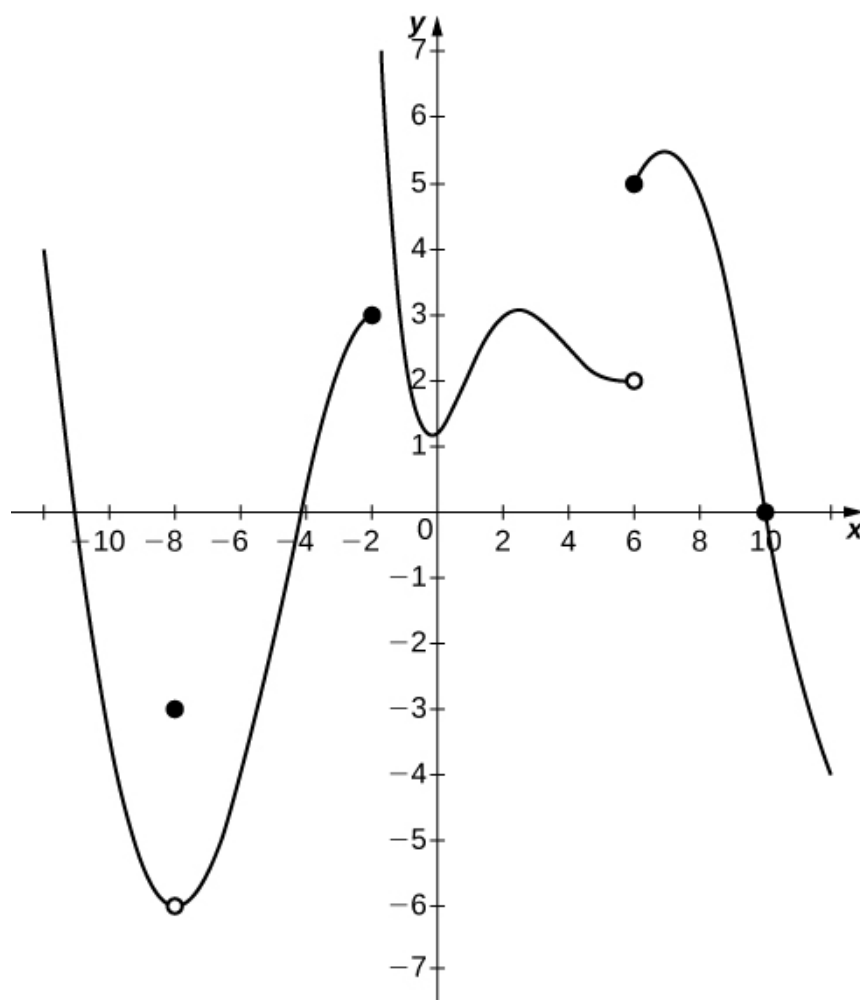
<i>a</i>	$\frac{1}{\alpha} \cos \left(\frac{\pi}{\alpha} \right)$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

Solution:

a. -10.00000; b. -100.00000; c. -1000.0000; d. -10,000.000; Guess:
 $\lim_{\alpha \rightarrow 0^+} \frac{1}{\alpha} \cos \left(\frac{\pi}{\alpha} \right) = \infty$, actual: DNE



In the following exercises, consider the graph of the function $y = f(x)$ shown here. Which of the statements about $y = f(x)$ are true and which are false? Explain why a statement is false.



Exercise:

Problem: $\lim_{x \rightarrow 10} f(x) = 0$

Exercise:

Problem: $\lim_{x \rightarrow -2^+} f(x) = 3$

Solution:

False; $\lim_{x \rightarrow -2^+} f(x) = +\infty$

Exercise:

Problem: $\lim_{x \rightarrow -8} f(x) = f(-8)$

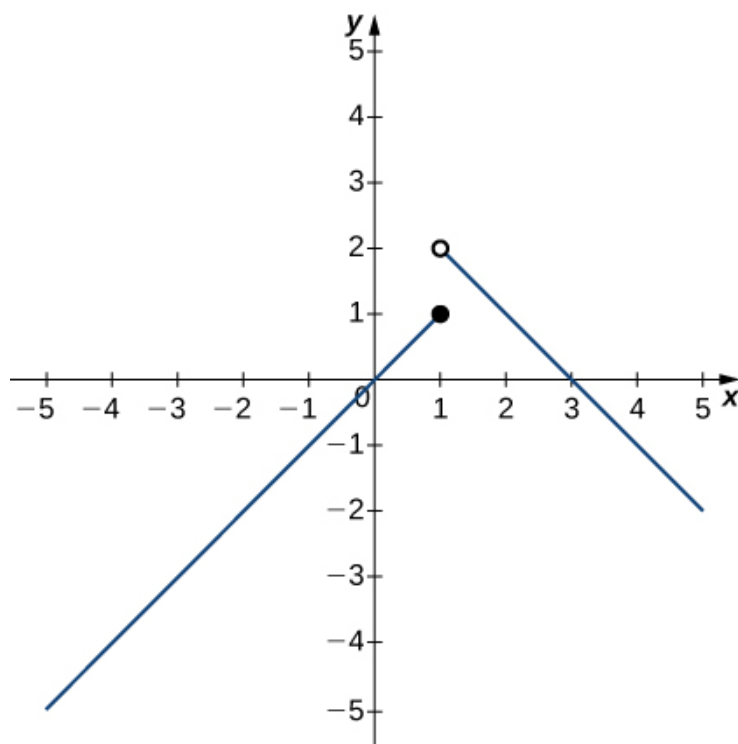
Exercise:

Problem: $\lim_{x \rightarrow 6} f(x) = 5$

Solution:

False; $\lim_{x \rightarrow 6} f(x)$ DNE since $\lim_{x \rightarrow 6^-} f(x) = 2$ and $\lim_{x \rightarrow 6^+} f(x) = 5$.

In the following exercises, use the following graph of the function $y = f(x)$ to find the values, if possible. Estimate when necessary.



Exercise:

Problem: $\lim_{x \rightarrow 1^-} f(x)$

Exercise:

Problem: $\lim_{x \rightarrow 1^+} f(x)$

Solution:

2

Exercise:

Problem: $\lim_{x \rightarrow 1} f(x)$

Exercise:

Problem: $\lim_{x \rightarrow 2} f(x)$

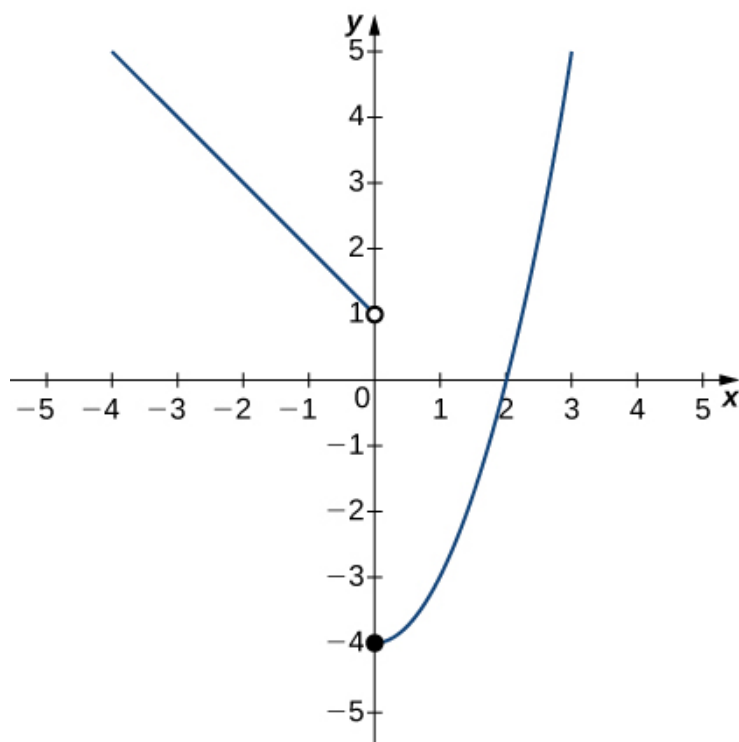
Solution:

1

Exercise:

Problem: $f(1)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



Exercise:

Problem: $\lim_{x \rightarrow 0^-} f(x)$

Solution:

1

Exercise:

Problem: $\lim_{x \rightarrow 0^+} f(x)$

Exercise:

Problem: $\lim_{x \rightarrow 0} f(x)$

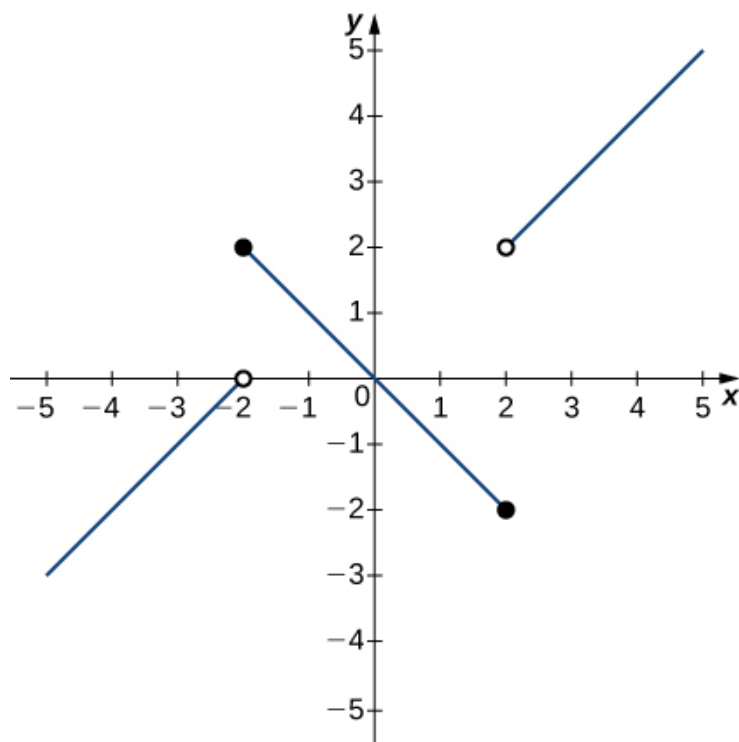
Solution:

DNE

Exercise:

Problem: $\lim_{x \rightarrow 2} f(x)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



Exercise:

Problem: $\lim_{x \rightarrow -2^-} f(x)$

Solution:

0

Exercise:

Problem: $\lim_{x \rightarrow -2^+} f(x)$

Exercise:

Problem: $\lim_{x \rightarrow -2} f(x)$

Solution:

DNE

Exercise:

Problem: $\lim_{x \rightarrow 2^-} f(x)$

Exercise:

Problem: $\lim_{x \rightarrow 2^+} f(x)$

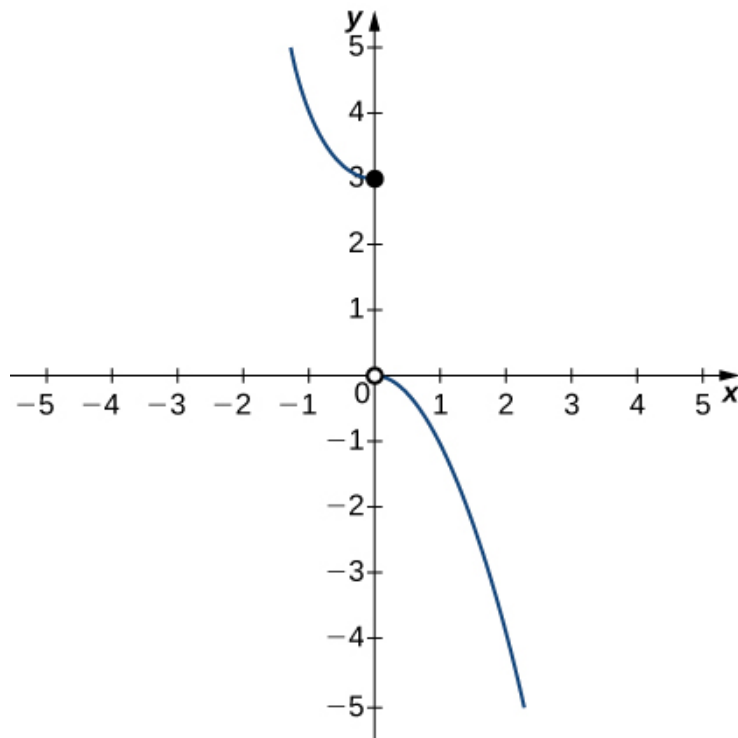
Solution:

2

Exercise:

Problem: $\lim_{x \rightarrow 2} f(x)$

In the following exercises, use the graph of the function $y = g(x)$ shown here to find the values, if possible. Estimate when necessary.



Exercise:

Problem: $\lim_{x \rightarrow 0^-} g(x)$

Solution:

3

Exercise:

Problem: $\lim_{x \rightarrow 0^+} g(x)$

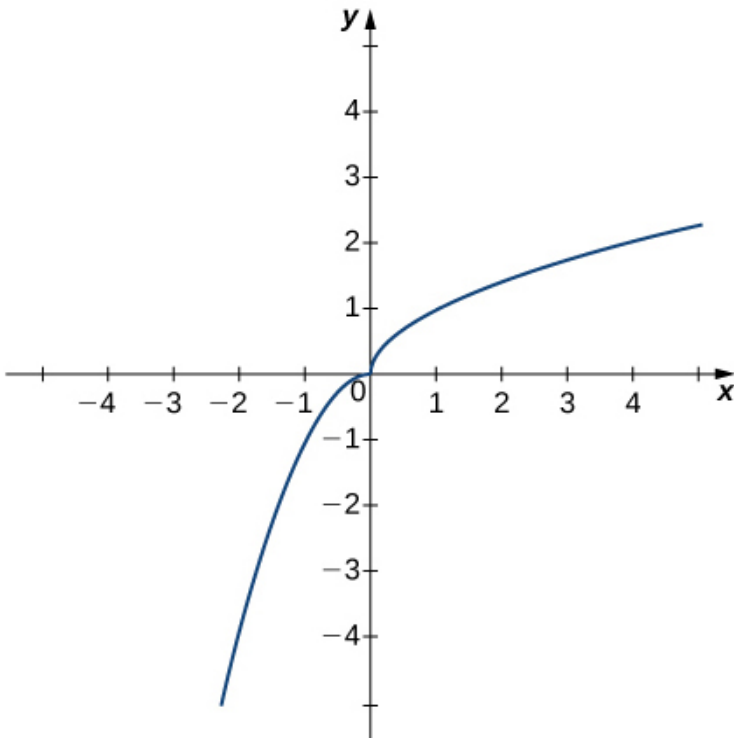
Exercise:

Problem: $\lim_{x \rightarrow 0} g(x)$

Solution:

DNE

In the following exercises, use the graph of the function $y = h(x)$ shown here to find the values, if possible. Estimate when necessary.



Exercise:

Problem: $\lim_{x \rightarrow 0^-} h(x)$

Exercise:

Problem: $\lim_{x \rightarrow 0^+} h(x)$

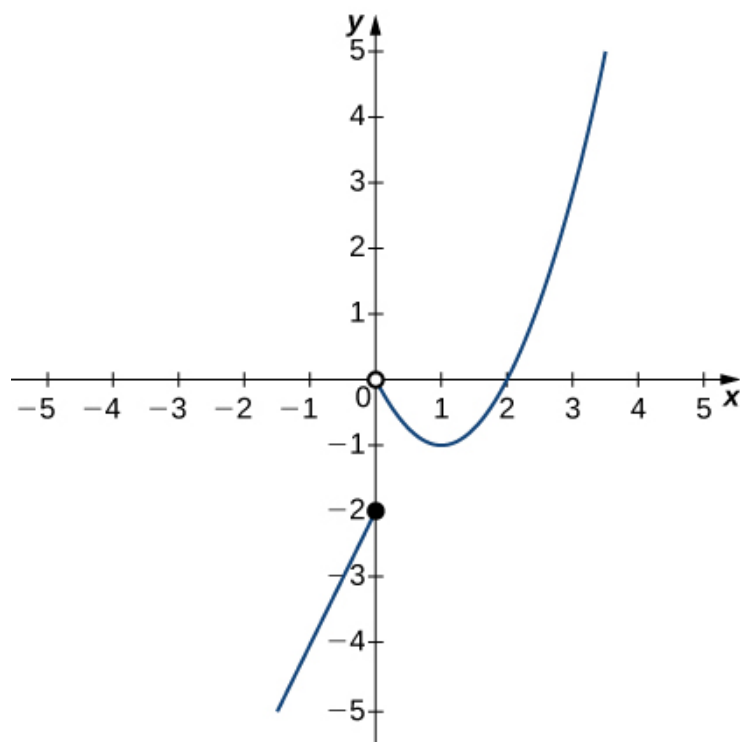
Solution:

0

Exercise:

Problem: $\lim_{x \rightarrow 0} h(x)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



Exercise:

Problem: $\lim_{x \rightarrow 0^-} f(x)$

Solution:

-2

Exercise:

Problem: $\lim_{x \rightarrow 0^+} f(x)$

Exercise:

Problem: $\lim_{x \rightarrow 0} f(x)$

Solution:

DNE

Exercise:

Problem: $\lim_{x \rightarrow 1} f(x)$

Exercise:

Problem: $\lim_{x \rightarrow 2} f(x)$

Solution:

0

In the following exercises, sketch the graph of a function with the given properties.

Exercise:

Problem: $\lim_{x \rightarrow 2} f(x) = 1$, $\lim_{x \rightarrow 4^-} f(x) = 3$, $\lim_{x \rightarrow 4^+} f(x) = 6$, $x = 4$ is not defined.

Exercise:

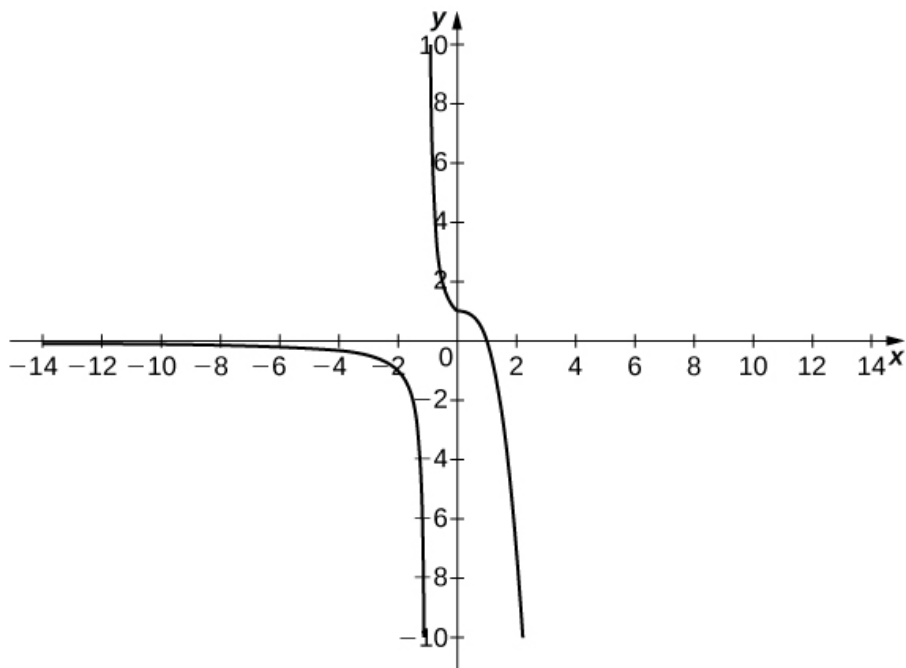
Problem:

$$\lim_{x \rightarrow -\infty} f(x) = 0, \quad \lim_{x \rightarrow -1^-} f(x) = -\infty,$$

$$\lim_{x \rightarrow -1^+} f(x) = \infty, \quad \lim_{x \rightarrow 0} f(x) = f(0), \quad f(0) = 1, \quad \lim_{x \rightarrow \infty} f(x) = -\infty$$

Solution:

Answers may vary.

**Exercise:****Problem:**

$$\lim_{x \rightarrow -\infty} f(x) = 2, \quad \lim_{x \rightarrow 3^-} f(x) = -\infty,$$

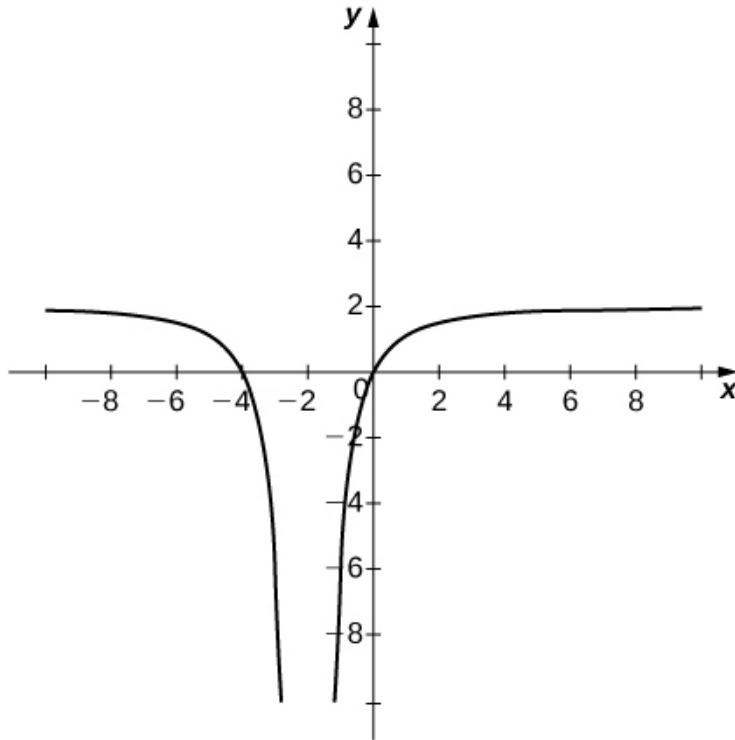
$$\lim_{x \rightarrow 3^+} f(x) = \infty, \quad \lim_{x \rightarrow \infty} f(x) = 2, \quad f(0) = \frac{-1}{3}$$

Exercise:

$$\textbf{Problem: } \lim_{x \rightarrow -\infty} f(x) = 2, \quad \lim_{x \rightarrow -2} f(x) = -\infty, \quad \lim_{x \rightarrow \infty} f(x) = 2, \quad f(0) = 0$$

Solution:

Answers may vary.



Exercise:

Problem:

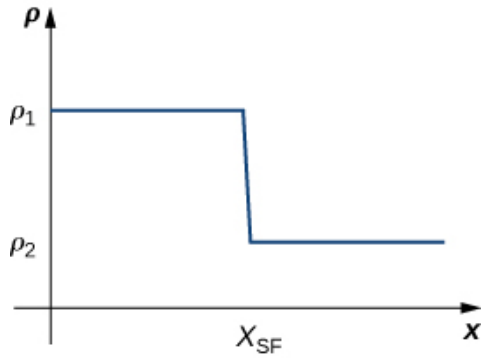
$$\lim_{x \rightarrow -\infty} f(x) = 0, \quad \lim_{x \rightarrow -1^-} f(x) = \infty, \quad \lim_{x \rightarrow -1^+} f(x) = -\infty,$$

$$f(0) = -1, \quad \lim_{x \rightarrow 1^-} f(x) = -\infty, \quad \lim_{x \rightarrow 1^+} f(x) = \infty, \quad \lim_{x \rightarrow \infty} f(x) = 0$$

Exercise:

Problem:

Shock waves arise in many physical applications, ranging from supernovas to detonation waves. A graph of the density of a shock wave with respect to distance, x , is shown here. We are mainly interested in the location of the front of the shock, labeled x_{SF} in the diagram.



- Evaluate $\lim_{x \rightarrow x_{SF}^+} \rho(x)$.
- Evaluate $\lim_{x \rightarrow x_{SF}^-} \rho(x)$.
- Evaluate $\lim_{x \rightarrow x_{SF}} \rho(x)$. Explain the physical meanings behind your answers.

Solution:

a. ρ_2 b. ρ_1 c. DNE unless $\rho_1 = \rho_2$. As you approach x_{SF} from the right, you are in the high-density area of the shock. When you approach from the left, you have not experienced the “shock” yet and are at a lower density.

Exercise:

Problem:

A track coach uses a camera with a fast shutter to estimate the position of a runner with respect to time. A table of the values of position of the athlete versus time is given here, where x is the position in meters of the runner and t is time in seconds. What is $\lim_{t \rightarrow 2} x(t)$? What does it mean physically?

t (sec)	x (m)
1.75	4.5
1.95	6.1
1.99	6.42

t (sec)	x (m)
2.01	6.58
2.05	6.9
2.25	8.5

Glossary

infinite limit

A function has an infinite limit at a point a if it either increases or decreases without bound as it approaches a

intuitive definition of the limit

If all values of the function $f(x)$ approach the real number L as the values of x ($\neq a$) approach a , $f(x)$ approaches L

one-sided limit

A one-sided limit of a function is a limit taken from either the left or the right

vertical asymptote

A function has a vertical asymptote at $x = a$ if the limit as x approaches a from the right or left is infinite

The Limit Laws

- Recognize the basic limit laws.
- Use the limit laws to evaluate the limit of a function.
- Evaluate the limit of a function by factoring.
- Use the limit laws to evaluate the limit of a polynomial or rational function.
- Evaluate the limit of a function by factoring or by using conjugates.
- Evaluate the limit of a function by using the squeeze theorem.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

Evaluating Limits with the Limit Laws

The first two limit laws were stated in [\[link\]](#) and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

Note:

Basic Limit Results

For any real number a and any constant c ,

i. Equation:

$$\lim_{x \rightarrow a} x = a$$

ii. Equation:

$$\lim_{x \rightarrow a} c = c$$

Example:

Exercise:

Problem:

Evaluating a Basic Limit

Evaluate each of the following limits using [\[link\]](#).

a. $\lim_{x \rightarrow 2} x$

b. $\lim_{x \rightarrow 2} 5$

Solution:

- a. The limit of x as x approaches a is a : $\lim_{x \rightarrow 2} x = 2$.
- b. The limit of a constant is that constant: $\lim_{x \rightarrow 2} 5 = 5$.

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

Note:

Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a . Assume that L and M are real numbers such that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

Sum law for limits: $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = L + M$

Difference law for limits: $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) = L - M$

Constant multiple law for limits: $\lim_{x \rightarrow a} cf(x) = c \cdot \lim_{x \rightarrow a} f(x) = cL$

Product law for limits: $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M$

Quotient law for limits: $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}$ for $M \neq 0$

Power law for limits: $\lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n = L^n$ for every positive integer n .

Root law for limits: $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even.

We now practice applying these limit laws to evaluate a limit.

Example:

Exercise:

Problem:

Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate $\lim_{x \rightarrow -3} (4x + 2)$.

Solution:

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned} \lim_{x \rightarrow -3} (4x + 2) &= \lim_{x \rightarrow -3} 4x + \lim_{x \rightarrow -3} 2 \\ &= 4 \cdot \lim_{x \rightarrow -3} x + \lim_{x \rightarrow -3} 2 \\ &= 4 \cdot (-3) + 2 = -10. \end{aligned}$$

Apply the sum law.

Apply the constant multiple law.

Apply the basic limit results and simplify.

Example:**Exercise:****Problem:****Using Limit Laws Repeatedly**

Use the limit laws to evaluate $\lim_{x \rightarrow 2} \frac{2x^2 - 3x + 1}{x^3 + 4}$.

Solution:

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{2x^2 - 3x + 1}{x^3 + 4} &= \frac{\lim_{x \rightarrow 2} (2x^2 - 3x + 1)}{\lim_{x \rightarrow 2} (x^3 + 4)} && \text{Apply the quotient law, making sure that } (2)^3 + 4 \neq 0 \\ &= \frac{2 \cdot \lim_{x \rightarrow 2} x^2 - 3 \cdot \lim_{x \rightarrow 2} x + \lim_{x \rightarrow 2} 1}{\lim_{x \rightarrow 2} x^3 + \lim_{x \rightarrow 2} 4} && \text{Apply the sum law and constant multiple law.} \\ &= \frac{2 \cdot \left(\lim_{x \rightarrow 2} x \right)^2 - 3 \cdot \lim_{x \rightarrow 2} x + \lim_{x \rightarrow 2} 1}{\left(\lim_{x \rightarrow 2} x \right)^3 + \lim_{x \rightarrow 2} 4} && \text{Apply the power law.} \\ &= \frac{2(4) - 3(2) + 1}{(2)^3 + 4} = \frac{1}{4}. && \text{Apply the basic limit laws and simplify.} \end{aligned}$$

Note:**Exercise:****Problem:**

Use the limit laws to evaluate $\lim_{x \rightarrow 6} (2x - 1)\sqrt{x + 4}$. In each step, indicate the limit law applied.

Solution:

$$11\sqrt{10}$$

Hint

Begin by applying the product law.

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x \rightarrow a} f(x) = f(a)$. This is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Note:**Limits of Polynomial and Rational Functions**

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,

Equation:

$$\lim_{x \rightarrow a} p(x) = p(a)$$

Equation:

$$\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

To see that this theorem holds, consider the polynomial $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$. By applying the sum, constant multiple, and power laws, we end up with

Equation:

$$\begin{aligned} \lim_{x \rightarrow a} p(x) &= \lim_{x \rightarrow a} (c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0) \\ &= c_n \left(\lim_{x \rightarrow a} x \right)^n + c_{n-1} \left(\lim_{x \rightarrow a} x \right)^{n-1} + \cdots + c_1 \left(\lim_{x \rightarrow a} x \right) + \lim_{x \rightarrow a} c_0 \\ &= c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0 \\ &= p(a). \end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a) \neq 0$, then

Equation:

$$\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

[\[link\]](#) applies this result.

Example:**Exercise:****Problem:****Evaluating a Limit of a Rational Function**

Evaluate the $\lim_{x \rightarrow 3} \frac{2x^2 - 3x + 1}{5x + 4}$.

Solution:

Since 3 is in the domain of the rational function $f(x) = \frac{2x^2 - 3x + 1}{5x + 4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

Equation:

$$\lim_{x \rightarrow 3} \frac{2x^2 - 3x + 1}{5x + 4} = \frac{10}{19}.$$

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow -2} (3x^3 - 2x + 7)$.

Solution:

-13;

Hint

Use [\[link\]](#)

Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x \rightarrow a} f(x)$ to exist when $f(a)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x \neq a$, $f(x) = g(x)$ over some open interval containing a , then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$.

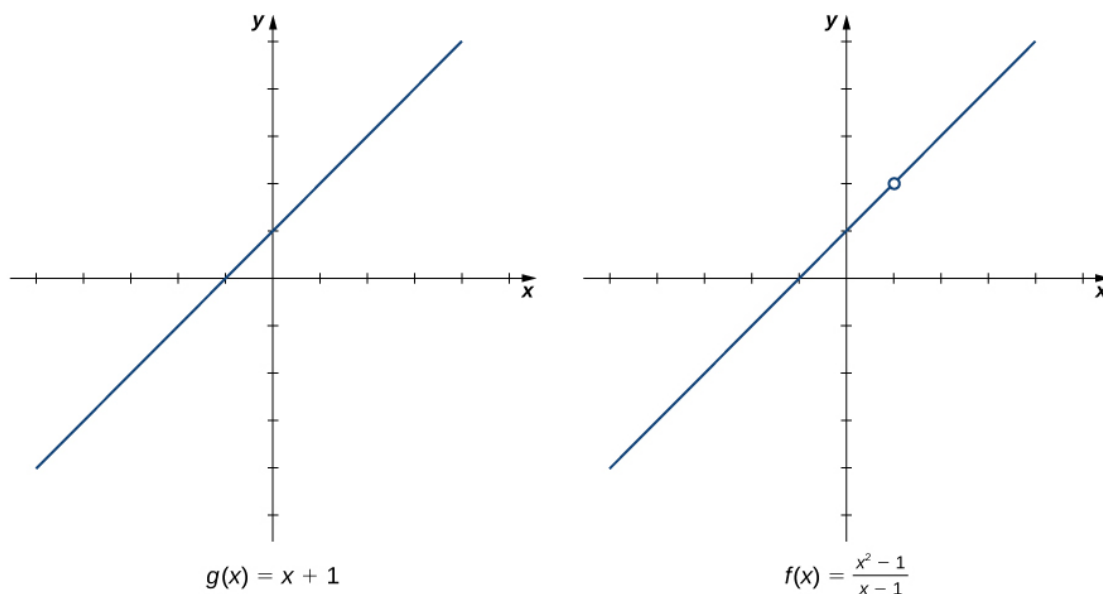
To understand this idea better, consider the limit $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

The function

Equation:

$$\begin{aligned} f(x) &= \frac{x^2 - 1}{x - 1} \\ &= \frac{(x - 1)(x + 1)}{x - 1} \end{aligned}$$

and the function $g(x) = x + 1$ are identical for all values of $x \neq 1$. The graphs of these two functions are shown in [\[link\]](#).



The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

Equation:

$$\begin{aligned}
 \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{x-1} \\
 &= \lim_{x \rightarrow 1} (x + 1) \\
 &= 2.
 \end{aligned}$$

The limit has the form $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$, where $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0$.) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

Note:

Problem-Solving Strategy: Calculating a Limit When $f(x)/g(x)$ has the Indeterminate Form $0/0$

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x) = f(x)/g(x)$ for all $x \neq a$ over some interval containing a . To do this, we may need to try one or more of the following steps:
 - a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
 - b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
 - c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.

3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. [\[link\]](#) illustrates the factor-and-cancel technique; [\[link\]](#) shows multiplying by a conjugate. In [\[link\]](#), we look at simplifying a complex fraction.

Example:

Exercise:

Problem:

Evaluating a Limit by Factoring and Canceling

Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 3x}{2x^2 - 5x - 3}$.

Solution:

Step 1. The function $f(x) = \frac{x^2 - 3x}{2x^2 - 5x - 3}$ is undefined for $x = 3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

Equation:

$$\lim_{x \rightarrow 3} \frac{x^2 - 3x}{2x^2 - 5x - 3} = \lim_{x \rightarrow 3} \frac{x(x - 3)}{(x - 3)(2x + 1)}$$

Step 2. For all $x \neq 3$, $\frac{x^2 - 3x}{2x^2 - 5x - 3} = \frac{x}{2x + 1}$. Therefore,

Equation:

$$\lim_{x \rightarrow 3} \frac{x(x - 3)}{(x - 3)(2x + 1)} = \lim_{x \rightarrow 3} \frac{x}{2x + 1}.$$

Step 3. Evaluate using the limit laws:

Equation:

$$\lim_{x \rightarrow 3} \frac{x}{2x + 1} = \frac{3}{7}.$$

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow -3} \frac{x^2 + 4x + 3}{x^2 - 9}$.

Solution:

$$\frac{1}{3}$$

Hint

Follow the steps in the Problem-Solving Strategy and [\[link\]](#).

Example:**Exercise:****Problem:****Evaluating a Limit by Multiplying by a Conjugate**

Evaluate $\lim_{x \rightarrow -1} \frac{\sqrt{x+2}-1}{x+1}$.

Solution:

Step 1. $\frac{\sqrt{x+2}-1}{x+1}$ has the form $0/0$ at -1 . Let's begin by multiplying by $\sqrt{x+2}+1$, the conjugate of $\sqrt{x+2}-1$, on the numerator and denominator:

Equation:

$$\lim_{x \rightarrow -1} \frac{\sqrt{x+2}-1}{x+1} = \lim_{x \rightarrow -1} \frac{\sqrt{x+2}-1}{x+1} \cdot \frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}.$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x+1)$ in the denominator cancels out in the end:

Equation:

$$= \lim_{x \rightarrow -1} \frac{x+1}{(x+1)(\sqrt{x+2}+1)}.$$

Step 3. Then we cancel:

Equation:

$$= \lim_{x \rightarrow -1} \frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:

Equation:

$$\lim_{x \rightarrow -1} \frac{1}{\sqrt{x+2}+1} = \frac{1}{2}.$$

Note:**Exercise:**

Problem: Evaluate $\lim_{x \rightarrow 5} \frac{\sqrt{x-1}-2}{x-5}$.

Solution:

$$\frac{1}{4}$$

Hint

Follow the steps in the Problem-Solving Strategy and [\[link\]](#).

Example:

Exercise:

Problem:

Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim_{x \rightarrow 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$.

Solution:

Step 1. $\frac{\frac{1}{x+1} - \frac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2(x+1)/2(x+1)$:

Equation:

$$\lim_{x \rightarrow 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1} = \lim_{x \rightarrow 1} \frac{\frac{1}{x+1} - \frac{1}{2}}{x-1} \cdot \frac{2(x+1)}{2(x+1)}.$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $(x-1)$:

Equation:

$$= \lim_{x \rightarrow 1} \frac{2 - (x+1)}{2(x-1)(x+1)}.$$

Step 3. Then, we simplify the numerator:

Equation:

$$= \lim_{x \rightarrow 1} \frac{-x+1}{2(x-1)(x+1)}.$$

Step 4. Now we factor out -1 from the numerator:

Equation:

$$= \lim_{x \rightarrow 1} \frac{-(x-1)}{2(x-1)(x+1)}.$$

Step 5. Then, we cancel the common factors of $(x - 1)$:

Equation:

$$= \lim_{x \rightarrow 1} \frac{-1}{2(x+1)}.$$

Step 6. Last, we evaluate using the limit laws:

Equation:

$$\lim_{x \rightarrow 1} \frac{-1}{2(x+1)} = -\frac{1}{4}.$$

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow -3} \frac{\frac{1}{x+2} + 1}{x+3}$.

Solution:

-1 ;

Hint

Follow the steps in the Problem-Solving Strategy and [\[link\]](#).

[\[link\]](#) does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

Example:

Exercise:

Problem:

Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim_{x \rightarrow 0} \left(\frac{1}{x} + \frac{5}{x(x-5)} \right)$.

Solution:

Both $1/x$ and $5/x(x-5)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

Equation:

$$\begin{aligned}\frac{1}{x} + \frac{5}{x(x-5)} &= \frac{x-5+5}{x(x-5)} \\ &= \frac{x}{x(x-5)}.\end{aligned}$$

Thus,

Equation:

$$\begin{aligned}\lim_{x \rightarrow 0} \left(\frac{1}{x} + \frac{5}{x(x-5)} \right) &= \lim_{x \rightarrow 0} \frac{x}{x(x-5)} \\ &= \lim_{x \rightarrow 0} \frac{1}{x-5} \\ &= -\frac{1}{5}.\end{aligned}$$

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow 3} \left(\frac{1}{x-3} - \frac{4}{x^2-2x-3} \right)$.

Solution:

$$\frac{1}{4}$$

Hint

Use the same technique as [\[link\]](#). Don't forget to factor $x^2 - 2x - 3$ before getting a common denominator.

Let's now revisit one-sided limits. Simple modifications in the limit laws allow us to apply them to one-sided limits. For example, to apply the limit laws to a limit of the form $\lim_{x \rightarrow a^-} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b, a) ; for a limit of the form $\lim_{x \rightarrow a^+} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a, c) . [\[link\]](#) illustrates this point.

Example:

Exercise:

Problem:

Evaluating a One-Sided Limit Using the Limit Laws

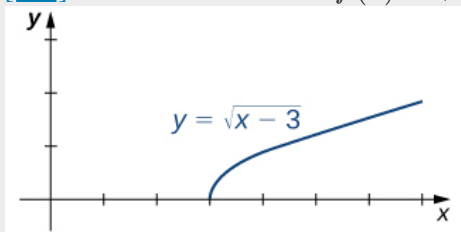
Evaluate each of the following limits, if possible.

a. $\lim_{x \rightarrow 3^-} \sqrt{x-3}$

b. $\lim_{x \rightarrow 3^+} \sqrt{x-3}$

Solution:

[\[link\]](#) illustrates the function $f(x) = \sqrt{x-3}$ and aids in our understanding of these limits.



The graph shows the function

$$f(x) = \sqrt{x-3}.$$

- The function $f(x) = \sqrt{x-3}$ is defined over the interval $[3, +\infty)$. Since this function is not defined to the left of 3, we cannot apply the limit laws to compute $\lim_{x \rightarrow 3^-} \sqrt{x-3}$. In fact, since $f(x) = \sqrt{x-3}$ is undefined to the left of 3, $\lim_{x \rightarrow 3^-} \sqrt{x-3}$ does not exist.
- Since $f(x) = \sqrt{x-3}$ is defined to the right of 3, the limit laws do apply to $\lim_{x \rightarrow 3^+} \sqrt{x-3}$. By applying these limit laws we obtain $\lim_{x \rightarrow 3^+} \sqrt{x-3} = 0$.

In [\[link\]](#) we look at one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

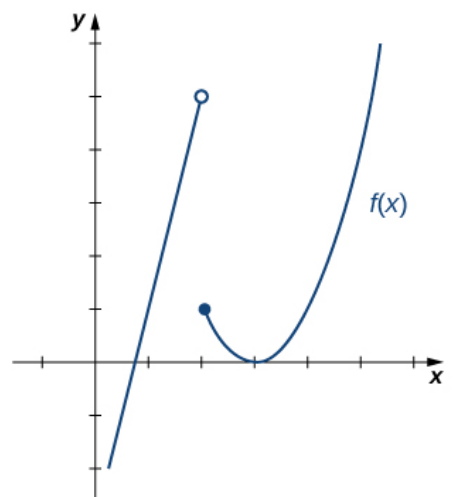
Example:**Exercise:****Problem:****Evaluating a Two-Sided Limit Using the Limit Laws**

For $f(x) = \begin{cases} 4x-3 & \text{if } x < 2 \\ (x-3)^2 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

- $\lim_{x \rightarrow 2^-} f(x)$
- $\lim_{x \rightarrow 2^+} f(x)$
- $\lim_{x \rightarrow 2} f(x)$

Solution:

[\[link\]](#) illustrates the function $f(x)$ and aids in our understanding of these limits.



This graph shows a function $f(x)$.

- a. Since $f(x) = 4x - 3$ for all x in $(-\infty, 2)$, replace $f(x)$ in the limit with $4x - 3$ and apply the limit laws:

Equation:

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (4x - 3) = 5.$$

- b. Since $f(x) = (x - 3)^2$ for all x in $(2, +\infty)$, replace $f(x)$ in the limit with $(x - 3)^2$ and apply the limit laws:

Equation:

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (x - 3)^2 = 1.$$

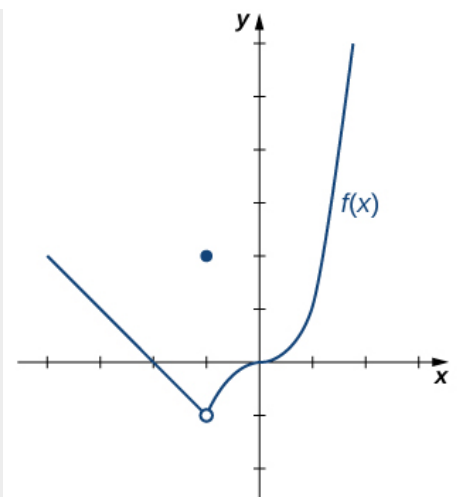
- c. Since $\lim_{x \rightarrow 2^-} f(x) = 5$ and $\lim_{x \rightarrow 2^+} f(x) = 1$, we conclude that $\lim_{x \rightarrow 2} f(x)$ does not exist.

Note:

Exercise:

Problem: Graph $f(x) = \begin{cases} -x - 2 & \text{if } x < -1 \\ 2 & \text{if } x = -1 \\ x^3 & \text{if } x > -1 \end{cases}$ and evaluate $\lim_{x \rightarrow -1^-} f(x)$.

Solution:



$$\lim_{x \rightarrow -1^-} f(x) = -1$$

Hint

Use the method in [\[link\]](#) to evaluate the limit.

We now turn our attention to evaluating a limit of the form $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$, where $\lim_{x \rightarrow a} f(x) = K$, where $K \neq 0$ and $\lim_{x \rightarrow a} g(x) = 0$. That is, $f(x)/g(x)$ has the form $K/0$, $K \neq 0$ at a .

Example:

Exercise:

Problem:

Evaluating a Limit of the Form $K/0$, $K \neq 0$ Using the Limit Laws

Evaluate $\lim_{x \rightarrow 2^-} \frac{x-3}{x^2-2x}$.

Solution:

Step 1. After substituting in $x = 2$, we see that this limit has the form $-1/0$. That is, as x approaches 2 from the left, the numerator approaches -1 ; and the denominator approaches 0. Consequently, the magnitude of $\frac{x-3}{x(x-2)}$ becomes infinite. To get a better idea of what the limit is, we need to factor the denominator:

Equation:

$$\lim_{x \rightarrow 2^-} \frac{x-3}{x^2-2x} = \lim_{x \rightarrow 2^-} \frac{x-3}{x(x-2)}.$$

Step 2. Since $x - 2$ is the only part of the denominator that is zero when 2 is substituted, we then separate $1/(x - 2)$ from the rest of the function:

Equation:

$$= \lim_{x \rightarrow 2^-} \frac{x-3}{x} \cdot \frac{1}{x-2}.$$

Step 3. $\lim_{x \rightarrow 2^-} \frac{x-3}{x} = -\frac{1}{2}$ and $\lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty$. Therefore, the product of $(x-3)/x$ and $1/(x-2)$ has a limit of $+\infty$:

Equation:

$$\lim_{x \rightarrow 2^-} \frac{x-3}{x^2-2x} = +\infty.$$

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow 1} \frac{x+2}{(x-1)^2}$.

Solution:

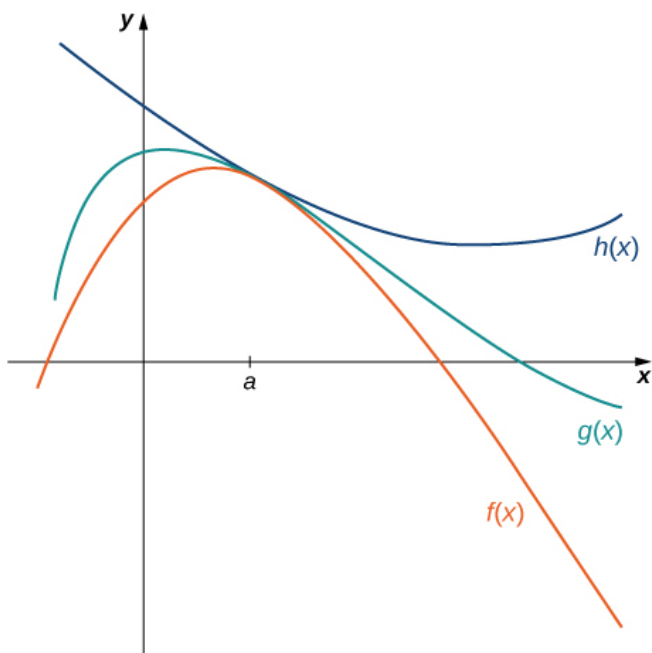
$+\infty$

Hint

Use the methods from [\[link\]](#).

The Squeeze Theorem

The techniques we have developed thus far work very well for algebraic functions, but we are still unable to evaluate limits of very basic trigonometric functions. The next theorem, called the **squeeze theorem**, proves very useful for establishing basic trigonometric limits. This theorem allows us to calculate limits by “squeezing” a function, with a limit at a point a that is unknown, between two functions having a common known limit at a . [\[link\]](#) illustrates this idea.



The Squeeze Theorem applies when $f(x) \leq g(x) \leq h(x)$ and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$.

Note:

The Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a . If

Equation:

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$ in an open interval containing a and

Equation:

$$\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$$

where L is a real number, then $\lim_{x \rightarrow a} g(x) = L$.

Example:

Exercise:

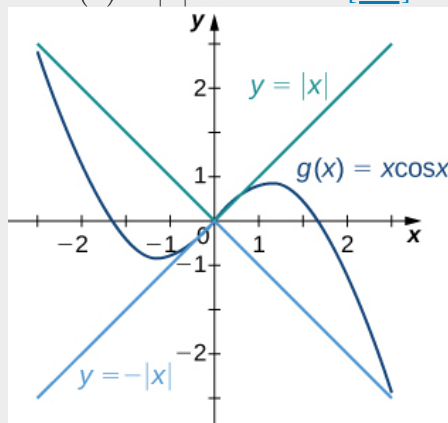
Problem:

Applying the Squeeze Theorem

Apply the squeeze theorem to evaluate $\lim_{x \rightarrow 0} x \cos x$.

Solution:

Because $-1 \leq \cos x \leq 1$ for all x , we have $-|x| \leq x \cos x \leq |x|$. Since $\lim_{x \rightarrow 0} (-|x|) = 0 = \lim_{x \rightarrow 0} |x|$, from the squeeze theorem, we obtain $\lim_{x \rightarrow 0} x \cos x = 0$. The graphs of $f(x) = -|x|$, $g(x) = x \cos x$, and $h(x) = |x|$ are shown in [\[link\]](#).



The graphs of $f(x)$, $g(x)$, and $h(x)$ are shown around the point $x = 0$.

Note:**Exercise:**

Problem: Use the squeeze theorem to evaluate $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x}$.

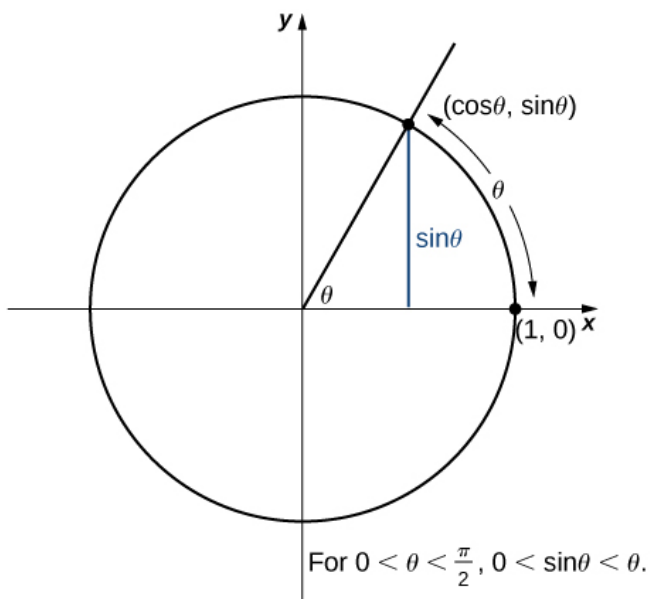
Solution:

0

Hint

Use the fact that $-x^2 \leq x^2 \sin (1/x) \leq x^2$ to help you find two functions such that $x^2 \sin (1/x)$ is squeezed between them.

We now use the squeeze theorem to tackle several very important limits. Although this discussion is somewhat lengthy, these limits prove invaluable for the development of the material in both the next section and the next chapter. The first of these limits is $\lim_{\theta \rightarrow 0} \sin \theta$. Consider the unit circle shown in [\[link\]](#). In the figure, we see that $\sin \theta$ is the y -coordinate on the unit circle and it corresponds to the line segment shown in blue. The radian measure of angle θ is the length of the arc it subtends on the unit circle. Therefore, we see that for $0 < \theta < \frac{\pi}{2}$, $0 < \sin \theta < \theta$.



The sine function is shown as a line on the unit circle.

Because $\lim_{\theta \rightarrow 0^+} 0 = 0$ and $\lim_{\theta \rightarrow 0^+} \theta = 0$, by using the squeeze theorem we conclude that

Equation:

$$\lim_{\theta \rightarrow 0^+} \sin \theta = 0.$$

To see that $\lim_{\theta \rightarrow 0^-} \sin \theta = 0$ as well, observe that for $-\frac{\pi}{2} < \theta < 0$, $0 < -\theta < \frac{\pi}{2}$ and hence, $0 < \sin(-\theta) < -\theta$. Consequently, $0 < -\sin \theta < -\theta$. It follows that $0 > \sin \theta > \theta$. An application of the squeeze theorem produces the desired limit. Thus, since $\lim_{\theta \rightarrow 0^+} \sin \theta = 0$ and $\lim_{\theta \rightarrow 0^-} \sin \theta = 0$,

Equation:

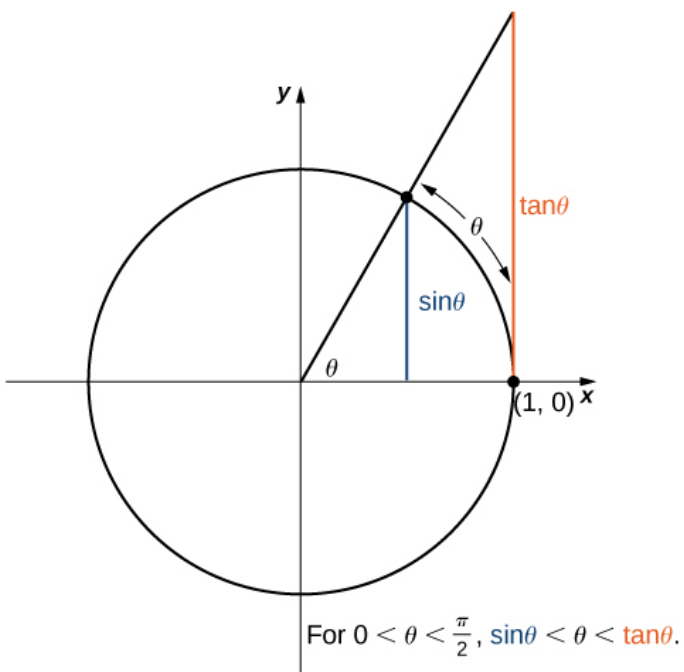
$$\lim_{\theta \rightarrow 0} \sin \theta = 0.$$

Next, using the identity $\cos \theta = \sqrt{1 - \sin^2 \theta}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we see that

Equation:

$$\lim_{\theta \rightarrow 0} \cos \theta = \lim_{\theta \rightarrow 0} \sqrt{1 - \sin^2 \theta} = 1.$$

We now take a look at a limit that plays an important role in later chapters—namely, $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}$. To evaluate this limit, we use the unit circle in [\[link\]](#). Notice that this figure adds one additional triangle to [\[link\]](#). We see that the length of the side opposite angle θ in this new triangle is $\tan \theta$. Thus, we see that for $0 < \theta < \frac{\pi}{2}$, $\sin \theta < \theta < \tan \theta$.



The sine and tangent functions are shown as lines on the unit circle.

By dividing by $\sin \theta$ in all parts of the inequality, we obtain

Equation:

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Equivalently, we have

Equation:

$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

Since $\lim_{\theta \rightarrow 0^+} 1 = 1 = \lim_{\theta \rightarrow 0^+} \cos \theta$, we conclude that $\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1$. By applying a manipulation similar to that used in demonstrating that $\lim_{\theta \rightarrow 0^-} \sin \theta = 0$, we can show that $\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1$. Thus,

Equation:

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

In [\[link\]](#) we use this limit to establish $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0$. This limit also proves useful in later chapters.

Example:

Exercise:

Problem:

Evaluating an Important Trigonometric Limit

Evaluate $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta}$.

Solution:

In the first step, we multiply by the conjugate so that we can use a trigonometric identity to convert the cosine in the numerator to a sine:

Equation:

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} \\&= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\theta(1 + \cos \theta)} \\&= \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta(1 + \cos \theta)} \\&= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{1 + \cos \theta} \\&= 1 \cdot \frac{0}{2} = 0.\end{aligned}$$

Therefore,

Equation:

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0.$$

Note:

Exercise:

Problem: Evaluate $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta}$.

Solution:

0

Hint

Multiply numerator and denominator by $1 + \cos \theta$.

Note:

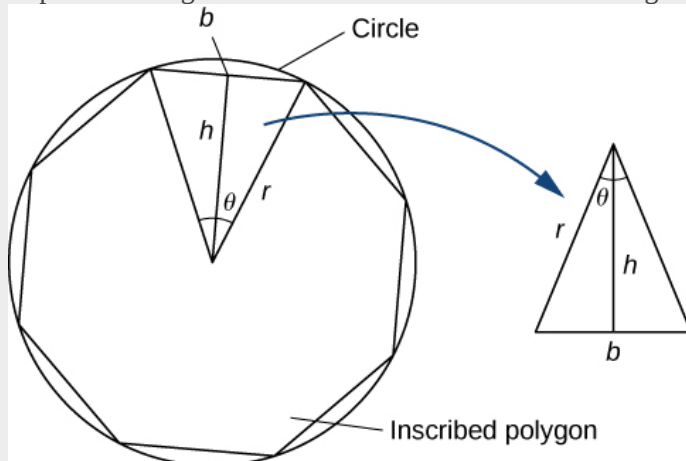
Deriving the Formula for the Area of a Circle

Some of the geometric formulas we take for granted today were first derived by methods that anticipate some of the methods of calculus. The Greek mathematician Archimedes (ca. 287–212; BCE) was

particularly inventive, using polygons inscribed within circles to approximate the area of the circle as the number of sides of the polygon increased. He never came up with the idea of a limit, but we can use this idea to see what his geometric constructions could have predicted about the limit.

We can estimate the area of a circle by computing the area of an inscribed regular polygon. Think of the regular polygon as being made up of n triangles. By taking the limit as the vertex angle of these triangles goes to zero, you can obtain the area of the circle. To see this, carry out the following steps:

1. Express the height h and the base b of the isosceles triangle in [\[link\]](#) in terms of θ and r .



2. Using the expressions that you obtained in step 1, express the area of the isosceles triangle in terms of θ and r .
(Substitute $(1/2)\sin\theta$ for $\sin(\theta/2)\cos(\theta/2)$ in your expression.)
3. If an n -sided regular polygon is inscribed in a circle of radius r , find a relationship between θ and n .
Solve this for n . Keep in mind there are 2π radians in a circle. (Use radians, not degrees.)
4. Find an expression for the area of the n -sided polygon in terms of r and θ .
5. To find a formula for the area of the circle, find the limit of the expression in step 4 as θ goes to zero.
(Hint: $\lim_{\theta \rightarrow 0} \frac{(\sin\theta)}{\theta} = 1$).

The technique of estimating areas of regions by using polygons is revisited in [Introduction to Integration](#).

Key Concepts

- The limit laws allow us to evaluate limits of functions without having to go through step-by-step processes each time.
- For polynomials and rational functions, $\lim_{x \rightarrow a} f(x) = f(a)$.
- You can evaluate the limit of a function by factoring and canceling, by multiplying by a conjugate, or by simplifying a complex fraction.
- The squeeze theorem allows you to find the limit of a function if the function is always greater than one function and less than another function with limits that are known.

Key Equations

- **Basic Limit Results**

$$\lim_{x \rightarrow a} x = a \quad \lim_{x \rightarrow a} c = c$$

- **Important Limits**

$$\lim_{\theta \rightarrow 0} \sin \theta = 0$$

$$\lim_{\theta \rightarrow 0} \cos \theta = 1$$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0$$

In the following exercises, use the limit laws to evaluate each limit. Justify each step by indicating the appropriate limit law(s).

Exercise:

Problem: $\lim_{x \rightarrow 0} (4x^2 - 2x + 3)$

Solution:

Use constant multiple law and difference law: $\lim_{x \rightarrow 0} (4x^2 - 2x + 3) = 4\lim_{x \rightarrow 0} x^2 - 2\lim_{x \rightarrow 0} x + \lim_{x \rightarrow 0} 3 = 3$

Exercise:

Problem: $\lim_{x \rightarrow 1} \frac{x^3 + 3x^2 + 5}{4 - 7x}$

Exercise:

Problem: $\lim_{x \rightarrow -2} \sqrt{x^2 - 6x + 3}$

Solution:

Use root law: $\lim_{x \rightarrow -2} \sqrt{x^2 - 6x + 3} = \sqrt{\lim_{x \rightarrow -2} (x^2 - 6x + 3)} = \sqrt{19}$

Exercise:

Problem: $\lim_{x \rightarrow -1} (9x + 1)^2$

In the following exercises, use direct substitution to evaluate each limit.

Exercise:

Problem: $\lim_{x \rightarrow 7} x^2$

Solution:

49

Exercise:

Problem: $\lim_{x \rightarrow -2} (4x^2 - 1)$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{1}{1 + \sin x}$

Solution:

$$1$$

Exercise:

Problem: $\lim_{x \rightarrow 2} e^{2x - x^2}$

Exercise:

Problem: $\lim_{x \rightarrow 1} \frac{2 - 7x}{x + 6}$

Solution:

$$-\frac{5}{7}$$

Exercise:

Problem: $\lim_{x \rightarrow 3} \ln e^{3x}$

In the following exercises, use direct substitution to show that each limit leads to the indeterminate form $0/0$. Then, evaluate the limit.

Exercise:

Problem: $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$

Solution:

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} = \frac{16 - 16}{4 - 4} = \frac{0}{0}; \text{ then, } \lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} = \lim_{x \rightarrow 4} \frac{(x + 4)(x - 4)}{x - 4} = 8$$

Exercise:

Problem: $\lim_{x \rightarrow 2} \frac{x - 2}{x^2 - 2x}$

Exercise:

Problem: $\lim_{x \rightarrow 6} \frac{3x - 18}{2x - 12}$

Solution:

$$\lim_{x \rightarrow 6} \frac{3x - 18}{2x - 12} = \frac{18 - 18}{12 - 12} = \frac{0}{0}; \text{ then, } \lim_{x \rightarrow 6} \frac{3x - 18}{2x - 12} = \lim_{x \rightarrow 6} \frac{3(x - 6)}{2(x - 6)} = \frac{3}{2}$$

Exercise:

Problem: $\lim_{h \rightarrow 0} \frac{(1 + h)^2 - 1}{h}$

Exercise:

Problem: $\lim_{t \rightarrow 9} \frac{t-9}{\sqrt{t}-3}$

Solution:

$$\lim_{x \rightarrow 9} \frac{t-9}{\sqrt{t}-3} = \frac{9-9}{3-3} = \frac{0}{0}; \text{ then, } \lim_{t \rightarrow 9} \frac{t-9}{\sqrt{t}-3} = \lim_{t \rightarrow 9} \frac{t-9}{\sqrt{t}-3} \frac{\sqrt{t}+3}{\sqrt{t}+3} = \lim_{t \rightarrow 9} (\sqrt{t}+3) = 6$$

Exercise:

Problem: $\lim_{h \rightarrow 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$, where a is a real-valued constant

Exercise:

Problem: $\lim_{\theta \rightarrow \pi} \frac{\sin \theta}{\tan \theta}$

Solution:

$$\lim_{\theta \rightarrow \pi} \frac{\sin \theta}{\tan \theta} = \frac{\sin \pi}{\tan \pi} = \frac{0}{0}; \text{ then, } \lim_{\theta \rightarrow \pi} \frac{\sin \theta}{\tan \theta} = \lim_{\theta \rightarrow \pi} \frac{\sin \theta}{\frac{\sin \theta}{\cos \theta}} = \lim_{\theta \rightarrow \pi} \cos \theta = -1$$

Exercise:

Problem: $\lim_{x \rightarrow 1} \frac{x^3-1}{x^2-1}$

Exercise:

Problem: $\lim_{x \rightarrow 1/2} \frac{2x^2+3x-2}{2x-1}$

Solution:

$$\lim_{x \rightarrow 1/2} \frac{2x^2+3x-2}{2x-1} = \frac{\frac{1}{2} + \frac{3}{2} - 2}{1-1} = \frac{0}{0}; \text{ then, } \lim_{x \rightarrow 1/2} \frac{2x^2+3x-2}{2x-1} = \lim_{x \rightarrow 1/2} \frac{(2x-1)(x+2)}{2x-1} = \frac{5}{2}$$

Exercise:

Problem: $\lim_{x \rightarrow -3} \frac{\sqrt{x+4}-1}{x+3}$

In the following exercises, use direct substitution to obtain an undefined expression. Then, use the method of [\[link\]](#) to simplify the function to help determine the limit.

Exercise:

Problem: $\lim_{x \rightarrow -2^-} \frac{2x^2+7x-4}{x^2+x-2}$

Solution:

$-\infty$

Exercise:

Problem: $\lim_{x \rightarrow -2^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

Exercise:

Problem: $\lim_{x \rightarrow 1^-} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

Solution:

$$-\infty$$

Exercise:

Problem: $\lim_{x \rightarrow 1^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

In the following exercises, assume that $\lim_{x \rightarrow 6} f(x) = 4$, $\lim_{x \rightarrow 6} g(x) = 9$, and $\lim_{x \rightarrow 6} h(x) = 6$. Use these three facts and the limit laws to evaluate each limit.

Exercise:

Problem: $\lim_{x \rightarrow 6} 2f(x)g(x)$

Solution:

$$\lim_{x \rightarrow 6} 2f(x)g(x) = 2 \lim_{x \rightarrow 6} f(x) \lim_{x \rightarrow 6} g(x) = 72$$

Exercise:

Problem: $\lim_{x \rightarrow 6} \frac{g(x) - 1}{f(x)}$

Exercise:

Problem: $\lim_{x \rightarrow 6} \left(f(x) + \frac{1}{3}g(x) \right)$

Solution:

$$\lim_{x \rightarrow 6} \left(f(x) + \frac{1}{3}g(x) \right) = \lim_{x \rightarrow 6} f(x) + \frac{1}{3} \lim_{x \rightarrow 6} g(x) = 7$$

Exercise:

Problem: $\lim_{x \rightarrow 6} \frac{(h(x))^3}{2}$

Exercise:

Problem: $\lim_{x \rightarrow 6} \sqrt{g(x) - f(x)}$

Solution:

$$\lim_{x \rightarrow 6} \sqrt{g(x) - f(x)} = \sqrt{\lim_{x \rightarrow 6} g(x) - \lim_{x \rightarrow 6} f(x)} = \sqrt{5}$$

Exercise:

Problem: $\lim_{x \rightarrow 6} x \cdot h(x)$

Exercise:

Problem: $\lim_{x \rightarrow 6} [(x + 1) \cdot f(x)]$

Solution:

$$\lim_{x \rightarrow 6} [(x + 1)f(x)] = \left(\lim_{x \rightarrow 6} (x + 1) \right) \left(\lim_{x \rightarrow 6} f(x) \right) = 28$$

Exercise:

Problem: $\lim_{x \rightarrow 6} (f(x) \cdot g(x) - h(x))$

[T] In the following exercises, use a calculator to draw the graph of each piecewise-defined function and study the graph to evaluate the given limits.

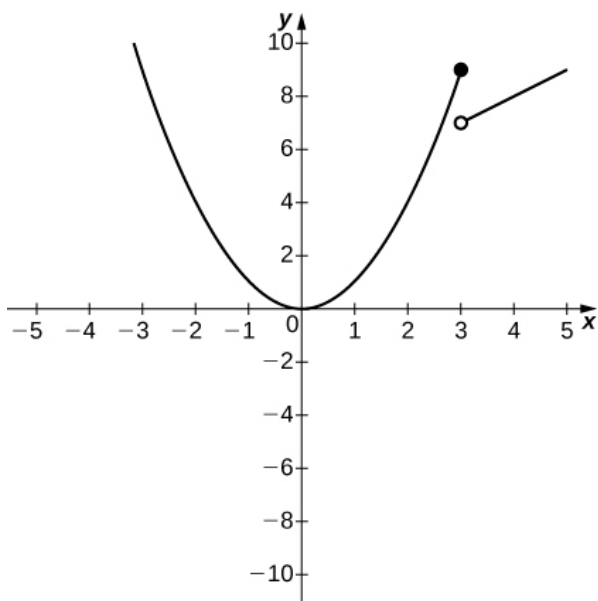
Exercise:

Problem: $f(x) = \begin{cases} x^2, & x \leq 3 \\ x + 4, & x > 3 \end{cases}$

a. $\lim_{x \rightarrow 3^-} f(x)$

b. $\lim_{x \rightarrow 3^+} f(x)$

Solution:



a. 9; b. 7

Exercise:

Problem: $g(x) = \begin{cases} x^3 - 1, & x \leq 0 \\ 1, & x > 0 \end{cases}$

a. $\lim_{x \rightarrow 0^-} g(x)$

b. $\lim_{x \rightarrow 0^+} g(x)$

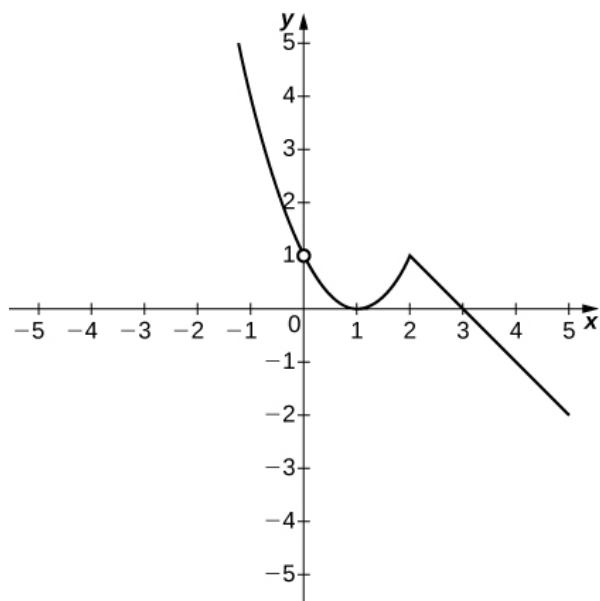
Exercise:

Problem: $h(x) = \begin{cases} x^2 - 2x + 1, & x < 2 \\ 3 - x, & x \geq 2 \end{cases}$

a. $\lim_{x \rightarrow 2^-} h(x)$

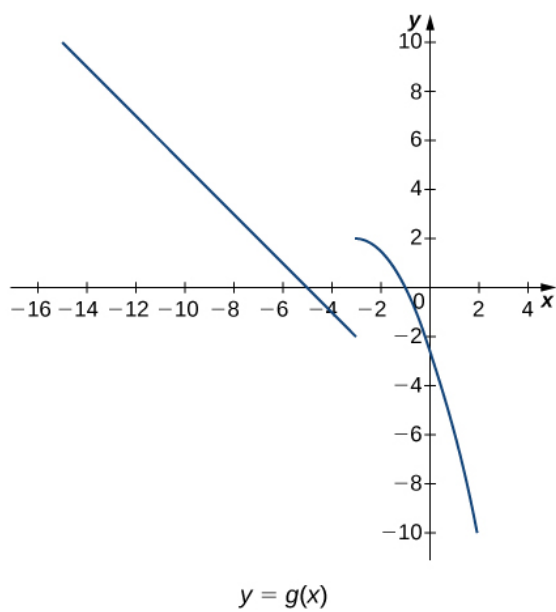
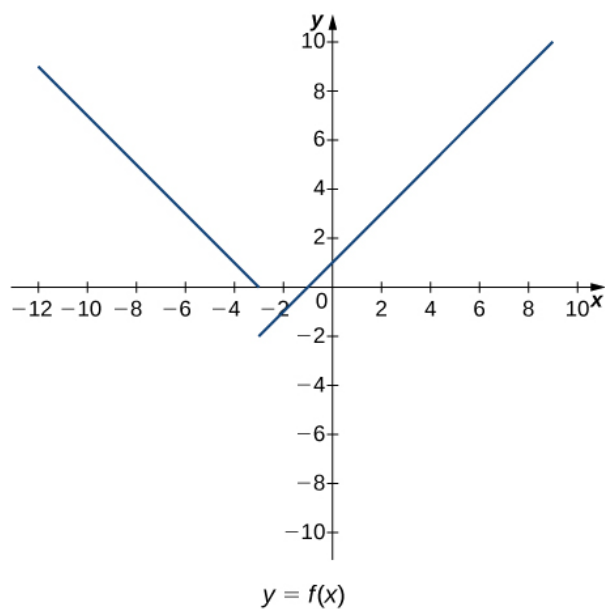
b. $\lim_{x \rightarrow 2^+} h(x)$

Solution:



a. 1; b. 1

In the following exercises, use the following graphs and the limit laws to evaluate each limit.



Exercise:

Problem: $\lim_{x \rightarrow -3^+} (f(x) + g(x))$

Exercise:

Problem: $\lim_{x \rightarrow -3^-} (f(x) - 3g(x))$

Solution:

$$\lim_{x \rightarrow -3^-} (f(x) - 3g(x)) = \lim_{x \rightarrow -3^-} f(x) - 3 \lim_{x \rightarrow -3^-} g(x) = 0 + 6 = 6$$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{f(x)g(x)}{3}$

Exercise:

Problem: $\lim_{x \rightarrow -5} \frac{2+g(x)}{f(x)}$

Solution:

$$\lim_{x \rightarrow -5} \frac{2+g(x)}{f(x)} = \frac{2+\left(\lim_{x \rightarrow -5} g(x)\right)}{\lim_{x \rightarrow -5} f(x)} = \frac{2+0}{2} = 1$$

Exercise:

Problem: $\lim_{x \rightarrow 1} (f(x))^2$

Exercise:

Problem: $\lim_{x \rightarrow 1} \sqrt{f(x) - g(x)}$

Solution:

$$\lim_{x \rightarrow 1} \sqrt[3]{f(x) - g(x)} = \sqrt[3]{\lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} g(x)} = \sqrt[3]{2 + 5} = \sqrt[3]{7}$$

Exercise:

Problem: $\lim_{x \rightarrow -7} (x \cdot g(x))$

Exercise:

Problem: $\lim_{x \rightarrow -9} [x \cdot f(x) + 2 \cdot g(x)]$

Solution:

$$\lim_{x \rightarrow -9} (xf(x) + 2g(x)) = \left(\lim_{x \rightarrow -9} x\right) \left(\lim_{x \rightarrow -9} f(x)\right) + 2 \lim_{x \rightarrow -9} (g(x)) = (-9)(6) + 2(4) = -46$$

For the following problems, evaluate the limit using the squeeze theorem. Use a calculator to graph the functions $f(x)$, $g(x)$, and $h(x)$ when possible.

Exercise:

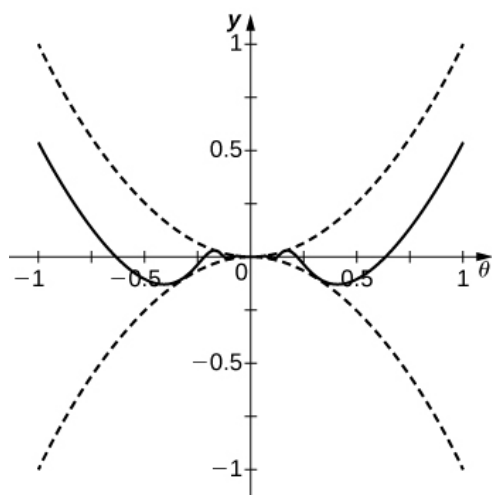
Problem: [T] True or False? If $2x - 1 \leq g(x) \leq x^2 - 2x + 3$, then $\lim_{x \rightarrow 2} g(x) = 0$.

Exercise:

Problem: [T] $\lim_{\theta \rightarrow 0} \theta^2 \cos\left(\frac{1}{\theta}\right)$

Solution:

The limit is zero.

**Exercise:**

Problem: $\lim_{x \rightarrow 0} f(x)$, where $f(x) = \begin{cases} 0, & x \text{ rational} \\ x^2, & x \text{ irrational} \end{cases}$

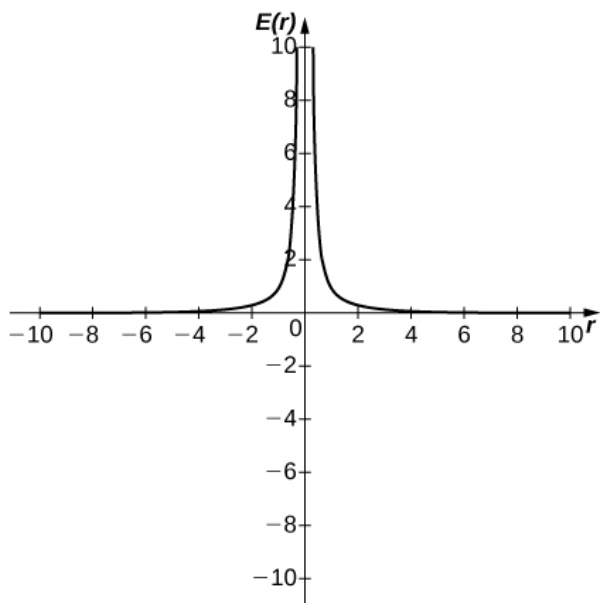
Exercise:**Problem:**

[T] In physics, the magnitude of an electric field generated by a point charge at a distance r in vacuum is governed by Coulomb's law: $E(r) = \frac{q}{4\pi\epsilon_0 r^2}$, where E represents the magnitude of the electric field, q is the charge of the particle, r is the distance between the particle and where the strength of the field is measured, and $\frac{1}{4\pi\epsilon_0}$ is Coulomb's constant: $8.988 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$.

- Use a graphing calculator to graph $E(r)$ given that the charge of the particle is $q = 10^{-10}$.
- Evaluate $\lim_{r \rightarrow 0^+} E(r)$. What is the physical meaning of this quantity? Is it physically relevant? Why are you evaluating from the right?

Solution:

a.



b. ∞ . The magnitude of the electric field as you approach the particle q becomes infinite. It does not make physical sense to evaluate negative distance.

Exercise:

Problem: [T] The density of an object is given by its mass divided by its volume: $\rho = m/V$.

- Use a calculator to plot the volume as a function of density ($V = m/\rho$), assuming you are examining something of mass 8 kg ($m = 8$).
- Evaluate $\lim_{\rho \rightarrow 0^+} V(\rho)$ and explain the physical meaning.

Glossary

constant multiple law for limits

$$\text{the limit law } \lim_{x \rightarrow a} c f(x) = c \cdot \lim_{x \rightarrow a} f(x) = cL$$

difference law for limits

$$\text{the limit law } \lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x) = L - M$$

limit laws

the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a ; assume that L and M are real numbers so that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$; let c be a constant

power law for limits

$$\text{the limit law } \lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n = L^n \text{ for every positive integer } n$$

product law for limits

the limit law $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = L \cdot M$

quotient law for limits

the limit law $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}$ for $M \neq 0$

root law for limits

the limit law $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even

squeeze theorem

states that if $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ over an open interval containing a and $\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$ where L is a real number, then $\lim_{x \rightarrow a} g(x) = L$

sum law for limits

The limit law $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = L + M$

Continuity

- Explain the three conditions for continuity at a point.
- Describe three kinds of discontinuities.
- Define continuity on an interval.
- State the theorem for limits of composite functions.
- Provide an example of the intermediate value theorem.

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called *continuous*. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a *discontinuity at a point* where a break occurs.

We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

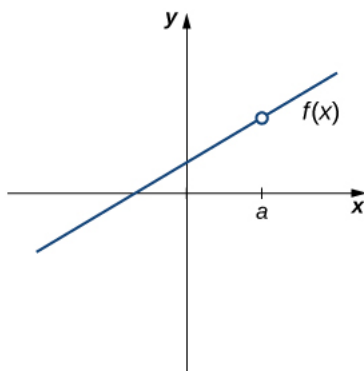
Continuity at a Point

Before we look at a formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in [\[link\]](#). We see that the graph of $f(x)$ has a hole at a . In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at a , we need the following condition:

Equation:

- i. $f(a)$ is defined.

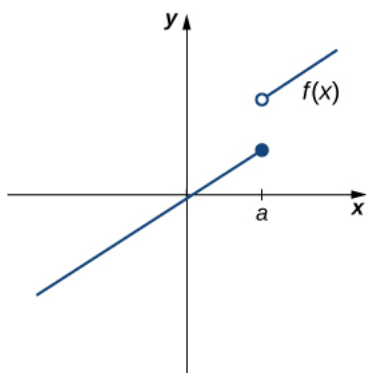


The function $f(x)$ is not continuous at a because $f(a)$ is undefined.

However, as we see in [\[link\]](#), this condition alone is insufficient to guarantee continuity at the point a . Although $f(a)$ is defined, the function has a gap at a . In this example, the gap exists because $\lim_{x \rightarrow a} f(x)$ does not exist. We must add another condition for continuity at a —namely,

Equation:

- ii. $\lim_{x \rightarrow a} f(x)$ exists.

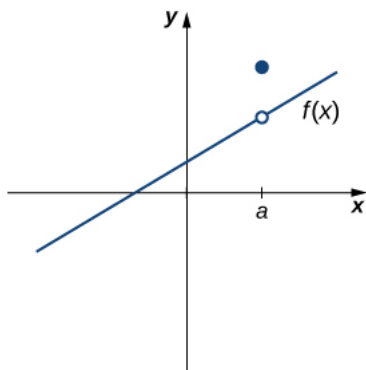


The function $f(x)$ is not continuous at a because $\lim_{x \rightarrow a} f(x)$ does not exist.

However, as we see in [\[link\]](#), these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at a . We must add a third condition to our list:

Equation:

$$\text{iii. } \lim_{x \rightarrow a} f(x) = f(a).$$



The function $f(x)$ is not continuous at a because $\lim_{x \rightarrow a} f(x) \neq f(a)$.

Now we put our list of conditions together and form a definition of continuity at a point.

Note:**Definition**

A function $f(x)$ is **continuous at a point** a if and only if the following three conditions are satisfied:

- i. $f(a)$ is defined
- ii. $\lim_{x \rightarrow a} f(x)$ exists
- iii. $\lim_{x \rightarrow a} f(x) = f(a)$

A function is **discontinuous at a point** a if it fails to be continuous at a .

The following procedure can be used to analyze the continuity of a function at a point using this definition.

Note:**Problem-Solving Strategy: Determining Continuity at a Point**

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at a . If $f(a)$ is defined, continue to step 2.
2. Compute $\lim_{x \rightarrow a} f(x)$. In some cases, we may need to do this by first computing $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$. If $\lim_{x \rightarrow a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at a and the problem is solved. If $\lim_{x \rightarrow a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim_{x \rightarrow a} f(x)$. If $\lim_{x \rightarrow a} f(x) \neq f(a)$, then the function is not continuous at a . If $\lim_{x \rightarrow a} f(x) = f(a)$, then the function is continuous at a .

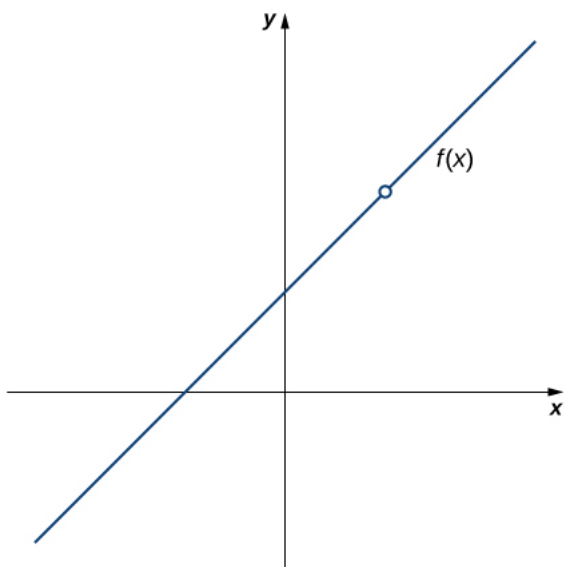
The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

Example:**Exercise:****Problem:****Determining Continuity at a Point, Condition 1**

Using the definition, determine whether the function $f(x) = (x^2 - 4)/(x - 2)$ is continuous at $x = 2$. Justify the conclusion.

Solution:

Let's begin by trying to calculate $f(2)$. We can see that $f(2) = 0/0$, which is undefined. Therefore, $f(x) = \frac{x^2 - 4}{x - 2}$ is discontinuous at 2 because $f(2)$ is undefined. The graph of $f(x)$ is shown in [\[link\]](#).



The function $f(x)$ is discontinuous at 2 because $f(2)$ is undefined.

Example:

Exercise:

Problem:

Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function $f(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$.

Justify the conclusion.

Solution:

Let's begin by trying to calculate $f(3)$.

Equation:

$$f(3) = -(3^2) + 4 = -5.$$

Thus, $f(3)$ is defined. Next, we calculate $\lim_{x \rightarrow 3} f(x)$. To do this, we must compute $\lim_{x \rightarrow 3^-} f(x)$ and $\lim_{x \rightarrow 3^+} f(x)$:

Equation:

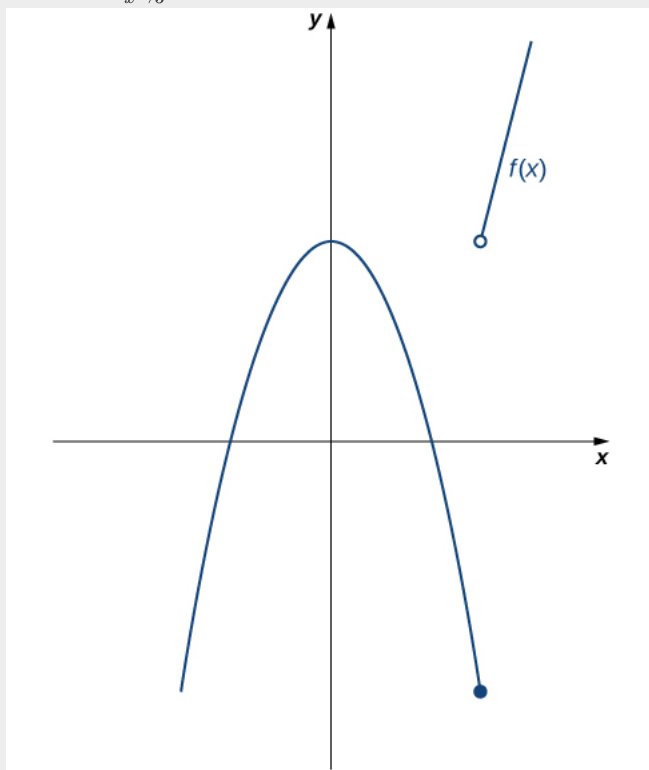
$$\lim_{x \rightarrow 3^-} f(x) = -(3^2) + 4 = -5$$

and

Equation:

$$\lim_{x \rightarrow 3^+} f(x) = 4(3) - 8 = 4.$$

Therefore, $\lim_{x \rightarrow 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3. The graph of $f(x)$ is shown in [\[link\]](#).



The function $f(x)$ is not continuous at 3 because $\lim_{x \rightarrow 3} f(x)$ does not exist.

Example:

Exercise:

Problem:

Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

Solution:

First, observe that

Equation:

$$f(0) = 1.$$

Next,

Equation:

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Last, compare $f(0)$ and $\lim_{x \rightarrow 1} f(x)$. We see that

Equation:

$$f(0) = 1 = \lim_{x \rightarrow 0} f(x).$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

Note:

Exercise:

Problem:

Using the definition, determine whether the function $f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x + 4 & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

Solution:

f is not continuous at 1 because $f(1) = 2 \neq 3 = \lim_{x \rightarrow 1} f(x)$.

Hint

Check each condition of the definition.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

Note:

Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x \rightarrow a} p(x) = p(a)$ for every polynomial $p(x)$ and $\lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

□

We now apply [\[link\]](#) to determine the points at which a given rational function is continuous.

Example:**Exercise:****Problem:****Continuity of a Rational Function**

For what values of x is $f(x) = \frac{x+1}{x-5}$ continuous?

Solution:

The rational function $f(x) = \frac{x+1}{x-5}$ is continuous for every value of x except $x = 5$.

Note:**Exercise:**

Problem: For what values of x is $f(x) = 3x^4 - 4x^2$ continuous?

Solution:

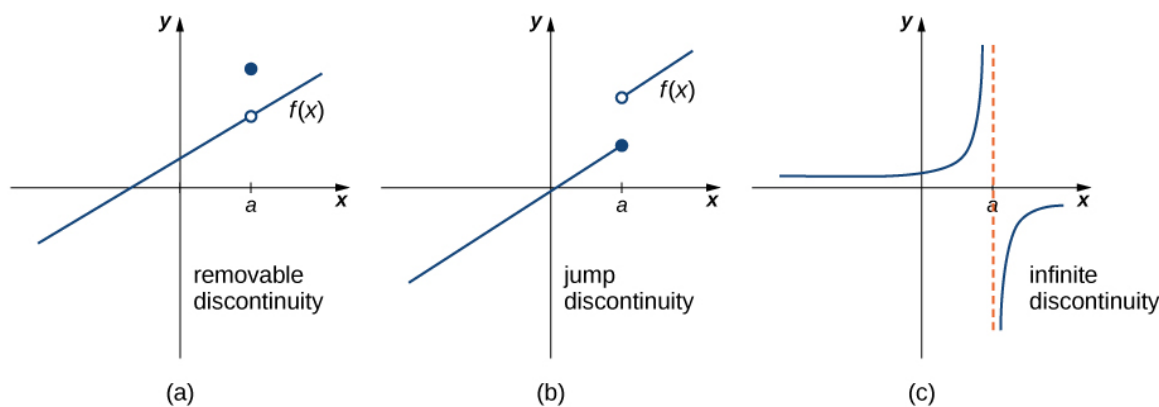
$f(x)$ is continuous at every real number.

Hint

Use [\[link\]](#)

Types of Discontinuities

As we have seen in [\[link\]](#) and [\[link\]](#), discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. [\[link\]](#) illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.



Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

These three discontinuities are formally defined as follows:

Note:

Definition

If $f(x)$ is discontinuous at a , then

1. f has a **removable discontinuity** at a if $\lim_{x \rightarrow a} f(x)$ exists. (Note: When we state that $\lim_{x \rightarrow a} f(x)$ exists, we mean that $\lim_{x \rightarrow a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ both exist, but $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$. (Note: When we state that $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim_{x \rightarrow a^-} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^+} f(x) = \pm\infty$.

Example:

Exercise:

Problem:

Classifying a Discontinuity

In [\[link\]](#), we showed that $f(x) = \frac{x^2-4}{x-2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

Solution:

To classify the discontinuity at 2 we must evaluate $\lim_{x \rightarrow 2} f(x)$:

Equation:

$$\begin{aligned}\lim_{x \rightarrow 2} f(x) &= \lim_{x \rightarrow 2} \frac{x^2-4}{x-2} \\ &= \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x \rightarrow 2} (x+2) \\ &= 4.\end{aligned}$$

Since f is discontinuous at 2 and $\lim_{x \rightarrow 2} f(x)$ exists, f has a removable discontinuity at $x = 2$.

Example:

Exercise:

Problem:

Classifying a Discontinuity

In [\[link\]](#), we showed that $f(x) = \begin{cases} -x^2 + 4 & \text{if } x \leq 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$ is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite.

Solution:

Earlier, we showed that f is discontinuous at 3 because $\lim_{x \rightarrow 3} f(x)$ does not exist. However, since $\lim_{x \rightarrow 3^-} f(x) = -5$ and $\lim_{x \rightarrow 3^+} f(x) = 4$ both exist, we conclude that the function has a jump discontinuity at 3.

Example:

Exercise:

Problem:

Classifying a Discontinuity

Determine whether $f(x) = \frac{x+2}{x+1}$ is continuous at -1 . If the function is discontinuous at -1 , classify the discontinuity as removable, jump, or infinite.

Solution:

The function value $f(-1)$ is undefined. Therefore, the function is not continuous at -1 . To determine the type of discontinuity, we must determine the limit at -1 . We see that $\lim_{x \rightarrow -1^-} \frac{x+2}{x+1} = -\infty$ and

$\lim_{x \rightarrow -1^+} \frac{x+2}{x+1} = +\infty$. Therefore, the function has an infinite discontinuity at -1 .

Note:

Exercise:

Problem:

For $f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$, decide whether f is continuous at 1. If f is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Solution:

Discontinuous at 1; removable

Hint

Follow the steps in [\[link\]](#). If the function is discontinuous at 1, look at $\lim_{x \rightarrow 1} f(x)$ and use the definition to determine the type of discontinuity.

Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in

the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by looking at the definition of what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Note:

Continuity from the Right and from the Left

A function $f(x)$ is said to be **continuous from the right** at a if $\lim_{x \rightarrow a^+} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the left** at a if $\lim_{x \rightarrow a^-} f(x) = f(a)$.

A function is continuous over an open interval if it is continuous at every point in the interval. A function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b . Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at b . Continuity over other types of intervals are defined in a similar fashion.

Requiring that $\lim_{x \rightarrow a^+} f(x) = f(a)$ and $\lim_{x \rightarrow b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim_{x \rightarrow a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a, b]$.

Example:

Exercise:

Problem:

Continuity on an Interval

State the interval(s) over which the function $f(x) = \frac{x-1}{x^2+2x}$ is continuous.

Solution:

Since $f(x) = \frac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain. The domain of $f(x)$ is the set $(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$. Thus, $f(x)$ is continuous over each of the intervals $(-\infty, -2)$, $(-2, 0)$, and $(0, +\infty)$.

Example:

Exercise:

Problem:

Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4-x^2}$ is continuous.

Solution:

From the limit laws, we know that $\lim_{x \rightarrow a} \sqrt{4-x^2} = \sqrt{4-a^2}$ for all values of a in $(-2, 2)$. We also know that $\lim_{x \rightarrow -2^+} \sqrt{4-x^2} = 0$ exists and $\lim_{x \rightarrow 2^-} \sqrt{4-x^2} = 0$ exists. Therefore, $f(x)$ is continuous over the interval $[-2, 2]$.

Note:

Exercise:

Problem: State the interval(s) over which the function $f(x) = \sqrt{x+3}$ is continuous.

Solution:

$$[-3, +\infty)$$

Hint

Use [\[link\]](#) as a guide for solving.

The [\[link\]](#) allows us to expand our ability to compute limits. In particular, this theorem ultimately allows us to demonstrate that trigonometric functions are continuous over their domains.

Note:

Composite Function Theorem

If $f(x)$ is continuous at L and $\lim_{x \rightarrow a} g(x) = L$, then

Equation:

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(L).$$

Before we move on to [\[link\]](#), recall that earlier, in the section on limit laws, we showed $\lim_{x \rightarrow 0} \cos x = 1 = \cos(0)$. Consequently, we know that $f(x) = \cos x$ is continuous at 0. In [\[link\]](#) we see how to combine this result with the composite function theorem.

Example:

Exercise:

Problem:

Limit of a Composite Cosine Function

Evaluate $\lim_{x \rightarrow \pi/2} \cos\left(x - \frac{\pi}{2}\right)$.

Solution:

The given function is a composite of $\cos x$ and $x - \frac{\pi}{2}$. Since $\lim_{x \rightarrow \pi/2} \left(x - \frac{\pi}{2}\right) = 0$ and $\cos x$ is continuous at 0, we may apply the composite function theorem. Thus,

Equation:

$$\lim_{x \rightarrow \pi/2} \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\lim_{x \rightarrow \pi/2} \left(x - \frac{\pi}{2}\right)\right) = \cos(0) = 1.$$

Note:
Exercise:

Problem: Evaluate $\lim_{x \rightarrow \pi} \sin(x - \pi)$.

Solution:

0

Hint

$f(x) = \sin x$ is continuous at 0. Use [\[link\]](#) as a guide.

The proof of the next theorem uses the composite function theorem as well as the continuity of $f(x) = \sin x$ and $g(x) = \cos x$ at the point 0 to show that trigonometric functions are continuous over their entire domains.

Note:
Continuity of Trigonometric Functions
Trigonometric functions are continuous over their entire domains.

Proof

We begin by demonstrating that $\cos x$ is continuous at every real number. To do this, we must show that $\lim_{x \rightarrow a} \cos x = \cos a$ for all values of a .

$$\begin{aligned} \lim_{x \rightarrow a} \cos x &= \lim_{x \rightarrow a} \cos((x - a) + a) && \text{rewrite } x = x - a + a \\ &= \lim_{x \rightarrow a} (\cos(x - a) \cos a - \sin(x - a) \sin a) && \text{apply the identity for the cosine of the} \\ &= \cos\left(\lim_{x \rightarrow a} (x - a)\right) \cos a - \sin\left(\lim_{x \rightarrow a} (x - a)\right) \sin a && \lim_{x \rightarrow a} (x - a) = 0, \text{ and } \sin x \text{ and } \cos x \text{ are} \\ &= \cos(0) \cos a - \sin(0) \sin a && \text{evaluate } \cos(0) \text{ and } \sin(0) \text{ and simplify} \\ &= 1 \cdot \cos a - 0 \cdot \sin a = \cos a. \end{aligned}$$

The proof that $\sin x$ is continuous at every real number is analogous. Because the remaining trigonometric functions may be expressed in terms of $\sin x$ and $\cos x$, their continuity follows from the quotient limit law.

□

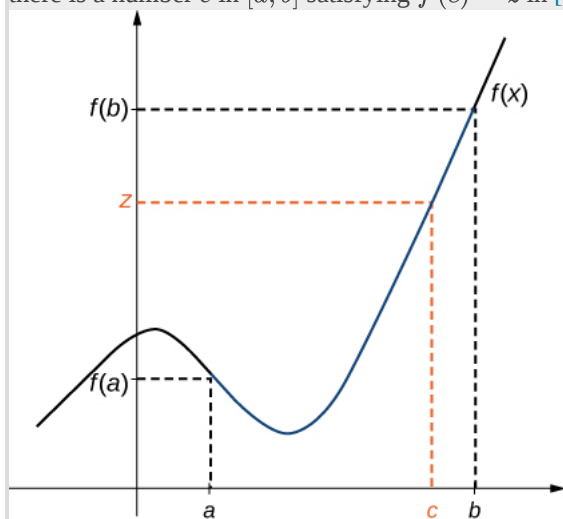
As you can see, the composite function theorem is invaluable in demonstrating the continuity of trigonometric functions. As we continue our study of calculus, we revisit this theorem many times.

The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b]$, where a and b are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem**.

Note:**The Intermediate Value Theorem**

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$ in [link](#).



There is a number $c \in [a, b]$ that satisfies
 $f(c) = z$.

Example:**Exercise:****Problem:****Application of the Intermediate Value Theorem**

Show that $f(x) = x - \cos x$ has at least one zero.

Solution:

Since $f(x) = x - \cos x$ is continuous over $(-\infty, +\infty)$, it is continuous over any closed interval of the form $[a, b]$. If you can find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number c in (a, b) that satisfies $f(c) = 0$. Note that

Equation:

$$f(0) = 0 - \cos(0) = -1 < 0$$

and

Equation:

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} > 0.$$

Using the Intermediate Value Theorem, we can see that there must be a real number c in $[0, \pi/2]$ that satisfies $f(c) = 0$. Therefore, $f(x) = x - \cos x$ has at least one zero.

Example:**Exercise:****Problem:****When Can You Apply the Intermediate Value Theorem?**

If $f(x)$ is continuous over $[0, 2]$, $f(0) > 0$ and $f(2) > 0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0, 2]$? Explain.

Solution:

No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f(0)$ and $f(2)$; it doesn't allow us to conclude that we can't find other values. To see this more clearly, consider the function $f(x) = (x - 1)^2$. It satisfies $f(0) = 1 > 0$, $f(2) = 1 > 0$, and $f(1) = 0$.

Example:**Exercise:****Problem:****When Can You Apply the Intermediate Value Theorem?**

For $f(x) = 1/x$, $f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

Solution:

No. The function is not continuous over $[-1, 1]$. The Intermediate Value Theorem does not apply here.

Note:**Exercise:**

Problem: Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

Solution:

$f(0) = 1 > 0$, $f(1) = -2 < 0$; $f(x)$ is continuous over $[0, 1]$. It must have a zero on this interval.

Hint

Find $f(0)$ and $f(1)$. Apply the Intermediate Value Theorem.

Key Concepts

- For a function to be continuous at a point, it must be defined at that point, its limit must exist at the point, and the value of the function at that point must equal the value of the limit at that point.
- Discontinuities may be classified as removable, jump, or infinite.
- A function is continuous over an open interval if it is continuous at every point in the interval. It is continuous over a closed interval if it is continuous at every point in its interior and is continuous at its endpoints.
- The composite function theorem states: If $f(x)$ is continuous at L and $\lim_{x \rightarrow a} g(x) = L$, then

$$\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(L).$$

- The Intermediate Value Theorem guarantees that if a function is continuous over a closed interval, then the function takes on every value between the values at its endpoints.

For the following exercises, determine the point(s), if any, at which each function is discontinuous. Classify any discontinuity as jump, removable, infinite, or other.

Exercise:

Problem: $f(x) = \frac{1}{\sqrt{x}}$

Solution:

The function is defined for all x in the interval $(0, \infty)$.

Exercise:

Problem: $f(x) = \frac{2}{x^2+1}$

Exercise:

Problem: $f(x) = \frac{x}{x^2-x}$

Solution:

Removable discontinuity at $x = 0$; infinite discontinuity at $x = 1$

Exercise:

Problem: $g(t) = t^{-1} + 1$

Exercise:

Problem: $f(x) = \frac{5}{e^x-2}$

Solution:

Infinite discontinuity at $x = \ln 2$

Exercise:

Problem: $f(x) = \frac{|x-2|}{x-2}$

Exercise:

Problem: $H(x) = \tan 2x$

Solution:

Infinite discontinuities at $x = \frac{(2k+1)\pi}{4}$, for $k = 0, \pm 1, \pm 2, \pm 3, \dots$

Exercise:

Problem: $f(t) = \frac{t+3}{t^2+5t+6}$

For the following exercises, decide if the function continuous at the given point. If it is discontinuous, what type of discontinuity is it?

Exercise:

Problem: $\frac{2x^2-5x+3}{x-1}$ at $x = 1$

Solution:

No. It is a removable discontinuity.

Exercise:

Problem: $h(\theta) = \frac{\sin\theta - \cos\theta}{\tan\theta}$ at $\theta = \pi$

Exercise:

Problem: $g(u) = \begin{cases} \frac{6u^2+u-2}{2u-1} & \text{if } u \neq \frac{1}{2} \\ \frac{7}{2} & \text{if } u = \frac{1}{2} \end{cases}, \text{ at } u = \frac{1}{2}$

Solution:

Yes. It is continuous.

Exercise:

Problem: $f(y) = \frac{\sin(\pi y)}{\tan(\pi y)}$, at $y = 1$

Exercise:

Problem: $f(x) = \begin{cases} x^2 - e^x & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}, \text{ at } x = 0$

Solution:

Yes. It is continuous.

Exercise:

Problem: $f(x) = \begin{cases} x \sin(x) & \text{if } x \leq \pi \\ x \tan(x) & \text{if } x > \pi \end{cases}, \text{ at } x = \pi$

In the following exercises, find the value(s) of k that makes each function continuous over the given interval.

Exercise:

Problem: $f(x) = \begin{cases} 3x + 2, & x < k \\ 2x - 3, & k \leq x \leq 8 \end{cases}$

Solution:

$$k = -5$$

Exercise:

Problem: $f(\theta) = \begin{cases} \sin\theta, & 0 \leq \theta < \frac{\pi}{2} \\ \cos(\theta + k), & \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} \frac{x^2+3x+2}{x+2}, & x \neq -2 \\ k, & x = -2 \end{cases}$

Solution:

$$k = -1$$

Exercise:

Problem: $f(x) = \begin{cases} e^{kx}, & 0 \leq x < 4 \\ x + 3, & 4 \leq x \leq 8 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} \sqrt{kx}, & 0 \leq x \leq 3 \\ x + 1, & 3 < x \leq 10 \end{cases}$

Solution:

$$k = \frac{16}{3}$$

In the following exercises, use the Intermediate Value Theorem (IVT).

Exercise:**Problem:**

Let $h(x) = \begin{cases} 3x^2 - 4, & x \leq 2 \\ 5 + 4x, & x > 2 \end{cases}$ Over the interval $[0, 4]$, there is no value of x such that $h(x) = 10$, although $h(0) < 10$ and $h(4) > 10$. Explain why this does not contradict the IVT.

Exercise:**Problem:**

A particle moving along a line has at each time t a position function $s(t)$, which is continuous. Assume $s(2) = 5$ and $s(5) = 2$. Another particle moves such that its position is given by $h(t) = s(t) - t$. Explain why there must be a value c for $2 < c < 5$ such that $h(c) = 0$.

Solution:

Since both s and $y = t$ are continuous everywhere, then $h(t) = s(t) - t$ is continuous everywhere and, in particular, it is continuous over the closed interval $[2, 5]$. Also, $h(2) = 3 > 0$ and $h(5) = -3 < 0$. Therefore, by the IVT, there is a value $x = c$ such that $h(c) = 0$.

Exercise:

Problem: [T] Use the statement “The cosine of t is equal to t cubed.”

- Write a mathematical equation of the statement.
- Prove that the equation in part a. has at least one real solution.
- Use a calculator to find an interval of length 0.01 that contains a solution.

Exercise:

Problem:

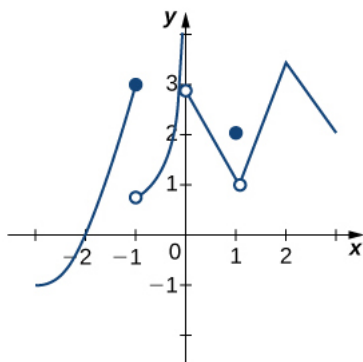
Apply the IVT to determine whether $2^x = x^3$ has a solution in one of the intervals $[1.25, 1.375]$ or $[1.375, 1.5]$. Briefly explain your response for each interval.

Solution:

The function $f(x) = 2^x - x^3$ is continuous over the interval $[1.25, 1.375]$ and has opposite signs at the endpoints.

Exercise:

Problem: Consider the graph of the function $y = f(x)$ shown in the following graph.



- Find all values for which the function is discontinuous.
- For each value in part a., state why the formal definition of continuity does not apply.
- Classify each discontinuity as either jump, removable, or infinite.

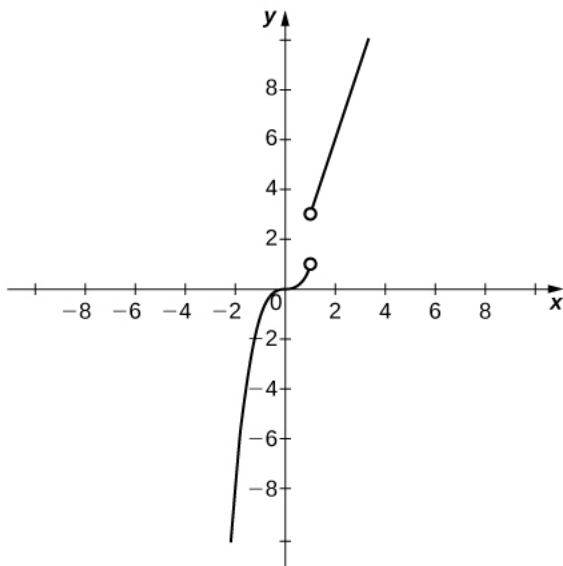
Exercise:

Problem: Let $f(x) = \begin{cases} 3x, & x > 1 \\ x^3, & x < 1 \end{cases}$.

- Sketch the graph of f .
 - Is it possible to find a value k such that $f(1) = k$, which makes $f(x)$ continuous for all real numbers? Briefly explain.
-

Solution:

-



b. It is not possible to redefine $f(1)$ since the discontinuity is a jump discontinuity.

Exercise:

Problem: Let $f(x) = \frac{x^4-1}{x^2-1}$ for $x \neq -1, 1$.

- Sketch the graph of f .
- Is it possible to find values k_1 and k_2 such that $f(-1) = k_1$ and $f(1) = k_2$, and that makes $f(x)$ continuous for all real numbers? Briefly explain.

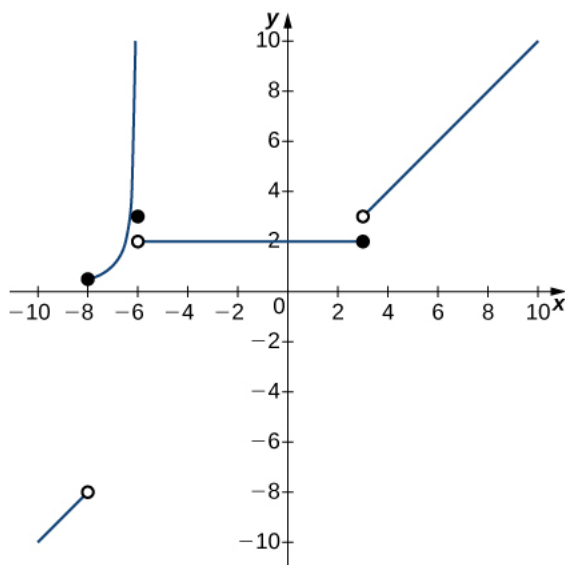
Exercise:

Problem: Sketch the graph of the function $y = f(x)$ with properties i. through vii.

- The domain of f is $(-\infty, +\infty)$.
- f has an infinite discontinuity at $x = -6$.
- $f(-6) = 3$
- $\lim_{x \rightarrow -3^-} f(x) = \lim_{x \rightarrow -3^+} f(x) = 2$
- $f(-3) = 3$
- f is left continuous but not right continuous at $x = 3$.
- $\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow +\infty} f(x) = +\infty$

Solution:

Answers may vary; see the following example:



Exercise:

Problem: Sketch the graph of the function $y = f(x)$ with properties i. through iv.

- i. The domain of f is $[0, 5]$.
- ii. $\lim_{x \rightarrow 1^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$ exist and are equal.
- iii. $f(x)$ is left continuous but not continuous at $x = 2$, and right continuous but not continuous at $x = 3$.
- iv. $f(x)$ has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and the following limits hold: $\lim_{x \rightarrow 3^-} f(x) = -\infty$ and $\lim_{x \rightarrow 3^+} f(x) = 2$.

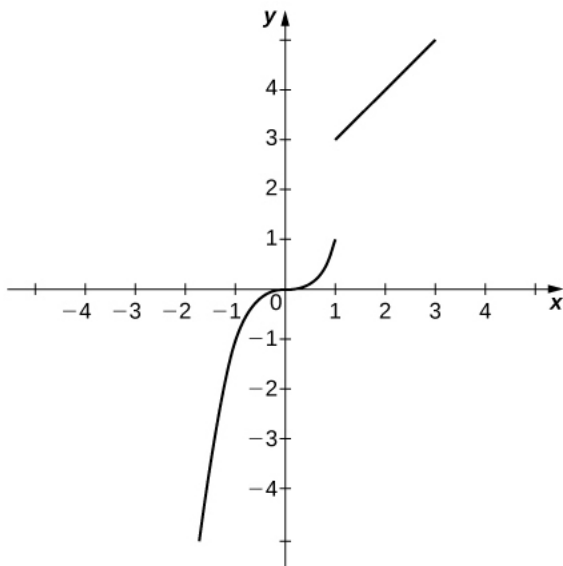
In the following exercises, suppose $y = f(x)$ is defined for all x . For each description, sketch a graph with the indicated property.

Exercise:

Problem: Discontinuous at $x = 1$ with $\lim_{x \rightarrow -1} f(x) = -1$ and $\lim_{x \rightarrow 2} f(x) = 4$

Solution:

Answers may vary; see the following example:



Exercise:

Problem: Discontinuous at $x = 2$ but continuous elsewhere with $\lim_{x \rightarrow 0} f(x) = \frac{1}{2}$

Determine whether each of the given statements is true. Justify your response with an explanation or counterexample.

Exercise:

Problem: $f(t) = \frac{2}{e^t - e^{-t}}$ is continuous everywhere.

Solution:

False. It is continuous over $(-\infty, 0) \cup (0, \infty)$.

Exercise:

Problem:

If the left- and right-hand limits of $f(x)$ as $x \rightarrow a$ exist and are equal, then f cannot be discontinuous at $x = a$.

Exercise:

Problem: If a function is not continuous at a point, then it is not defined at that point.

Solution:

False. Consider $f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 4 & \text{if } x = 0 \end{cases}$.

Exercise:

Problem: According to the IVT, $\cos x - \sin x - x = 2$ has a solution over the interval $[-1, 1]$.

Exercise:

Problem:

If $f(x)$ is continuous such that $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has exactly one solution in $[a, b]$.

Solution:

False. Consider $f(x) = \cos(x)$ on $[-\pi, 2\pi]$.

Exercise:

Problem: The function $f(x) = \frac{x^2 - 4x + 3}{x^2 - 1}$ is continuous over the interval $[0, 3]$.

Exercise:**Problem:**

If $f(x)$ is continuous everywhere and $f(a), f(b) > 0$, then there is no root of $f(x)$ in the interval $[a, b]$.

Solution:

False. The IVT does *not* work in reverse! Consider $(x - 1)^2$ over the interval $[-2, 2]$.

[T] The following problems consider the scalar form of Coulomb's law, which describes the electrostatic force between two point charges, such as electrons. It is given by the equation $F(r) = k_e \frac{|q_1 q_2|}{r^2}$, where k_e is Coulomb's constant, q_i are the magnitudes of the charges of the two particles, and r is the distance between the two particles.

Exercise:**Problem:**

To simplify the calculation of a model with many interacting particles, after some threshold value $r = R$, we approximate F as zero.

- Explain the physical reasoning behind this assumption.
- What is the force equation?
- Evaluate the force F using both Coulomb's law and our approximation, assuming two protons with a charge magnitude of 1.6022×10^{-19} coulombs (C), and the Coulomb constant $k_e = 8.988 \times 10^9 \text{ Nm}^2/\text{C}^2$ are 1 m apart. Also, assume $R < 1$ m. How much inaccuracy does our approximation generate? Is our approximation reasonable?
- Is there any finite value of R for which this system remains continuous at R ?

Exercise:**Problem:**

Instead of making the force 0 at R , instead we let the force be 10^{-20} for $r \geq R$. Assume two protons, which have a magnitude of charge 1.6022×10^{-19} C, and the Coulomb constant $k_e = 8.988 \times 10^9 \text{ Nm}^2/\text{C}^2$. Is there a value R that can make this system continuous? If so, find it.

Solution:

$R = 0.0001519$ m

Recall the discussion on spacecraft from the chapter opener. The following problems consider a rocket launch from Earth's surface. The force of gravity on the rocket is given by $F(d) = -mk/d^2$, where m is the mass of the rocket, d is the distance of the rocket from the center of Earth, and k is a constant.

Exercise:

Problem:

[T] Determine the value and units of k given that the mass of the rocket on Earth is 3 million kg. (*Hint:* The distance from the center of Earth to its surface is 6378 km.)

Exercise:**Problem:**

[T] After a certain distance D has passed, the gravitational effect of Earth becomes quite negligible, so we can approximate the force function by $F(d) = \begin{cases} -\frac{mk}{d^2} & \text{if } d < D \\ 10,000 & \text{if } d \geq D \end{cases}$. Find the necessary condition D such that the force function remains continuous.

Solution:

$$D = 63.78 \text{ km}$$

Exercise:**Problem:**

As the rocket travels away from Earth's surface, there is a distance D where the rocket sheds some of its mass, since it no longer needs the excess fuel storage. We can write this function as

$$F(d) = \begin{cases} -\frac{m_1 k}{d^2} & \text{if } d < D \\ -\frac{m_2 k}{d^2} & \text{if } d \geq D \end{cases}. \text{ Is there a } D \text{ value such that this function is continuous, assuming } m_1 \neq m_2?$$

Prove the following functions are continuous everywhere

Exercise:

$$\textbf{Problem: } f(\theta) = \sin \theta$$

Solution:

For all values of a , $f(a)$ is defined, $\lim_{\theta \rightarrow a} f(\theta)$ exists, and $\lim_{\theta \rightarrow a} f(\theta) = f(a)$. Therefore, $f(\theta)$ is continuous everywhere.

Exercise:

$$\textbf{Problem: } g(x) = |x|$$

Exercise:

$$\textbf{Problem: } \text{Where is } f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases} \text{ continuous?}$$

Solution:

Nowhere

Glossary

continuity at a point

A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim_{x \rightarrow a} f(x)$ exists, and (3) $\lim_{x \rightarrow a} f(x) = f(a)$

continuity from the left

A function is continuous from the left at b if $\lim_{x \rightarrow b^-} f(x) = f(b)$

continuity from the right

A function is continuous from the right at a if $\lim_{x \rightarrow a^+} f(x) = f(a)$

continuity over an interval

a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) , and it is continuous from the right at a and from the left at b

discontinuity at a point

A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

infinite discontinuity

An infinite discontinuity occurs at a point a if $\lim_{x \rightarrow a^-} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^+} f(x) = \pm\infty$

Intermediate Value Theorem

Let f be continuous over a closed bounded interval $[a, b]$; if z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$

jump discontinuity

A jump discontinuity occurs at a point a if $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ both exist, but $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$

removable discontinuity

A removable discontinuity occurs at a point a if $f(x)$ is discontinuous at a , but $\lim_{x \rightarrow a} f(x)$ exists

Limits at Infinity and Asymptotes

- Calculate the limit of a function as x increases or decreases without bound.
- Recognize a horizontal asymptote on the graph of a function.
- Estimate the end behavior of a function as x increases or decreases without bound.
- Recognize an oblique asymptote on the graph of a function.
- Analyze a function and its derivatives to draw its graph.

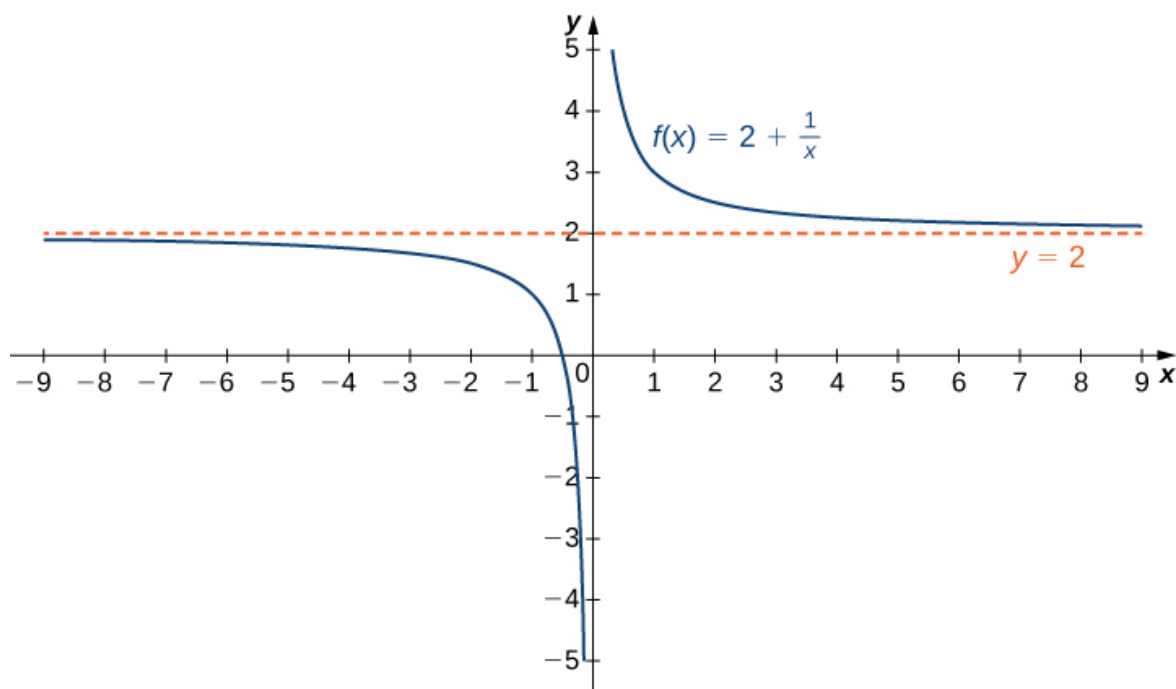
We have shown how to use the first and second derivatives of a function to describe the shape of a graph. To graph a function f defined on an unbounded domain, we also need to know the behavior of f as $x \rightarrow \pm\infty$. In this section, we define limits at infinity and show how these limits affect the graph of a function. At the end of this section, we outline a strategy for graphing an arbitrary function f .

Limits at Infinity

We begin by examining what it means for a function to have a finite limit at infinity. Then we study the idea of a function with an infinite limit at infinity. Back in [Introduction to Functions and Graphs](#), we looked at vertical asymptotes; in this section we deal with horizontal and oblique asymptotes.

Limits at Infinity and Horizontal Asymptotes

Recall that $\lim_{x \rightarrow a} f(x) = L$ means $f(x)$ becomes arbitrarily close to L as long as x is sufficiently close to a . We can extend this idea to limits at infinity. For example, consider the function $f(x) = 2 + \frac{1}{x}$. As can be seen graphically in [\[link\]](#) and numerically in [\[link\]](#), as the values of x get larger, the values of $f(x)$ approach 2. We say the limit as x approaches ∞ of $f(x)$ is 2 and write $\lim_{x \rightarrow \infty} f(x) = 2$. Similarly, for $x < 0$, as the values $|x|$ get larger, the values of $f(x)$ approach 2. We say the limit as x approaches $-\infty$ of $f(x)$ is 2 and write $\lim_{x \rightarrow -\infty} f(x) = 2$.



The function approaches the asymptote $y = 2$ as x approaches $\pm\infty$.

x	10	100	1,000	10,000
$2 + \frac{1}{x}$	2.1	2.01	2.001	2.0001
x	-10	-100	-1000	-10,000
$2 + \frac{1}{x}$	1.9	1.99	1.999	1.9999

Values of a function f as $x \rightarrow \pm\infty$

More generally, for any function f , we say the limit as $x \rightarrow \infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as x is sufficiently large. In that case, we write $\lim_{x \rightarrow \infty} f(x) = L$.

Similarly, we say the limit as $x \rightarrow -\infty$ of $f(x)$ is L if $f(x)$ becomes arbitrarily close to L as long as $x < 0$ and $|x|$ is sufficiently large. In that case, we write $\lim_{x \rightarrow -\infty} f(x) = L$. We now

look at the definition of a function having a limit at infinity.

Note:**Definition**

(Informal) If the values of $f(x)$ become arbitrarily close to L as x becomes sufficiently large, we say the function f has a **limit at infinity** and write

Equation:

$$\lim_{x \rightarrow \infty} f(x) = L.$$

If the values of $f(x)$ become arbitrarily close to L for $x < 0$ as $|x|$ becomes sufficiently large, we say that the function f has a limit at negative infinity and write

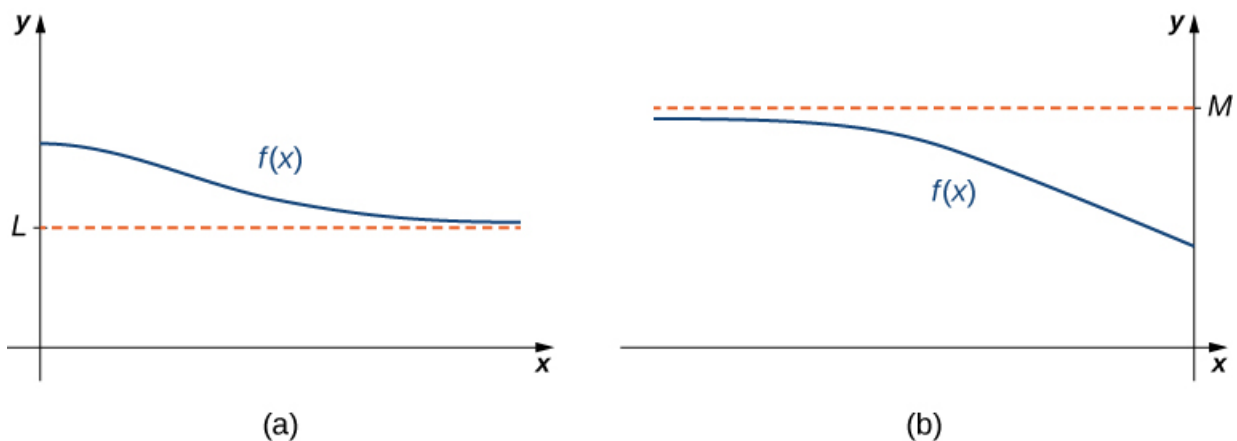
Equation:

$$\lim_{x \rightarrow -\infty} f(x) = L.$$

If the values $f(x)$ are getting arbitrarily close to some finite value L as $x \rightarrow \infty$ or $x \rightarrow -\infty$, the graph of f approaches the line $y = L$. In that case, the line $y = L$ is a horizontal asymptote of f ([link](#)). For example, for the function $f(x) = \frac{1}{x}$, since $\lim_{x \rightarrow \infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of $f(x) = \frac{1}{x}$.

Note:**Definition**

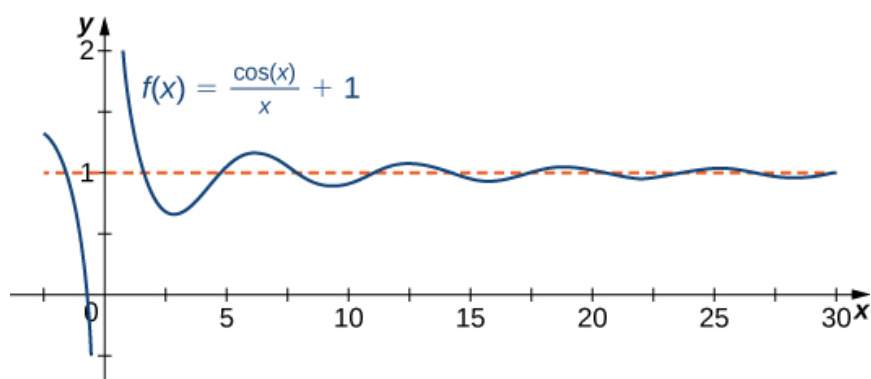
If $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote** of f .



(a) As $x \rightarrow \infty$, the values of f are getting arbitrarily close to L . The line $y = L$ is a

horizontal asymptote of f . (b) As $x \rightarrow -\infty$, the values of f are getting arbitrarily close to M . The line $y = M$ is a horizontal asymptote of f .

A function cannot cross a vertical asymptote because the graph must approach infinity (or $-\infty$) from at least one direction as x approaches the vertical asymptote. However, a function may cross a horizontal asymptote. In fact, a function may cross a horizontal asymptote an unlimited number of times. For example, the function $f(x) = \frac{\cos(x)}{x} + 1$ shown in [\[link\]](#) intersects the horizontal asymptote $y = 1$ an infinite number of times as it oscillates around the asymptote with ever-decreasing amplitude.



The graph of $f(x) = (\cos x)/x + 1$ crosses its horizontal asymptote $y = 1$ an infinite number of times.

The algebraic limit laws and squeeze theorem we introduced in [Introduction to Limits](#) also apply to limits at infinity. We illustrate how to use these laws to compute several limits at infinity.

Example:

Exercise:

Problem:

Computing Limits at Infinity

For each of the following functions f , evaluate $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$. Determine the horizontal asymptote(s) for f .

a. $f(x) = 5 - \frac{2}{x^2}$

b. $f(x) = \frac{\sin x}{x}$

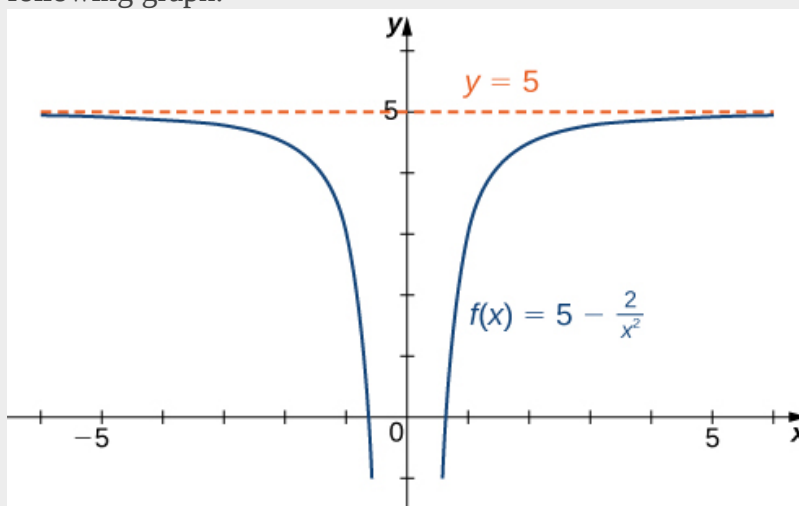
c. $f(x) = \tan^{-1}(x)$

Solution:

a. Using the algebraic limit laws, we have

$$\lim_{x \rightarrow \infty} \left(5 - \frac{2}{x^2} \right) = \lim_{x \rightarrow \infty} 5 - 2 \left(\lim_{x \rightarrow \infty} \frac{1}{x} \right) \cdot \left(\lim_{x \rightarrow \infty} \frac{1}{x} \right) = 5 - 2 \cdot 0 = 5.$$

Similarly, $\lim_{x \rightarrow -\infty} f(x) = 5$. Therefore, $f(x) = 5 - \frac{2}{x^2}$ has a horizontal asymptote of $y = 5$ and f approaches this horizontal asymptote as $x \rightarrow \pm\infty$ as shown in the following graph.



This function approaches a horizontal asymptote as $x \rightarrow \pm\infty$.

b. Since $-1 \leq \sin x \leq 1$ for all x , we have

Equation:

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

for all $x \neq 0$. Also, since

Equation:

$$\lim_{x \rightarrow \infty} \frac{-1}{x} = 0 = \lim_{x \rightarrow \infty} \frac{1}{x},$$

we can apply the squeeze theorem to conclude that

Equation:

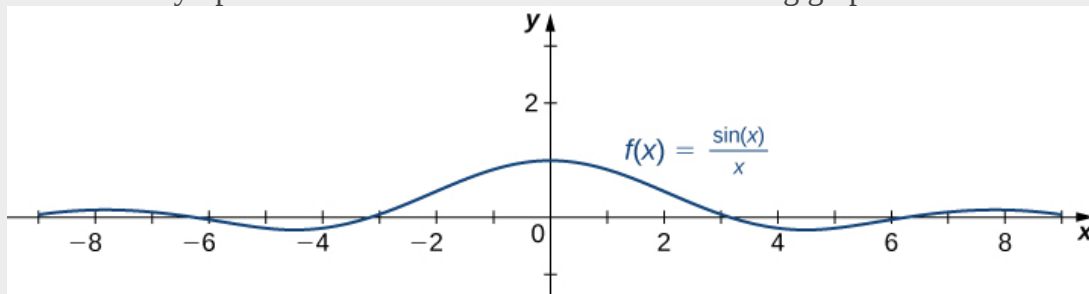
$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0.$$

Similarly,

Equation:

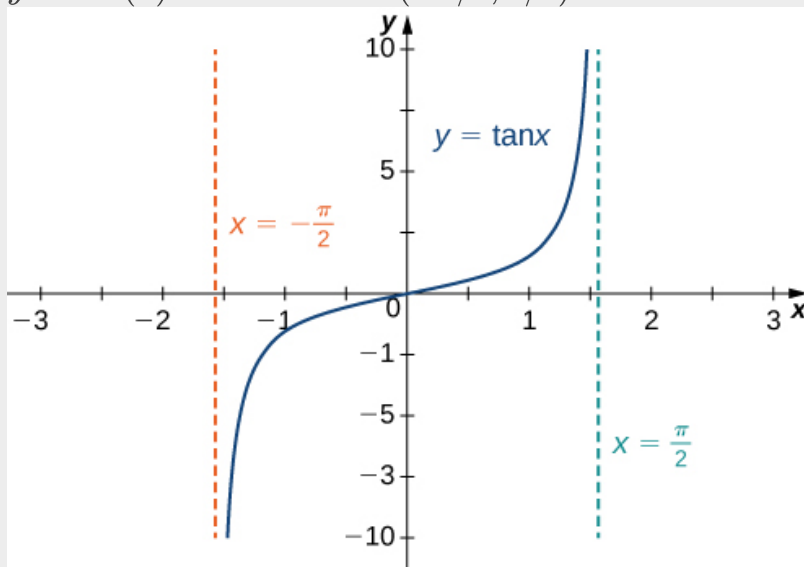
$$\lim_{x \rightarrow -\infty} \frac{\sin x}{x} = 0.$$

Thus, $f(x) = \frac{\sin x}{x}$ has a horizontal asymptote of $y = 0$ and $f(x)$ approaches this horizontal asymptote as $x \rightarrow \pm\infty$ as shown in the following graph.



This function crosses its horizontal asymptote multiple times.

- c. To evaluate $\lim_{x \rightarrow \infty} \tan^{-1}(x)$ and $\lim_{x \rightarrow -\infty} \tan^{-1}(x)$, we first consider the graph of $y = \tan(x)$ over the interval $(-\pi/2, \pi/2)$ as shown in the following graph.



The graph of $\tan x$ has vertical asymptotes at $x = \pm \frac{\pi}{2}$

Since

Equation:

$$\lim_{x \rightarrow (\pi/2)^-} \tan x = \infty,$$

it follows that

Equation:

$$\lim_{x \rightarrow \infty} \tan^{-1}(x) = \frac{\pi}{2}.$$

Similarly, since

Equation:

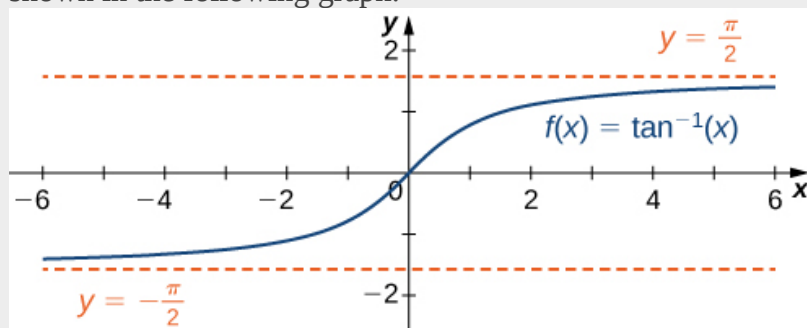
$$\lim_{x \rightarrow (\pi/2)^+} \tan x = -\infty,$$

it follows that

Equation:

$$\lim_{x \rightarrow -\infty} \tan^{-1}(x) = -\frac{\pi}{2}.$$

As a result, $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes of $f(x) = \tan^{-1}(x)$ as shown in the following graph.



This function has two horizontal asymptotes.

Note:**Exercise:****Problem:**

Evaluate $\lim_{x \rightarrow -\infty} \left(3 + \frac{4}{x}\right)$ and $\lim_{x \rightarrow \infty} \left(3 + \frac{4}{x}\right)$. Determine the horizontal asymptotes of $f(x) = 3 + \frac{4}{x}$, if any.

Solution:

Both limits are 3. The line $y = 3$ is a horizontal asymptote.

Hint

$$\lim_{x \rightarrow \pm\infty} 1/x = 0$$

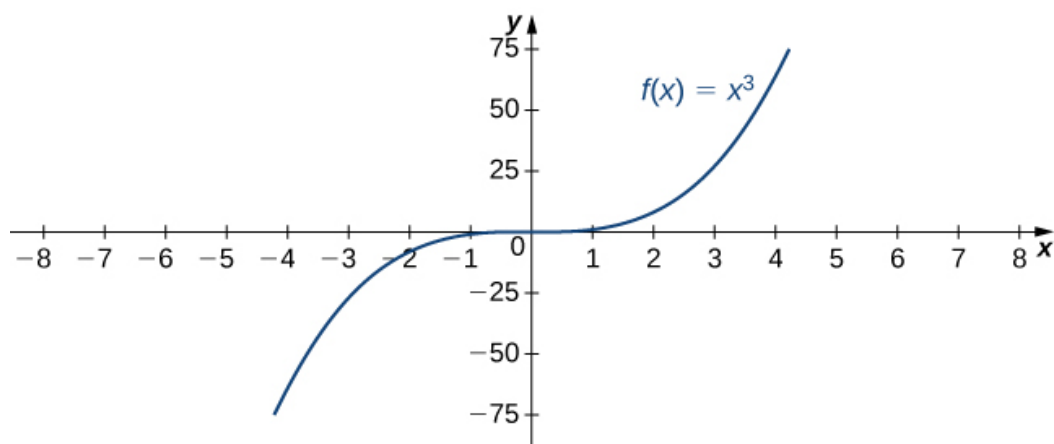
Infinite Limits at Infinity

Sometimes the values of a function f become arbitrarily large as $x \rightarrow \infty$ (or as $x \rightarrow -\infty$). In this case, we write $\lim_{x \rightarrow \infty} f(x) = \infty$ (or $\lim_{x \rightarrow -\infty} f(x) = \infty$). On the other hand, if the values of f are negative but become arbitrarily large in magnitude as $x \rightarrow \infty$ (or as $x \rightarrow -\infty$), we write $\lim_{x \rightarrow \infty} f(x) = -\infty$ (or $\lim_{x \rightarrow -\infty} f(x) = -\infty$).

For example, consider the function $f(x) = x^3$. As seen in [\[link\]](#) and [\[link\]](#), as $x \rightarrow \infty$ the values $f(x)$ become arbitrarily large. Therefore, $\lim_{x \rightarrow \infty} x^3 = \infty$. On the other hand, as $x \rightarrow -\infty$, the values of $f(x) = x^3$ are negative but become arbitrarily large in magnitude. Consequently, $\lim_{x \rightarrow -\infty} x^3 = -\infty$.

x	10	20	50	100	1000
x^3	1000	8000	125,000	1,000,000	1,000,000,000
x	-10	-20	-50	-100	-1000
x^3	-1000	-8000	-125,000	-1,000,000	-1,000,000,000

Values of a power function as $x \rightarrow \pm\infty$



For this function, the functional values approach infinity as $x \rightarrow \pm\infty$.

Note:

Definition

(Informal) We say a function f has an infinite limit at infinity and write

Equation:

$$\lim_{x \rightarrow \infty} f(x) = \infty.$$

if $f(x)$ becomes arbitrarily large for x sufficiently large. We say a function has a negative infinite limit at infinity and write

Equation:

$$\lim_{x \rightarrow \infty} f(x) = -\infty.$$

if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large for x sufficiently large. Similarly, we can define infinite limits as $x \rightarrow -\infty$.

Formal Definitions

Earlier, we used the terms *arbitrarily close*, *arbitrarily large*, and *sufficiently large* to define limits at infinity informally. Although these terms provide accurate descriptions of limits at infinity, they are not precise mathematically. Here are more formal definitions of limits at infinity. We then look at how to use these definitions to prove results involving limits at infinity.

Note:**Definition**

(Formal) We say a function f has a **limit at infinity**, if there exists a real number L such that for all $\varepsilon > 0$, there exists $N > 0$ such that

Equation:

$$|f(x) - L| < \varepsilon$$

for all $x > N$. In that case, we write

Equation:

$$\lim_{x \rightarrow \infty} f(x) = L$$

(see [\[link\]](#)).

We say a function f has a limit at negative infinity if there exists a real number L such that for all $\varepsilon > 0$, there exists $N < 0$ such that

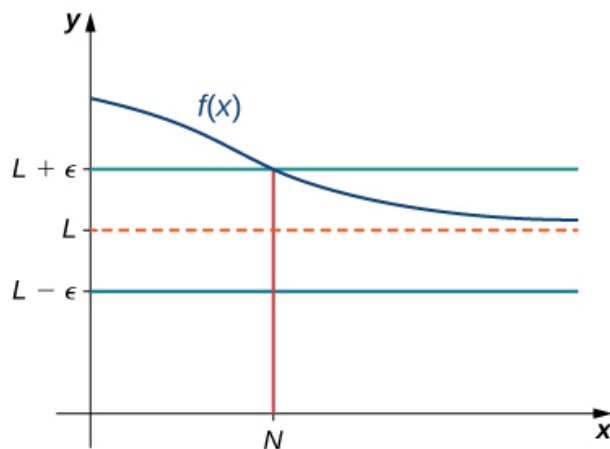
Equation:

$$|f(x) - L| < \varepsilon$$

for all $x < N$. In that case, we write

Equation:

$$\lim_{x \rightarrow -\infty} f(x) = L.$$



For a function with a limit at infinity, for all $x > N$, $|f(x) - L| < \varepsilon$.

Earlier in this section, we used graphical evidence in [\[link\]](#) and numerical evidence in [\[link\]](#) to conclude that $\lim_{x \rightarrow \infty} \left(\frac{2+1}{x} \right) = 2$. Here we use the formal definition of limit at infinity to prove this result rigorously.

Example:

A Finite Limit at Infinity Example

Exercise:

Problem: Use the formal definition of limit at infinity to prove that $\lim_{x \rightarrow \infty} \left(2 + \frac{1}{x} \right) = 2$.

Solution:

Let $\varepsilon > 0$. Let $N = \frac{1}{\varepsilon}$. Therefore, for all $x > N$, we have

Equation:

$$\left| 2 + \frac{1}{x} - 2 \right| = \left| \frac{1}{x} \right| = \frac{1}{x} < \frac{1}{N} = \varepsilon.$$

Note:

Exercise:

Problem: Use the formal definition of limit at infinity to prove that $\lim_{x \rightarrow \infty} \left(3 - \frac{1}{x^2} \right) = 3$.

Solution:

Let $\varepsilon > 0$. Let $N = \frac{1}{\sqrt{\varepsilon}}$. Therefore, for all $x > N$, we have

$$\left| 3 - \frac{1}{x^2} - 3 \right| = \frac{1}{x^2} < \frac{1}{N^2} = \varepsilon$$

Therefore, $\lim_{x \rightarrow \infty} \left(3 - 1/x^2 \right) = 3$.

Hint

Let $N = \frac{1}{\sqrt{\varepsilon}}$.

We now turn our attention to a more precise definition for an infinite limit at infinity.

Note:**Definition**

(Formal) We say a function f has an **infinite limit at infinity** and write

Equation:

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

if for all $M > 0$, there exists an $N > 0$ such that

Equation:

$$f(x) > M$$

for all $x > N$ (see [\[link\]](#)).

We say a function has a negative infinite limit at infinity and write

Equation:

$$\lim_{x \rightarrow \infty} f(x) = -\infty$$

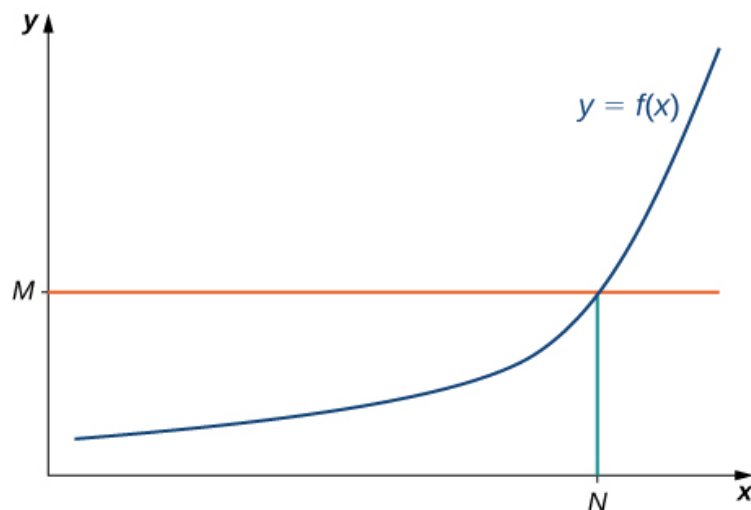
if for all $M < 0$, there exists an $N > 0$ such that

Equation:

$$f(x) < M$$

for all $x > N$.

Similarly we can define limits as $x \rightarrow -\infty$.



For a function with an infinite limit at infinity, for
all $x > N$, $f(x) > M$.

Earlier, we used graphical evidence ([link](#)) and numerical evidence ([link](#)) to conclude that $\lim_{x \rightarrow \infty} x^3 = \infty$. Here we use the formal definition of infinite limit at infinity to prove that result.

Example:

An Infinite Limit at Infinity

Exercise:

Problem:

Use the formal definition of infinite limit at infinity to prove that $\lim_{x \rightarrow \infty} x^3 = \infty$.

Solution:

Let $M > 0$. Let $N = \sqrt[3]{M}$. Then, for all $x > N$, we have

Equation:

$$x^3 > N^3 = \left(\sqrt[3]{M}\right)^3 = M.$$

Therefore, $\lim_{x \rightarrow \infty} x^3 = \infty$.

Note:

Exercise:

Problem:

Use the formal definition of infinite limit at infinity to prove that $\lim_{x \rightarrow \infty} 3x^2 = \infty$.

Solution:

Let $M > 0$. Let $N = \sqrt{\frac{M}{3}}$. Then, for all $x > N$, we have

$$3x^2 > 3N^2 = 3\left(\sqrt{\frac{M}{3}}\right)^2 = \frac{3M}{3} = M$$

Hint

Let $N = \sqrt{\frac{M}{3}}$.

End Behavior

The behavior of a function as $x \rightarrow \pm\infty$ is called the function's **end behavior**. At each of the function's ends, the function could exhibit one of the following types of behavior:

1. The function $f(x)$ approaches a horizontal asymptote $y = L$.
2. The function $f(x) \rightarrow \infty$ or $f(x) \rightarrow -\infty$.
3. The function does not approach a finite limit, nor does it approach ∞ or $-\infty$. In this case, the function may have some oscillatory behavior.

Let's consider several classes of functions here and look at the different types of end behaviors for these functions.

End Behavior for Polynomial Functions

Consider the power function $f(x) = x^n$ where n is a positive integer. From [\[link\]](#) and [\[link\]](#), we see that

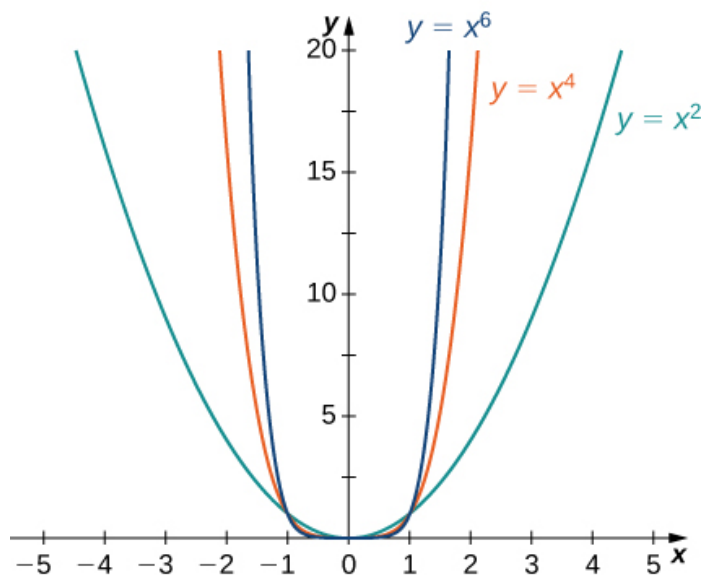
Equation:

$$\lim_{x \rightarrow \infty} x^n = \infty; n = 1, 2, 3, \dots$$

and

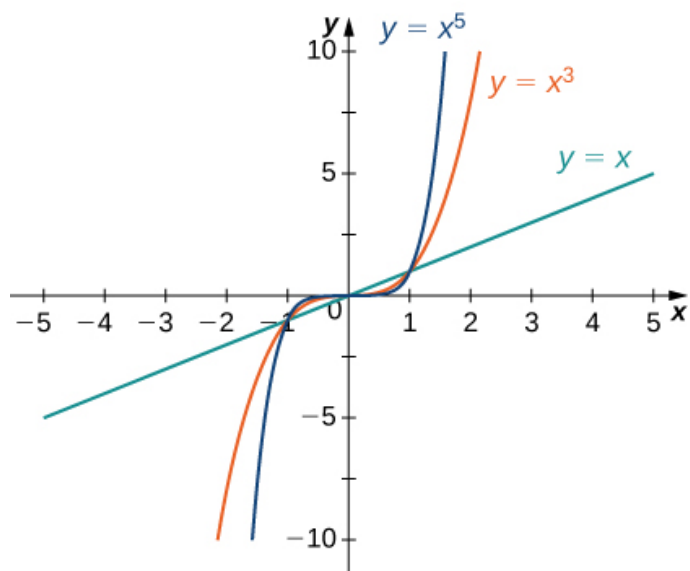
Equation:

$$\lim_{x \rightarrow -\infty} x^n = \begin{cases} \infty; n = 2, 4, 6, \dots \\ -\infty; n = 1, 3, 5, \dots \end{cases}$$



For power functions with an even power of n ,

$$\lim_{x \rightarrow \infty} x^n = \infty = \lim_{x \rightarrow -\infty} x^n.$$



For power functions with an odd power of n ,

$$\lim_{x \rightarrow \infty} x^n = \infty \text{ and } \lim_{x \rightarrow -\infty} x^n = -\infty.$$

Using these facts, it is not difficult to evaluate $\lim_{x \rightarrow \infty} cx^n$ and $\lim_{x \rightarrow -\infty} cx^n$, where c is any constant and n is a positive integer. If $c > 0$, the graph of $y = cx^n$ is a vertical stretch or compression of $y = x^n$, and therefore

Equation:

$$\lim_{x \rightarrow \infty} cx^n = \lim_{x \rightarrow \infty} x^n \text{ and } \lim_{x \rightarrow -\infty} cx^n = \lim_{x \rightarrow -\infty} x^n \text{ if } c > 0.$$

If $c < 0$, the graph of $y = cx^n$ is a vertical stretch or compression combined with a reflection about the x -axis, and therefore

Equation:

$$\lim_{x \rightarrow \infty} cx^n = -\lim_{x \rightarrow \infty} x^n \text{ and } \lim_{x \rightarrow -\infty} cx^n = -\lim_{x \rightarrow -\infty} x^n \text{ if } c < 0.$$

If $c = 0$, $y = cx^n = 0$, in which case $\lim_{x \rightarrow \infty} cx^n = 0 = \lim_{x \rightarrow -\infty} cx^n$.

Example:

Exercise:

Problem:

Limits at Infinity for Power Functions

For each function f , evaluate $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$.

- a. $f(x) = -5x^3$
- b. $f(x) = 2x^4$

Solution:

- a. Since the coefficient of x^3 is -5 , the graph of $f(x) = -5x^3$ involves a vertical stretch and reflection of the graph of $y = x^3$ about the x -axis. Therefore, $\lim_{x \rightarrow \infty} (-5x^3) = -\infty$ and $\lim_{x \rightarrow -\infty} (-5x^3) = \infty$.
- b. Since the coefficient of x^4 is 2 , the graph of $f(x) = 2x^4$ is a vertical stretch of the graph of $y = x^4$. Therefore, $\lim_{x \rightarrow \infty} 2x^4 = \infty$ and $\lim_{x \rightarrow -\infty} 2x^4 = \infty$.

Note:

Exercise:

Problem: Let $f(x) = -3x^4$. Find $\lim_{x \rightarrow \infty} f(x)$.

Solution:

$-\infty$

Hint

The coefficient -3 is negative.

We now look at how the limits at infinity for power functions can be used to determine $\lim_{x \rightarrow \pm\infty} f(x)$ for any polynomial function f . Consider a polynomial function

Equation:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

of degree $n \geq 1$ so that $a_n \neq 0$. Factoring, we see that

Equation:

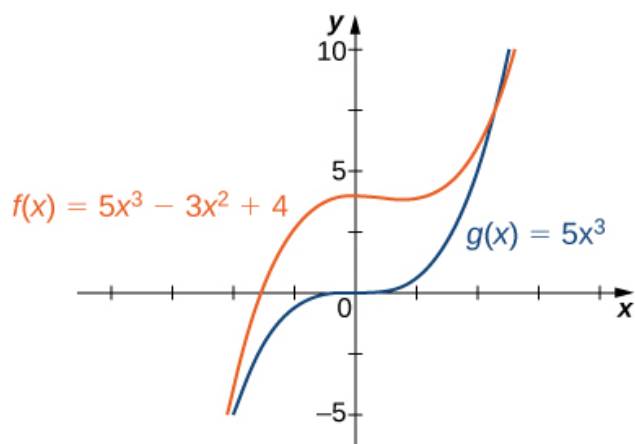
$$f(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n} \frac{1}{x} + \dots + \frac{a_1}{a_n} \frac{1}{x^{n-1}} + \frac{a_0}{a_n} \frac{1}{x^n} \right).$$

As $x \rightarrow \pm\infty$, all the terms inside the parentheses approach zero except the first term. We conclude that

Equation:

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} a_n x^n.$$

For example, the function $f(x) = 5x^3 - 3x^2 + 4$ behaves like $g(x) = 5x^3$ as $x \rightarrow \pm\infty$ as shown in [\[link\]](#) and [\[link\]](#).



The end behavior of a polynomial is determined by the behavior of the term with the largest exponent.

x	10	100	1000
$f(x) = 5x^3 - 3x^2 + 4$	4704	4,970,004	4,997,000,004
$g(x) = 5x^3$	5000	5,000,000	5,000,000,000
x	-10	-100	-1000
$f(x) = 5x^3 - 3x^2 + 4$	-5296	-5,029,996	-5,002,999,996
$g(x) = 5x^3$	-5000	-5,000,000	-5,000,000,000

A polynomial's end behavior is determined by the term with the largest exponent.

End Behavior for Algebraic Functions

The end behavior for rational functions and functions involving radicals is a little more complicated than for polynomials. In [\[link\]](#), we show that the limits at infinity of a rational function $f(x) = \frac{p(x)}{q(x)}$ depend on the relationship between the degree of the numerator and the degree of the denominator. To evaluate the limits at infinity for a rational function, we

divide the numerator and denominator by the highest power of x appearing in the denominator. This determines which term in the overall expression dominates the behavior of the function at large values of x .

Example:

Exercise:

Problem:

Determining End Behavior for Rational Functions

For each of the following functions, determine the limits as $x \rightarrow \infty$ and $x \rightarrow -\infty$. Then, use this information to describe the end behavior of the function.

- $f(x) = \frac{3x-1}{2x+5}$ (Note: The degree of the numerator and the denominator are the same.)
- $f(x) = \frac{3x^2+2x}{4x^3-5x+7}$ (Note: The degree of numerator is less than the degree of the denominator.)
- $f(x) = \frac{3x^2+4x}{x+2}$ (Note: The degree of numerator is greater than the degree of the denominator.)

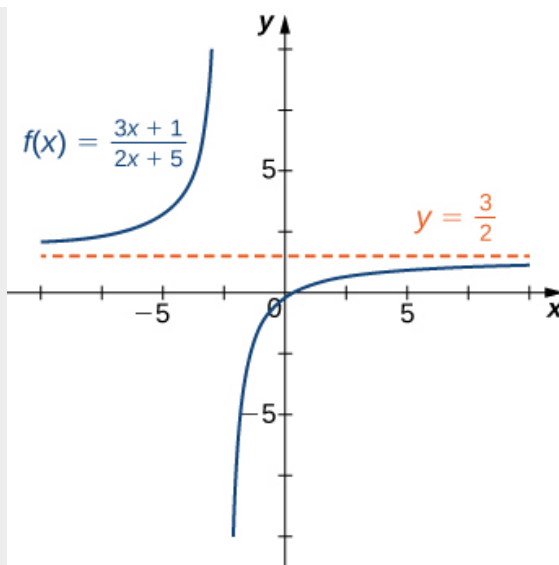
Solution:

- The highest power of x in the denominator is x . Therefore, dividing the numerator and denominator by x and applying the algebraic limit laws, we see that

Equation:

$$\begin{aligned} \lim_{x \rightarrow \pm\infty} \frac{3x-1}{2x+5} &= \lim_{x \rightarrow \pm\infty} \frac{3-1/x}{2+5/x} \\ &= \frac{\lim_{x \rightarrow \pm\infty} (3-1/x)}{\lim_{x \rightarrow \pm\infty} (2+5/x)} \\ &= \frac{\lim_{x \rightarrow \pm\infty} 3 - \lim_{x \rightarrow \pm\infty} 1/x}{\lim_{x \rightarrow \pm\infty} 2 + \lim_{x \rightarrow \pm\infty} 5/x} \\ &= \frac{3-0}{2+0} = \frac{3}{2}. \end{aligned}$$

Since $\lim_{x \rightarrow \pm\infty} f(x) = \frac{3}{2}$, we know that $y = \frac{3}{2}$ is a horizontal asymptote for this function as shown in the following graph.



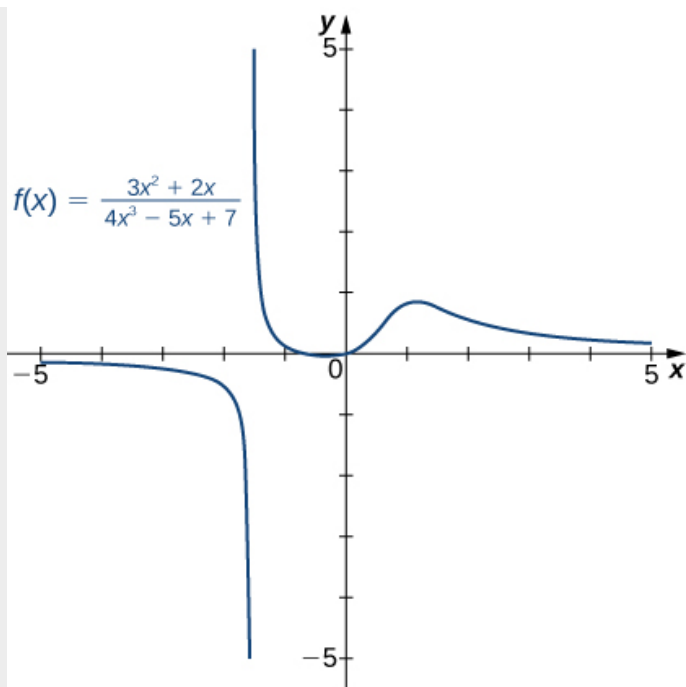
The graph of this rational function approaches a horizontal asymptote as $x \rightarrow \pm\infty$.

- b. Since the largest power of x appearing in the denominator is x^3 , divide the numerator and denominator by x^3 . After doing so and applying algebraic limit laws, we obtain

Equation:

$$\lim_{x \rightarrow \pm\infty} \frac{3x^2 + 2x}{4x^3 - 5x + 7} = \lim_{x \rightarrow \pm\infty} \frac{3/x + 2/x^2}{4 - 5/x^2 + 7/x^3} = \frac{3(0) + 2(0)}{4 - 5(0) + 7(0)} = 0.$$

Therefore f has a horizontal asymptote of $y = 0$ as shown in the following graph.



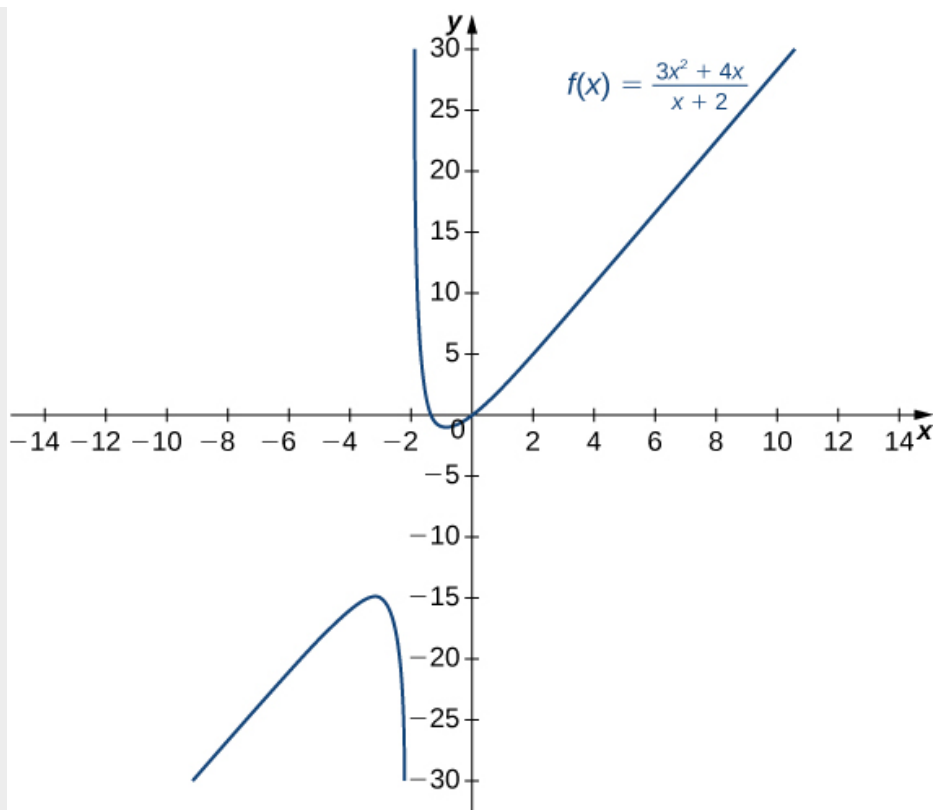
The graph of this rational function approaches the horizontal asymptote $y = 0$ as $x \rightarrow \pm\infty$.

c. Dividing the numerator and denominator by x , we have

Equation:

$$\lim_{x \rightarrow \pm\infty} \frac{3x^2 + 4x}{x + 2} = \lim_{x \rightarrow \pm\infty} \frac{3x + 4}{1 + 2/x}.$$

As $x \rightarrow \pm\infty$, the denominator approaches 1. As $x \rightarrow \infty$, the numerator approaches $+\infty$. As $x \rightarrow -\infty$, the numerator approaches $-\infty$. Therefore $\lim_{x \rightarrow \infty} f(x) = \infty$, whereas $\lim_{x \rightarrow -\infty} f(x) = -\infty$ as shown in the following figure.



As $x \rightarrow \infty$, the values $f(x) \rightarrow \infty$. As $x \rightarrow -\infty$, the values $f(x) \rightarrow -\infty$.

Note:

Exercise:

Problem:

Evaluate $\lim_{x \rightarrow \pm\infty} \frac{3x^2 + 2x - 1}{5x^2 - 4x + 7}$ and use these limits to determine the end behavior of $f(x) = \frac{3x^2 + 2x - 1}{5x^2 - 4x + 7}$.

Solution:

$$\frac{3}{5}$$

Hint

Divide the numerator and denominator by x^2 .

Before proceeding, consider the graph of $f(x) = \frac{(3x^2+4x)}{(x+2)}$ shown in [\[link\]](#). As $x \rightarrow \infty$ and $x \rightarrow -\infty$, the graph of f appears almost linear. Although f is certainly not a linear function, we now investigate why the graph of f seems to be approaching a linear function. First, using long division of polynomials, we can write

Equation:

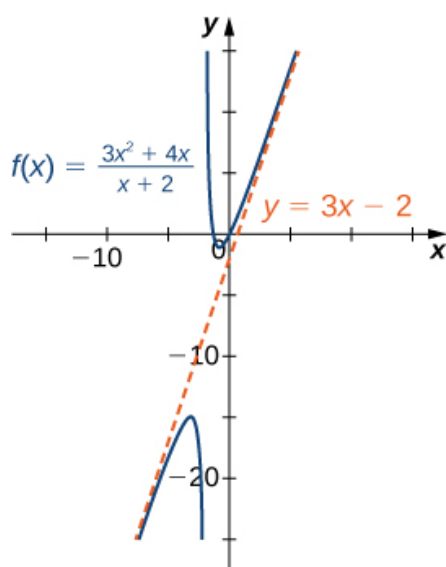
$$f(x) = \frac{3x^2 + 4x}{x + 2} = 3x - 2 + \frac{4}{x + 2}.$$

Since $\frac{4}{(x+2)} \rightarrow 0$ as $x \rightarrow \pm\infty$, we conclude that

Equation:

$$\lim_{x \rightarrow \pm\infty} (f(x) - (3x - 2)) = \lim_{x \rightarrow \pm\infty} \frac{4}{x + 2} = 0.$$

Therefore, the graph of f approaches the line $y = 3x - 2$ as $x \rightarrow \pm\infty$. This line is known as an **oblique asymptote** for f ([\[link\]](#)).



The graph of the rational
function
 $f(x) = (3x^2 + 4x) / (x + 2)$
approaches the oblique
asymptote
 $y = 3x - 2$ as $x \rightarrow \pm\infty$.

We can summarize the results of [\[link\]](#) to make the following conclusion regarding end behavior for rational functions. Consider a rational function

Equation:

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0},$$

where $a_n \neq 0$ and $b_m \neq 0$.

1. If the degree of the numerator is the same as the degree of the denominator ($n = m$), then f has a horizontal asymptote of $y = a_n/b_m$ as $x \rightarrow \pm\infty$.
2. If the degree of the numerator is less than the degree of the denominator ($n < m$), then f has a horizontal asymptote of $y = 0$ as $x \rightarrow \pm\infty$.
3. If the degree of the numerator is greater than the degree of the denominator ($n > m$), then f does not have a horizontal asymptote. The limits at infinity are either positive or negative infinity, depending on the signs of the leading terms. In addition, using long division, the function can be rewritten as

Equation:

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)},$$

where the degree of $r(x)$ is less than the degree of $q(x)$. As a result,

$\lim_{x \rightarrow \pm\infty} r(x)/q(x) = 0$. Therefore, the values of $[f(x) - g(x)]$ approach zero as $x \rightarrow \pm\infty$. If the degree of $p(x)$ is exactly one more than the degree of $q(x)$ ($n = m + 1$), the function $g(x)$ is a linear function. In this case, we call $g(x)$ an oblique asymptote.

Now let's consider the end behavior for functions involving a radical.

Example:

Exercise:

Problem:

Determining End Behavior for a Function Involving a Radical

Find the limits as $x \rightarrow \infty$ and $x \rightarrow -\infty$ for $f(x) = \frac{3x-2}{\sqrt{4x^2+5}}$ and describe the end behavior of f .

Solution:

Let's use the same strategy as we did for rational functions: divide the numerator and denominator by a power of x . To determine the appropriate power of x , consider the expression $\sqrt{4x^2 + 5}$ in the denominator. Since

Equation:

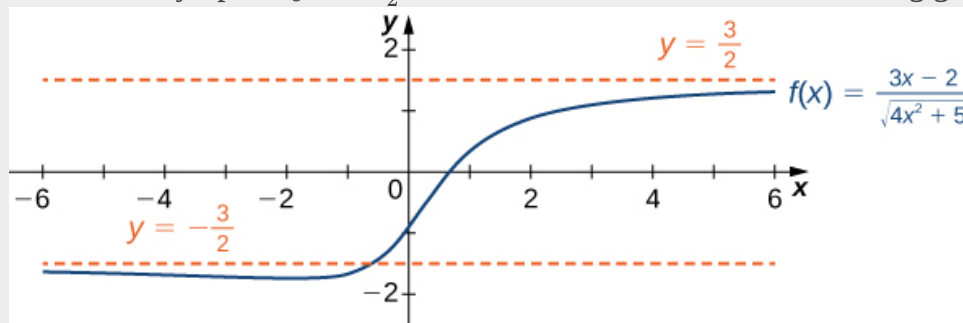
$$\sqrt{4x^2 + 5} \approx \sqrt{4x^2} = 2|x|$$

for large values of x in effect x appears just to the first power in the denominator. Therefore, we divide the numerator and denominator by $|x|$. Then, using the fact that $|x| = x$ for $x > 0$, $|x| = -x$ for $x < 0$, and $|x| = \sqrt{x^2}$ for all x , we calculate the limits as follows:

Equation:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x-2}{\sqrt{4x^2+5}} &= \lim_{x \rightarrow \infty} \frac{(1/|x|)(3x-2)}{(1/|x|)\sqrt{4x^2+5}} \\ &= \lim_{x \rightarrow \infty} \frac{(1/x)(3x-2)}{\sqrt{(1/x^2)(4x^2+5)}} \\ &= \lim_{x \rightarrow \infty} \frac{3-2/x}{\sqrt{4+5/x^2}} = \frac{3}{\sqrt{4}} = \frac{3}{2} \\ \lim_{x \rightarrow -\infty} \frac{3x-2}{\sqrt{4x^2+5}} &= \lim_{x \rightarrow -\infty} \frac{(1/|x|)(3x-2)}{(1/|x|)\sqrt{4x^2+5}} \\ &= \lim_{x \rightarrow -\infty} \frac{(-1/x)(3x-2)}{\sqrt{(1/x^2)(4x^2+5)}} \\ &= \lim_{x \rightarrow -\infty} \frac{-3+2/x}{\sqrt{4+5/x^2}} = \frac{-3}{\sqrt{4}} = \frac{-3}{2}. \end{aligned}$$

Therefore, $f(x)$ approaches the horizontal asymptote $y = \frac{3}{2}$ as $x \rightarrow \infty$ and the horizontal asymptote $y = -\frac{3}{2}$ as $x \rightarrow -\infty$ as shown in the following graph.



This function has two horizontal asymptotes and it crosses one of the asymptotes.

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow \infty} \frac{\sqrt{3x^2+4}}{x+6}$.

Solution:

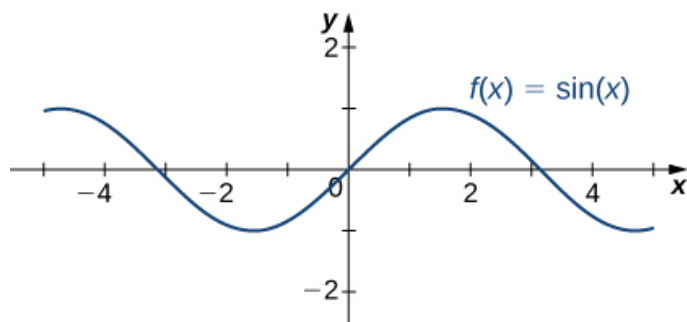
$$\pm\sqrt{3}$$

Hint

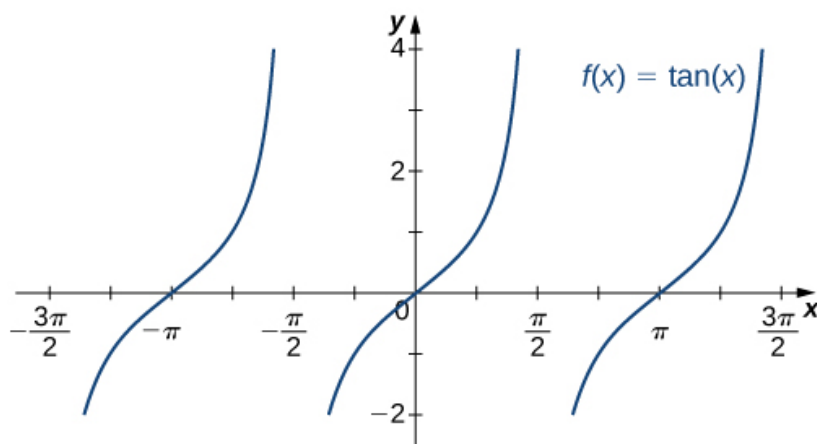
Divide the numerator and denominator by $|x|$.

Determining End Behavior for Transcendental Functions

The six basic trigonometric functions are periodic and do not approach a finite limit as $x \rightarrow \pm\infty$. For example, $\sin x$ oscillates between 1 and -1 ([\[link\]](#)). The tangent function x has an infinite number of vertical asymptotes as $x \rightarrow \pm\infty$; therefore, it does not approach a finite limit nor does it approach $\pm\infty$ as $x \rightarrow \pm\infty$ as shown in [\[link\]](#).



The function $f(x) = \sin x$ oscillates between 1 and -1 as $x \rightarrow \pm\infty$

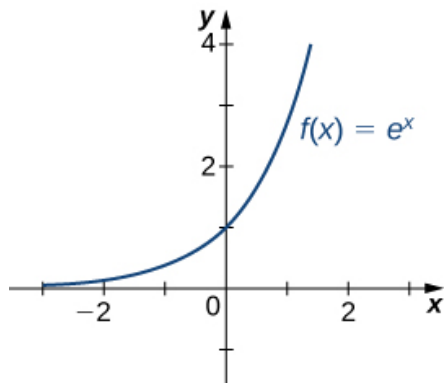


The function $f(x) = \tan x$ does not approach a limit
and does not approach $\pm\infty$ as $x \rightarrow \pm\infty$

Recall that for any base $b > 0, b \neq 1$, the function $y = b^x$ is an exponential function with domain $(-\infty, \infty)$ and range $(0, \infty)$. If $b > 1$, $y = b^x$ is increasing over \mathbb{R} . If $0 < b < 1$, $y = b^x$ is decreasing over \mathbb{R} . For the natural exponential function $f(x) = e^x$, $e \approx 2.718 > 1$. Therefore, $f(x) = e^x$ is increasing on \mathbb{R} and the range is $(0, \infty)$. The exponential function $f(x) = e^x$ approaches ∞ as $x \rightarrow \infty$ and approaches 0 as $x \rightarrow -\infty$ as shown in [\[link\]](#) and [\[link\]](#).

x	-5	-2	0	2	5
e^x	0.00674	0.135	1	7.389	148.413

End behavior of the natural exponential function

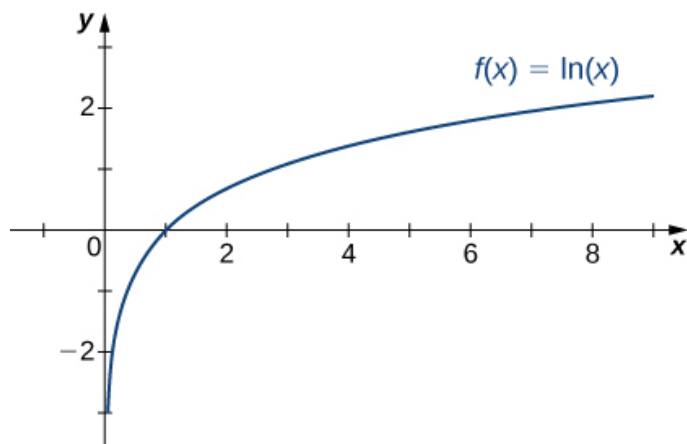


The exponential function
approaches zero as
 $x \rightarrow -\infty$ and approaches
 ∞ as $x \rightarrow \infty$.

Recall that the natural logarithm function $f(x) = \ln(x)$ is the inverse of the natural exponential function $y = e^x$. Therefore, the domain of $f(x) = \ln(x)$ is $(0, \infty)$ and the range is $(-\infty, \infty)$. The graph of $f(x) = \ln(x)$ is the reflection of the graph of $y = e^x$ about the line $y = x$. Therefore, $\ln(x) \rightarrow -\infty$ as $x \rightarrow 0^+$ and $\ln(x) \rightarrow \infty$ as $x \rightarrow \infty$ as shown in [\[link\]](#) and [\[link\]](#).

x	0.01	0.1	1	10	100
$\ln(x)$	-4.605	-2.303	0	2.303	4.605

End behavior of the natural logarithm function



The natural logarithm function approaches ∞
as $x \rightarrow \infty$.

Example:

Exercise:

Problem:

Determining End Behavior for a Transcendental Function

Find the limits as $x \rightarrow \infty$ and $x \rightarrow -\infty$ for $f(x) = \frac{(2+3e^x)}{(7-5e^x)}$ and describe the end behavior of f .

Solution:

To find the limit as $x \rightarrow \infty$, divide the numerator and denominator by e^x :

Equation:

$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{2+3e^x}{7-5e^x} \\ &= \lim_{x \rightarrow \infty} \frac{(2/e^x)+3}{(7/e^x)-5}.\end{aligned}$$

As shown in [\[link\]](#), $e^x \rightarrow \infty$ as $x \rightarrow \infty$. Therefore,

Equation:

$$\lim_{x \rightarrow \infty} \frac{2}{e^x} = 0 = \lim_{x \rightarrow \infty} \frac{7}{e^x}.$$

We conclude that $\lim_{x \rightarrow \infty} f(x) = -\frac{3}{5}$, and the graph of f approaches the horizontal asymptote $y = -\frac{3}{5}$ as $x \rightarrow \infty$. To find the limit as $x \rightarrow -\infty$, use the fact that $e^x \rightarrow 0$ as $x \rightarrow -\infty$ to conclude that $\lim_{x \rightarrow -\infty} f(x) = \frac{2}{7}$, and therefore the graph of f approaches the horizontal asymptote $y = \frac{2}{7}$ as $x \rightarrow -\infty$.

Note:

Exercise:

Problem: Find the limits as $x \rightarrow \infty$ and $x \rightarrow -\infty$ for $f(x) = \frac{(3e^x - 4)}{(5e^x + 2)}$.

Solution:

$$\lim_{x \rightarrow \infty} f(x) = \frac{3}{5}, \quad \lim_{x \rightarrow -\infty} f(x) = -2$$

Hint

$$\lim_{x \rightarrow \infty} e^x = \infty \text{ and } \lim_{x \rightarrow -\infty} e^x = 0.$$

Guidelines for Drawing the Graph of a Function

We now have enough analytical tools to draw graphs of a wide variety of algebraic and transcendental functions. Before showing how to graph specific functions, let's look at a general strategy to use when graphing any function.

Note:

Problem-Solving Strategy: Drawing the Graph of a Function

Given a function f , use the following steps to sketch a graph of f :

1. Determine the domain of the function.
2. Locate the x - and y -intercepts.
3. Evaluate $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$ to determine the end behavior. If either of these limits is a finite number L , then $y = L$ is a horizontal asymptote. If either of these limits is ∞ or $-\infty$, determine whether f has an oblique asymptote. If f is a rational function such that $f(x) = \frac{p(x)}{q(x)}$, where the degree of the numerator is greater than the degree of the denominator, then f can be written as

Equation:

$$f(x) = \frac{p(x)}{q(x)} = g(x) + \frac{r(x)}{q(x)},$$

where the degree of $r(x)$ is less than the degree of $q(x)$. The values of $f(x)$ approach the values of $g(x)$ as $x \rightarrow \pm\infty$. If $g(x)$ is a linear function, it is known as an *oblique asymptote*.

4. Determine whether f has any vertical asymptotes.
5. Calculate f' . Find all critical points and determine the intervals where f is increasing and where f is decreasing. Determine whether f has any local extrema.
6. Calculate f'' . Determine the intervals where f is concave up and where f is concave down. Use this information to determine whether f has any inflection points. The second derivative can also be used as an alternate means to determine or verify that f has a local extremum at a critical point.

Now let's use this strategy to graph several different functions. We start by graphing a polynomial function.

Example:

Exercise:

Problem:

Sketching a Graph of a Polynomial

Sketch a graph of $f(x) = (x - 1)^2(x + 2)$.

Solution:

Step 1. Since f is a polynomial, the domain is the set of all real numbers.

Step 2. When $x = 0$, $f(x) = 2$. Therefore, the y -intercept is $(0, 2)$. To find the x -intercepts, we need to solve the equation $(x - 1)^2(x + 2) = 0$, gives us the x -intercepts $(1, 0)$ and $(-2, 0)$

Step 3. We need to evaluate the end behavior of f . As $x \rightarrow \infty$, $(x - 1)^2 \rightarrow \infty$ and $(x + 2) \rightarrow \infty$. Therefore, $\lim_{x \rightarrow \infty} f(x) = \infty$. As $x \rightarrow -\infty$, $(x - 1)^2 \rightarrow \infty$ and $(x + 2) \rightarrow -\infty$. Therefore, $\lim_{x \rightarrow -\infty} f(x) = -\infty$. To get even more information about the end behavior of f , we can multiply the factors of f . When doing so, we see that

Equation:

$$f(x) = (x - 1)^2(x + 2) = x^3 - 3x + 2.$$

Since the leading term of f is x^3 , we conclude that f behaves like $y = x^3$ as $x \rightarrow \pm\infty$.

Step 4. Since f is a polynomial function, it does not have any vertical asymptotes.

Step 5. The first derivative of f is

Equation:

$$f'(x) = 3x^2 - 3.$$

Therefore, f has two critical points: $x = 1, -1$. Divide the interval $(-\infty, \infty)$ into the three smaller intervals: $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. Then, choose test points $x = -2$, $x = 0$, and $x = 2$ from these intervals and evaluate the sign of $f'(x)$ at each of these test points, as shown in the following table.

Interval	Test Point	Sign of Derivative $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$	Conclusion
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f is increasing.
$(-1, 1)$	$x = 0$	$(+)(-)(+) = -$	f is decreasing.
$(1, \infty)$	$x = 2$	$(+)(+)(+) = +$	f is increasing.

From the table, we see that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. Evaluating $f(x)$ at those two points, we find that the local maximum value is $f(-1) = 4$ and the local minimum value is $f(1) = 0$.

Step 6. The second derivative of f is

Equation:

$$f''(x) = 6x.$$

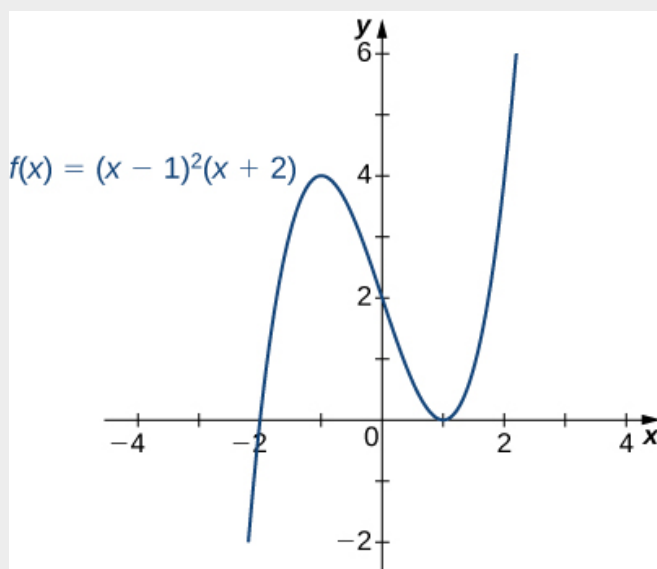
The second derivative is zero at $x = 0$. Therefore, to determine the concavity of f , divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$ and $(0, \infty)$, and

choose test points $x = -1$ and $x = 1$ to determine the concavity of f on each of these smaller intervals as shown in the following table.

Interval	Test Point	Sign of $f''(x) = 6x$	Conclusion
$(-\infty, 0)$	$x = -1$	—	f is concave down.
$(0, \infty)$	$x = 1$	+	f is concave up.

We note that the information in the preceding table confirms the fact, found in step 5, that f has a local maximum at $x = -1$ and a local minimum at $x = 1$. In addition, the information found in step 5—namely, f has a local maximum at $x = -1$ and a local minimum at $x = 1$, and $f'(x) = 0$ at those points—combined with the fact that f'' changes sign only at $x = 0$ confirms the results found in step 6 on the concavity of f .

Combining this information, we arrive at the graph of $f(x) = (x - 1)^2(x + 2)$ shown in the following graph.

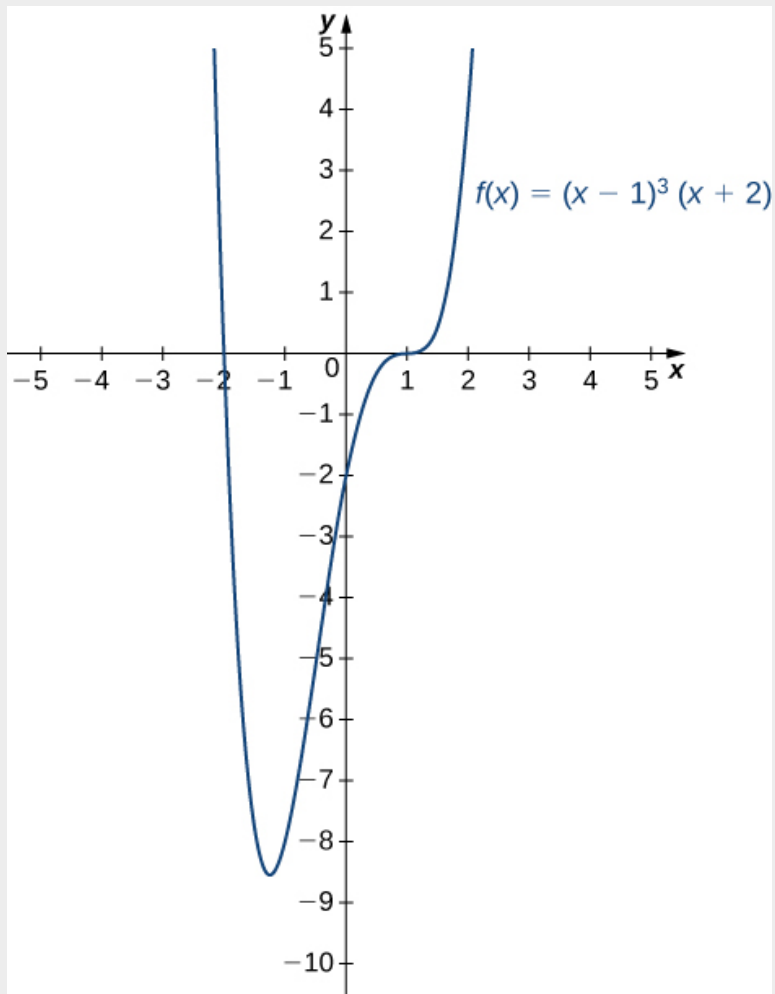


Note:

Exercise:

Problem: Sketch a graph of $f(x) = (x - 1)^3(x + 2)$.

Solution:



Hint

f is a fourth-degree polynomial.

Example:

Exercise:

Problem:
Sketching a Rational Function

Sketch the graph of $f(x) = \frac{x^2}{(1-x^2)}$.

Solution:

Step 1. The function f is defined as long as the denominator is not zero. Therefore, the domain is the set of all real numbers x except $x = \pm 1$.

Step 2. Find the intercepts. If $x = 0$, then $f(x) = 0$, so 0 is an intercept. If $y = 0$, then $\frac{x^2}{(1-x^2)} = 0$, which implies $x = 0$. Therefore, $(0, 0)$ is the only intercept.

Step 3. Evaluate the limits at infinity. Since f is a rational function, divide the numerator and denominator by the highest power in the denominator: x^2 . We obtain

Equation:

$$\lim_{x \rightarrow \pm\infty} \frac{x^2}{1-x^2} = \lim_{x \rightarrow \pm\infty} \frac{1}{\frac{1}{x^2} - 1} = -1.$$

Therefore, f has a horizontal asymptote of $y = -1$ as $x \rightarrow \infty$ and $x \rightarrow -\infty$.

Step 4. To determine whether f has any vertical asymptotes, first check to see whether the denominator has any zeroes. We find the denominator is zero when $x = \pm 1$. To determine whether the lines $x = 1$ or $x = -1$ are vertical asymptotes of f , evaluate $\lim_{x \rightarrow 1} f(x)$ and $\lim_{x \rightarrow -1} f(x)$. By looking at each one-sided limit as $x \rightarrow 1$, we see that

Equation:

$$\lim_{x \rightarrow 1^+} \frac{x^2}{1-x^2} = -\infty \text{ and } \lim_{x \rightarrow 1^-} \frac{x^2}{1-x^2} = \infty.$$

In addition, by looking at each one-sided limit as $x \rightarrow -1$, we find that

Equation:

$$\lim_{x \rightarrow -1^+} \frac{x^2}{1-x^2} = \infty \text{ and } \lim_{x \rightarrow -1^-} \frac{x^2}{1-x^2} = -\infty.$$

Step 5. Calculate the first derivative:

Equation:

$$f'(x) = \frac{(1-x^2)(2x) - x^2(-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}.$$

Critical points occur at points x where $f'(x) = 0$ or $f'(x)$ is undefined. We see that $f'(x) = 0$ when $x = 0$. The derivative f' is not undefined at any point in the domain of f . However, $x = \pm 1$ are not in the domain of f . Therefore, to determine where f is increasing and where f is decreasing, divide the interval $(-\infty, \infty)$ into four smaller intervals: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$, and choose a test point in each interval to determine the sign of $f'(x)$ in each of these intervals. The values $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are good choices for test points as shown in the following table.

Interval	Test Point	Sign of $f'(x) = \frac{2x}{(1-x^2)^2}$	Conclusion
$(-\infty, -1)$	$x = -2$	$-/+ = -$	f is decreasing.
$(-1, 0)$	$x = -1/2$	$-/+ = -$	f is decreasing.
$(0, 1)$	$x = 1/2$	$+/+ = +$	f is increasing.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing.

From this analysis, we conclude that f has a local minimum at $x = 0$ but no local maximum.

Step 6. Calculate the second derivative:

Equation:

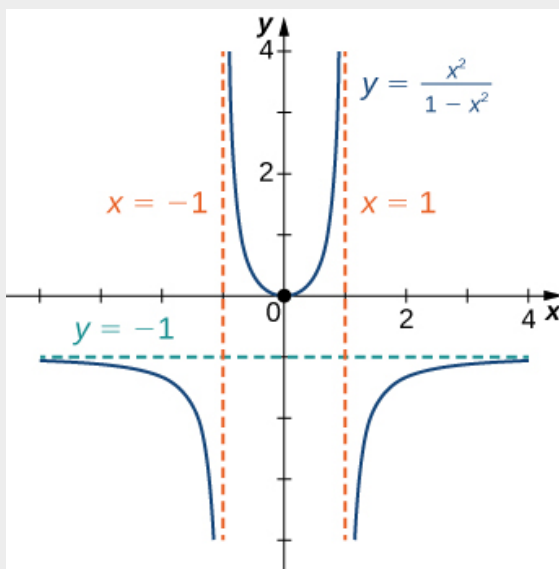
$$\begin{aligned}
 f''(x) &= \frac{(1-x^2)^2(2) - 2x(2(1-x^2)(-2x))}{(1-x^2)^4} \\
 &= \frac{(1-x^2)[2(1-x^2) + 8x^2]}{(1-x^2)^4} \\
 &= \frac{2(1-x^2) + 8x^2}{(1-x^2)^3} \\
 &= \frac{6x^2 + 2}{(1-x^2)^3}.
 \end{aligned}$$

To determine the intervals where f is concave up and where f is concave down, we first need to find all points x where $f''(x) = 0$ or $f''(x)$ is undefined. Since the numerator $6x^2 + 2 \neq 0$ for any x , $f''(x)$ is never zero. Furthermore, f'' is not undefined for any x in the domain of f . However, as discussed earlier, $x = \pm 1$ are not in the domain of f . Therefore, to determine the concavity of f , we divide the interval $(-\infty, \infty)$ into the

three smaller intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$, and choose a test point in each of these intervals to evaluate the sign of $f''(x)$. The values $x = -2$, $x = 0$, and $x = 2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{6x^2+2}{(1-x^2)^3}$	Conclusion
$(-\infty, -1)$	$x = -2$	$+/- = -$	f is concave down.
$(-1, 1)$	$x = 0$	$+/+ = +$	f is concave up.
$(1, \infty)$	$x = 2$	$+/- = -$	f is concave down.

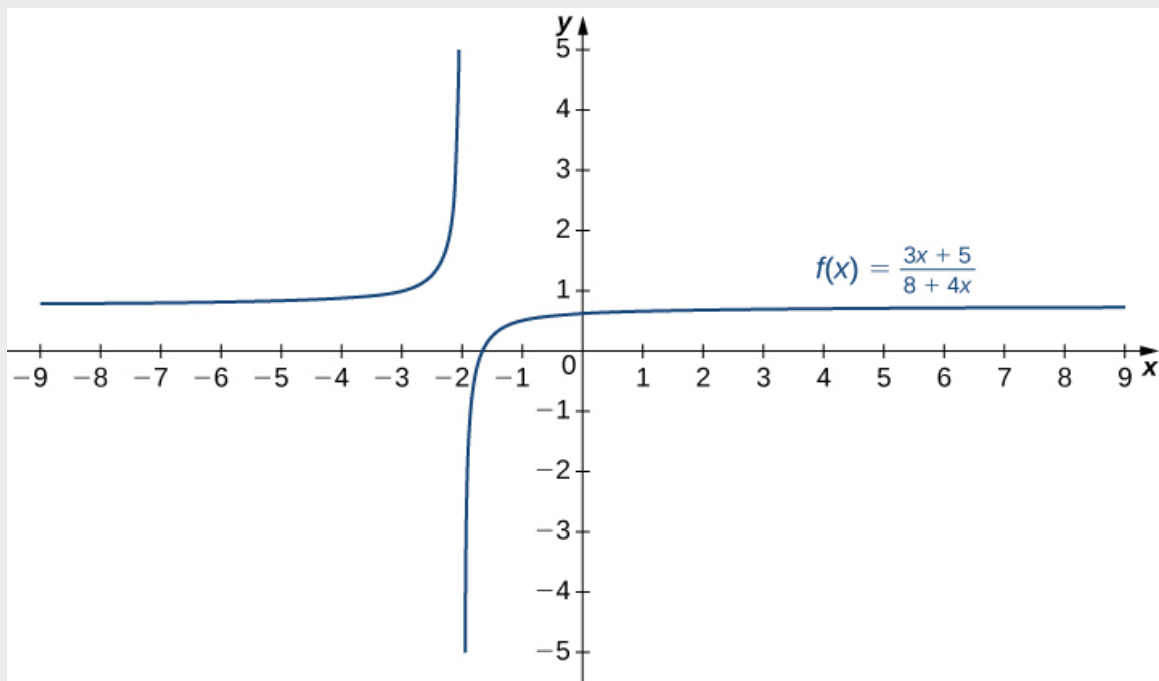
Combining all this information, we arrive at the graph of f shown below. Note that, although f changes concavity at $x = -1$ and $x = 1$, there are no inflection points at either of these places because f is not continuous at $x = -1$ or $x = 1$.



Note:
Exercise:

Problem: Sketch a graph of $f(x) = \frac{(3x+5)}{(8+4x)}$.

Solution:



Hint

A line $y = L$ is a horizontal asymptote of f if the limit as $x \rightarrow \infty$ or the limit as $x \rightarrow -\infty$ of $f(x)$ is L . A line $x = a$ is a vertical asymptote if at least one of the one-sided limits of f as $x \rightarrow a$ is ∞ or $-\infty$.

Example:

Exercise:

Problem:

Sketching a Rational Function with an Oblique Asymptote

Sketch the graph of $f(x) = \frac{x^2}{(x-1)}$

Solution:

Step 1. The domain of f is the set of all real numbers x except $x = 1$.

Step 2. Find the intercepts. We can see that when $x = 0$, $f(x) = 0$, so $(0, 0)$ is the only intercept.

Step 3. Evaluate the limits at infinity. Since the degree of the numerator is one more than the degree of the denominator, f must have an oblique asymptote. To find the oblique asymptote, use long division of polynomials to write

Equation:

$$f(x) = \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}.$$

Since $1/(x-1) \rightarrow 0$ as $x \rightarrow \pm\infty$, $f(x)$ approaches the line $y = x + 1$ as $x \rightarrow \pm\infty$. The line $y = x + 1$ is an oblique asymptote for f .

Step 4. To check for vertical asymptotes, look at where the denominator is zero. Here the denominator is zero at $x = 1$. Looking at both one-sided limits as $x \rightarrow 1$, we find

Equation:

$$\lim_{x \rightarrow 1^+} \frac{x^2}{x-1} = \infty \text{ and } \lim_{x \rightarrow 1^-} \frac{x^2}{x-1} = -\infty.$$

Therefore, $x = 1$ is a vertical asymptote, and we have determined the behavior of f as x approaches 1 from the right and the left.

Step 5. Calculate the first derivative:

Equation:

$$f'(x) = \frac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}.$$

We have $f'(x) = 0$ when $x^2 - 2x = x(x-2) = 0$. Therefore, $x = 0$ and $x = 2$ are critical points. Since f is undefined at $x = 1$, we need to divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, and $(2, \infty)$, and choose a test point from each interval to evaluate the sign of $f'(x)$ in each of these smaller intervals. For example, let $x = -1$, $x = \frac{1}{2}$, $x = \frac{3}{2}$, and $x = 3$ be the test points as shown in the following table.

Interval	Test Point	Sign of $f'(x) = \frac{x^2-2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$	Conclusion
$(-\infty, 0)$	$x = -1$	$(-)(-)/+ = +$	f is increasing.
$(0, 1)$	$x = 1/2$	$(+)(-)/+ = -$	f is decreasing.
$(1, 2)$	$x = 3/2$	$(+)(-)/+ = -$	f is decreasing.
$(2, \infty)$	$x = 3$	$(+)(+)/+ = +$	f is increasing.

From this table, we see that f has a local maximum at $x = 0$ and a local minimum at $x = 2$. The value of f at the local maximum is $f(0) = 0$ and the value of f at the local minimum is $f(2) = 4$. Therefore, $(0, 0)$ and $(2, 4)$ are important points on the graph.

Step 6. Calculate the second derivative:

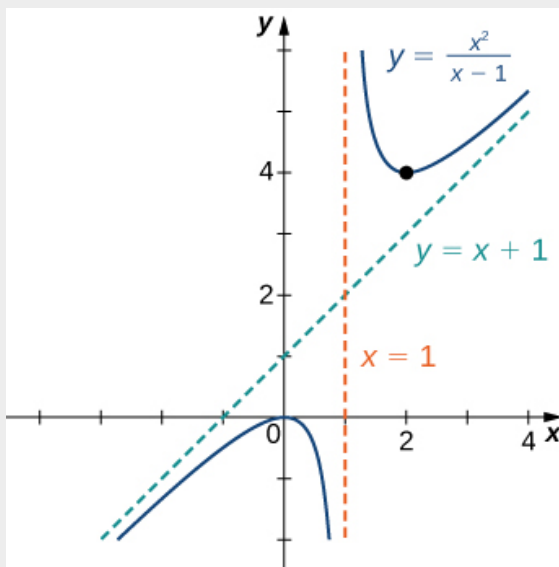
Equation:

$$\begin{aligned}
 f''(x) &= \frac{(x-1)^2(2x-2) - (x^2-2x)(2(x-1))}{(x-1)^4} \\
 &= \frac{(x-1)[(x-1)(2x-2) - 2(x^2-2x)]}{(x-1)^4} \\
 &= \frac{(x-1)(2x-2) - 2(x^2-2x)}{(x-1)^3} \\
 &= \frac{2x^2-4x+2 - (2x^2-4x)}{(x-1)^3} \\
 &= \frac{2}{(x-1)^3}.
 \end{aligned}$$

We see that $f''(x)$ is never zero or undefined for x in the domain of f . Since f is undefined at $x = 1$, to check concavity we just divide the interval $(-\infty, \infty)$ into the two smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose a test point from each interval to evaluate the sign of $f''(x)$ in each of these intervals. The values $x = 0$ and $x = 2$ are possible test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) \equiv \frac{2}{(x-1)^3}$	Conclusion
$(-\infty, 1)$	$x = 0$	$+/- = -$	f is concave down.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is concave up.

From the information gathered, we arrive at the following graph for f .



Note:

Exercise:

Problem: Find the oblique asymptote for $f(x) = \frac{(3x^3 - 2x + 1)}{(2x^2 - 4)}$.

Solution:

$$y = \frac{3}{2}x$$

Hint

Use long division of polynomials.

Example:

Exercise:

Problem:
Sketching the Graph of a Function with a Cusp

Sketch a graph of $f(x) = (x - 1)^{2/3}$.

Solution:

Step 1: Since the cube-root function is defined for all real numbers x and $(x - 1)^{2/3} = \left(\sqrt[3]{x - 1}\right)^2$, the domain of f is all real numbers.

Step 2: To find the y -intercept, evaluate $f(0)$. Since $f(0) = 1$, the y -intercept is $(0, 1)$. To find the x -intercept, solve $(x - 1)^{2/3} = 0$. The solution of this equation is $x = 1$, so the x -intercept is $(1, 0)$.

Step 3: Since $\lim_{x \rightarrow \pm\infty} (x - 1)^{2/3} = \infty$, the function continues to grow without bound as $x \rightarrow \infty$ and $x \rightarrow -\infty$.

Step 4: The function has no vertical asymptotes.

Step 5: To determine where f is increasing or decreasing, calculate f' . We find

Equation:

$$f'(x) = \frac{2}{3}(x - 1)^{-1/3} = \frac{2}{3(x - 1)^{1/3}}.$$

This function is not zero anywhere, but it is undefined when $x = 1$. Therefore, the only critical point is $x = 1$. Divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points in each of these intervals to determine the sign of $f'(x)$ in each of these smaller intervals. Let $x = 0$ and $x = 2$ be the test points as shown in the following table.

Interval	Test Point	Sign of $f'(x) = \frac{2}{3(x-1)^{1/3}}$	Conclusion
$(-\infty, 1)$	$x = 0$	$+/- = -$	f is decreasing.
$(1, \infty)$	$x = 2$	$+/+ = +$	f is increasing.

We conclude that f has a local minimum at $x = 1$. Evaluating f at $x = 1$, we find that the value of f at the local minimum is zero. Note that $f'(1)$ is undefined, so to determine the behavior of the function at this critical point, we need to examine $\lim_{x \rightarrow 1} f'(x)$. Looking at the one-sided limits, we have

Equation:

$$\lim_{x \rightarrow 1^+} \frac{2}{3(x-1)^{1/3}} = \infty \text{ and } \lim_{x \rightarrow 1^-} \frac{2}{3(x-1)^{1/3}} = -\infty.$$

Therefore, f has a cusp at $x = 1$.

Step 6: To determine concavity, we calculate the second derivative of f :

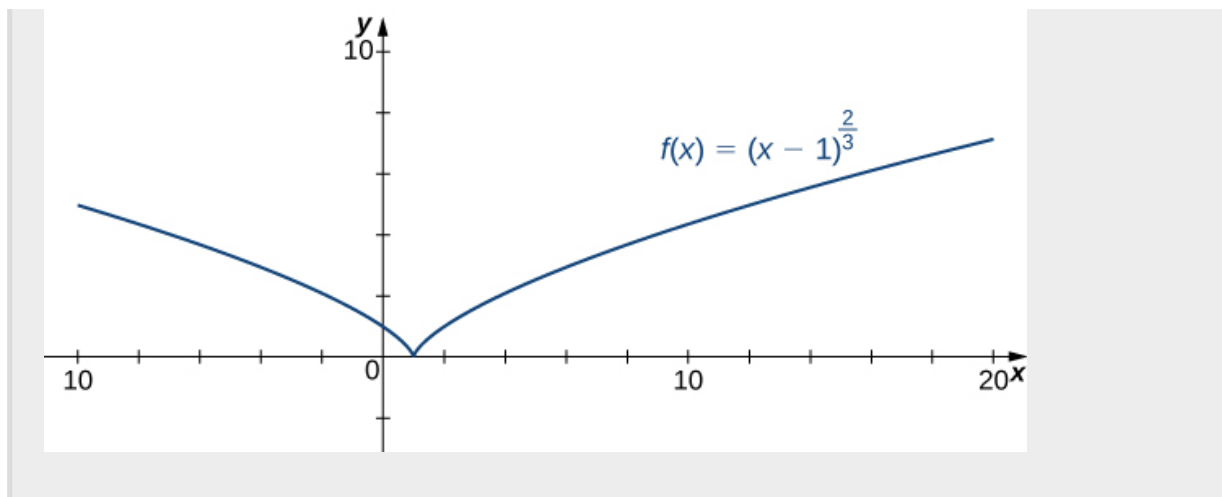
Equation:

$$f''(x) = -\frac{2}{9}(x-1)^{-4/3} = \frac{-2}{9(x-1)^{4/3}}.$$

We find that $f''(x)$ is defined for all x , but is undefined when $x = 1$. Therefore, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, 1)$ and $(1, \infty)$, and choose test points to evaluate the sign of $f''(x)$ in each of these intervals. As we did earlier, let $x = 0$ and $x = 2$ be test points as shown in the following table.

Interval	Test Point	Sign of $f''(x) = \frac{-2}{9(x-1)^{4/3}}$	Conclusion
$(-\infty, 1)$	$x = 0$	$-/+ = -$	f is concave down.
$(1, \infty)$	$x = 2$	$-/+ = -$	f is concave down.

From this table, we conclude that f is concave down everywhere. Combining all of this information, we arrive at the following graph for f .



Note:

Exercise:

Problem:

Consider the function $f(x) = 5 - x^{2/3}$. Determine the point on the graph where a cusp is located. Determine the end behavior of f .

Solution:

The function f has a cusp at $(0, 5)$ $\lim_{x \rightarrow 0^-} f'(x) = \infty$, $\lim_{x \rightarrow 0^+} f'(x) = -\infty$. For end behavior, $\lim_{x \rightarrow \pm\infty} f(x) = -\infty$.

Hint

A function f has a cusp at a point a if $f(a)$ exists, $f'(a)$ is undefined, one of the one-sided limits as $x \rightarrow a$ of $f'(x)$ is $+\infty$, and the other one-sided limit is $-\infty$.

Key Concepts

- The limit of $f(x)$ is L as $x \rightarrow \infty$ (or as $x \rightarrow -\infty$) if the values $f(x)$ become arbitrarily close to L as x becomes sufficiently large.
- The limit of $f(x)$ is ∞ as $x \rightarrow \infty$ if $f(x)$ becomes arbitrarily large as x becomes sufficiently large. The limit of $f(x)$ is $-\infty$ as $x \rightarrow \infty$ if $f(x) < 0$ and $|f(x)|$ becomes arbitrarily large as x becomes sufficiently large. We can define the limit of $f(x)$ as x approaches $-\infty$ similarly.
- For a polynomial function $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$, where $a_n \neq 0$, the end behavior is determined by the leading term $a_n x^n$. If $n \neq 0$, $p(x)$ approaches ∞

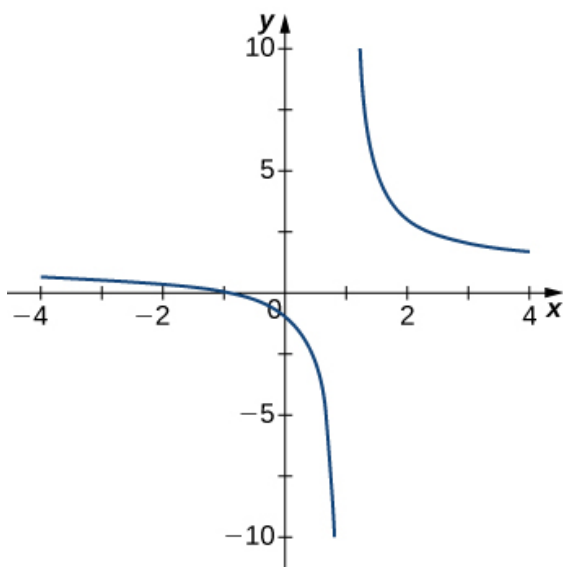
or $-\infty$ at each end.

- For a rational function $f(x) = \frac{p(x)}{q(x)}$, the end behavior is determined by the relationship between the degree of p and the degree of q . If the degree of p is less than the degree of q , the line $y = 0$ is a horizontal asymptote for f . If the degree of p is equal to the degree of q , then the line $y = \frac{a_n}{b_n}$ is a horizontal asymptote, where a_n and b_n are the leading coefficients of p and q , respectively. If the degree of p is greater than the degree of q , then f approaches ∞ or $-\infty$ at each end.

For the following exercises, examine the graphs. Identify where the vertical asymptotes are located.

Exercise:

Problem:

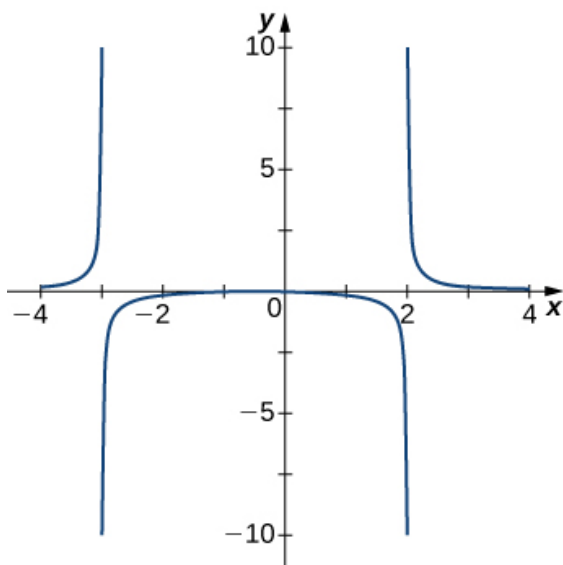


Solution:

$$x = 1$$

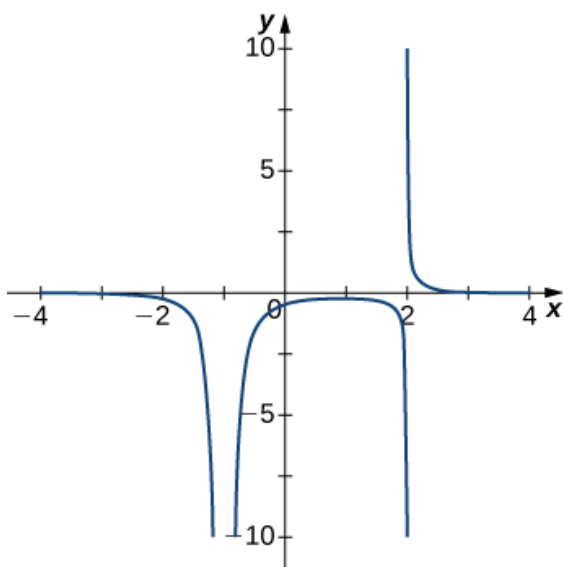
Exercise:

Problem:



Exercise:

Problem:

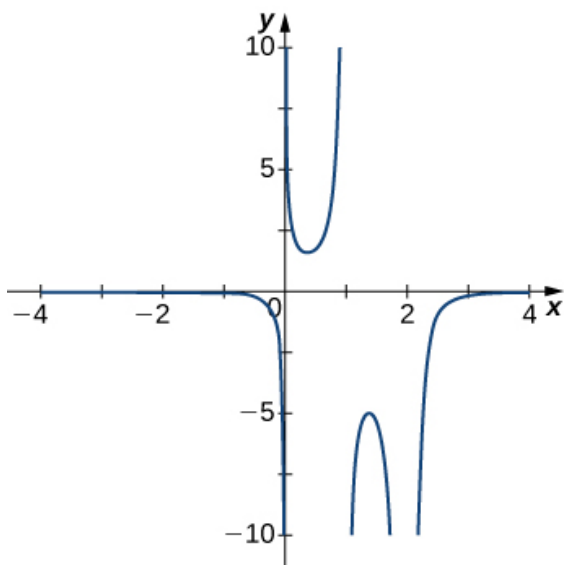


Solution:

$$x = -1, x = 2$$

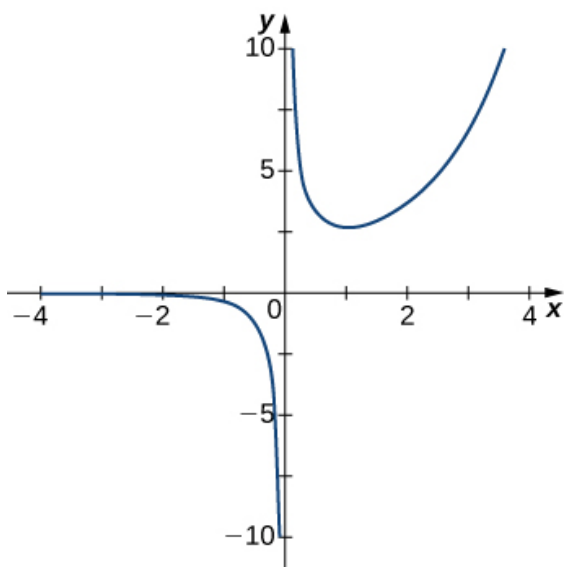
Exercise:

Problem:



Exercise:

Problem:



Solution:

$$x = 0$$

For the following functions $f(x)$, determine whether there is an asymptote at $x = a$. Justify your answer without graphing on a calculator.

Exercise:

Problem: $f(x) = \frac{x+1}{x^2+5x+4}, a = -1$

Exercise:

Problem: $f(x) = \frac{x}{x-2}, a = 2$

Solution:

Yes, there is a vertical asymptote

Exercise:

Problem: $f(x) = (x+2)^{3/2}, a = -2$

Exercise:

Problem: $f(x) = (x-1)^{-1/3}, a = 1$

Solution:

Yes, there is vertical asymptote

Exercise:

Problem: $f(x) = 1 + x^{-2/5}, a = 1$

For the following exercises, evaluate the limit.

Exercise:

Problem: $\lim_{x \rightarrow \infty} \frac{1}{3x+6}$

Solution:

0

Exercise:

Problem: $\lim_{x \rightarrow \infty} \frac{2x-5}{4x}$

Exercise:

Problem: $\lim_{x \rightarrow \infty} \frac{x^2-2x+5}{x+2}$

Solution:

$$\infty$$

Exercise:

Problem: $\lim_{x \rightarrow -\infty} \frac{3x^3 - 2x}{x^2 + 2x + 8}$

Exercise:

Problem: $\lim_{x \rightarrow -\infty} \frac{x^4 - 4x^3 + 1}{2 - 2x^2 - 7x^4}$

Solution:

$$-\frac{1}{7}$$

Exercise:

Problem: $\lim_{x \rightarrow \infty} \frac{3x}{\sqrt{x^2 + 1}}$

Exercise:

Problem: $\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 - 1}}{x + 2}$

Solution:

$$-2$$

Exercise:

Problem: $\lim_{x \rightarrow \infty} \frac{4x}{\sqrt{x^2 - 1}}$

Exercise:

Problem: $\lim_{x \rightarrow -\infty} \frac{4x}{\sqrt{x^2 - 1}}$

Solution:

$$-4$$

Exercise:

Problem: $\lim_{x \rightarrow \infty} \frac{2\sqrt{x}}{x - \sqrt{x} + 1}$

For the following exercises, find the horizontal and vertical asymptotes.

Exercise:

Problem: $f(x) = x - \frac{9}{x}$

Solution:

Horizontal: none, vertical: $x = 0$

Exercise:

Problem: $f(x) = \frac{1}{1-x^2}$

Exercise:

Problem: $f(x) = \frac{x^3}{4-x^2}$

Solution:

Horizontal: none, vertical: $x = \pm 2$

Exercise:

Problem: $f(x) = \frac{x^2+3}{x^2+1}$

Exercise:

Problem: $f(x) = \sin(x) \sin(2x)$

Solution:

Horizontal: none, vertical: none

Exercise:

Problem: $f(x) = \cos x + \cos(3x) + \cos(5x)$

Exercise:

Problem: $f(x) = \frac{x \sin(x)}{x^2-1}$

Solution:

Horizontal: $y = 0$, vertical: $x = \pm 1$

Exercise:

Problem: $f(x) = \frac{x}{\sin(x)}$

Exercise:

Problem: $f(x) = \frac{1}{x^3+x^2}$

Solution:

Horizontal: $y = 0$, vertical: $x = 0$ and $x = -1$

Exercise:

Problem: $f(x) = \frac{1}{x-1} - 2x$

Exercise:

Problem: $f(x) = \frac{x^3+1}{x^3-1}$

Solution:

Horizontal: $y = 1$, vertical: $x = 1$

Exercise:

Problem: $f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$

Exercise:

Problem: $f(x) = x - \sin x$

Solution:

Horizontal: none, vertical: none

Exercise:

Problem: $f(x) = \frac{1}{x} - \sqrt{x}$

For the following exercises, construct a function $f(x)$ that has the given asymptotes.

Exercise:

Problem: $x = 1$ and $y = 2$

Solution:

Answers will vary, for example: $y = \frac{2x}{x-1}$

Exercise:

Problem: $x = 1$ and $y = 0$

Exercise:

Problem: $y = 4, x = -1$

Solution:

Answers will vary, for example: $y = \frac{4x}{x+1}$

Exercise:

Problem: $x = 0$

For the following exercises, graph the function on a graphing calculator on the window $x = [-5, 5]$ and estimate the horizontal asymptote or limit. Then, calculate the actual horizontal asymptote or limit.

Exercise:

Problem: [T] $f(x) = \frac{1}{x+10}$

Solution:

$$y = 0$$

Exercise:

Problem: [T] $f(x) = \frac{x+1}{x^2+7x+6}$

Exercise:

Problem: [T] $\lim_{x \rightarrow -\infty} x^2 + 10x + 25$

Solution:

$$\infty$$

Exercise:

Problem: [T] $\lim_{x \rightarrow -\infty} \frac{x+2}{x^2+7x+6}$

Exercise:

Problem: [T] $\lim_{x \rightarrow \infty} \frac{3x+2}{x+5}$

Solution:

$$y = 3$$

For the following exercises, draw a graph of the functions without using a calculator. Be sure to notice all important features of the graph: local maxima and minima, inflection points, and asymptotic behavior.

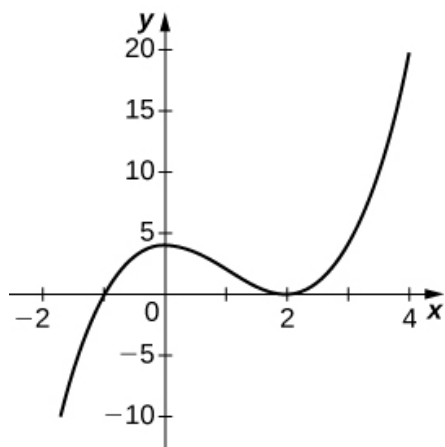
Exercise:

Problem: $y = 3x^2 + 2x + 4$

Exercise:

Problem: $y = x^3 - 3x^2 + 4$

Solution:



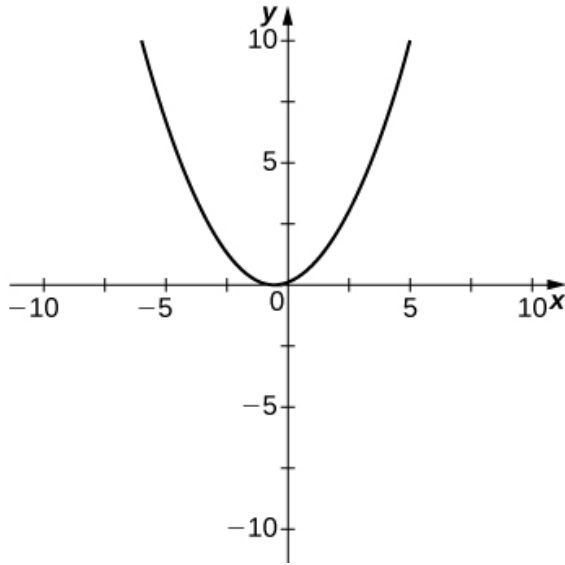
Exercise:

Problem: $y = \frac{2x+1}{x^2+6x+5}$

Exercise:

Problem: $y = \frac{x^3+4x^2+3x}{3x+9}$

Solution:



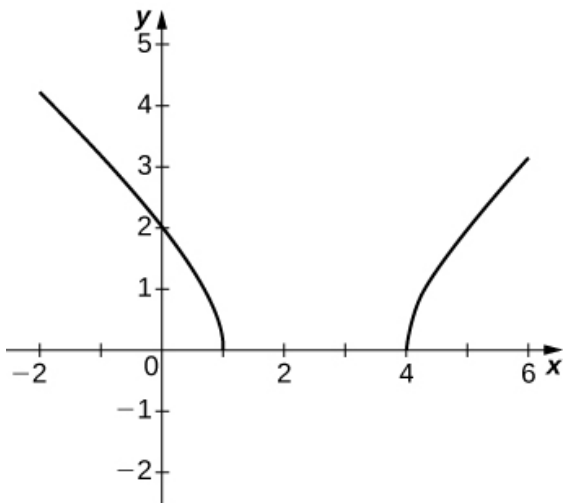
Exercise:

Problem: $y = \frac{x^2+x-2}{x^2-3x-4}$

Exercise:

Problem: $y = \sqrt{x^2 - 5x + 4}$

Solution:



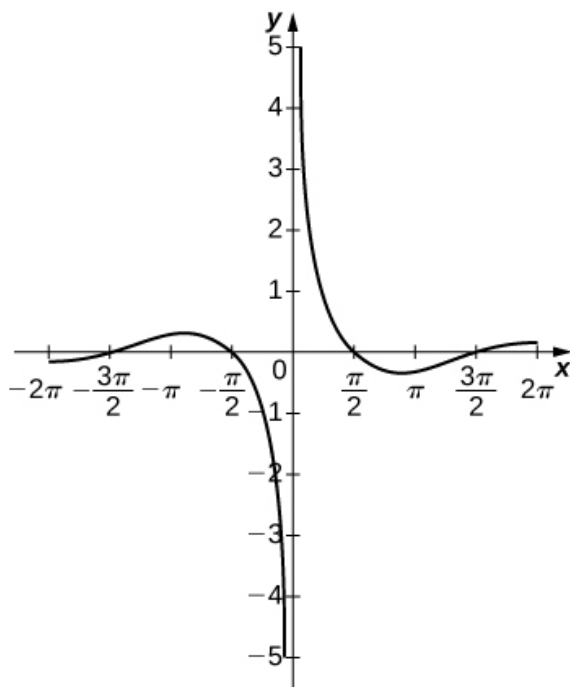
Exercise:

Problem: $y = 2x\sqrt{16 - x^2}$

Exercise:

Problem: $y = \frac{\cos x}{x}$, on $x = [-2\pi, 2\pi]$

Solution:



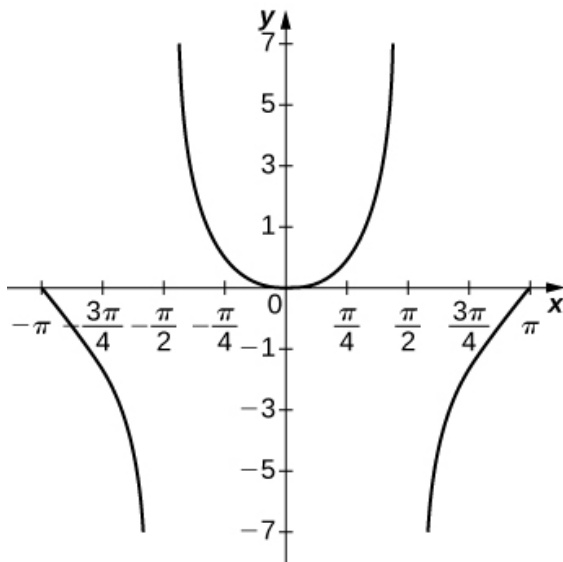
Exercise:

Problem: $y = e^x - x^3$

Exercise:

Problem: $y = x \tan x$, $x = [-\pi, \pi]$

Solution:



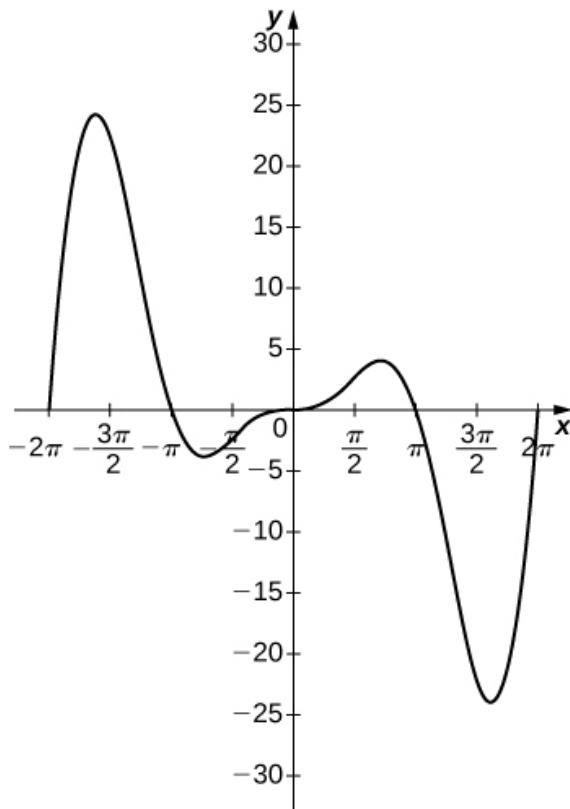
Exercise:

Problem: $y = x \ln(x), x > 0$

Exercise:

Problem: $y = x^2 \sin(x), x = [-2\pi, 2\pi]$

Solution:



Exercise:

Problem:

For $f(x) = \frac{P(x)}{Q(x)}$ to have an asymptote at $y = 2$ then the polynomials $P(x)$ and $Q(x)$ must have what relation?

Exercise:

Problem:

For $f(x) = \frac{P(x)}{Q(x)}$ to have an asymptote at $x = 0$, then the polynomials $P(x)$ and $Q(x)$ must have what relation?

Solution:

The degree of $Q(x)$ must be greater than the degree of $P(x)$.

Exercise:

Problem: If $f'(x)$ has asymptotes at $y = 3$ and $x = 1$, then $f(x)$ has what asymptotes?

Exercise:

Problem:

Both $f(x) = \frac{1}{(x-1)}$ and $g(x) = \frac{1}{(x-1)^2}$ have asymptotes at $x = 1$ and $y = 0$. What is the most obvious difference between these two functions?

Solution:

$$\lim_{x \rightarrow 1^-} f(x) = -\infty \text{ and } \lim_{x \rightarrow 1^-} g(x) = \infty$$

Exercise:

Problem: True or false: Every ratio of polynomials has vertical asymptotes.

Glossary

end behavior

the behavior of a function as $x \rightarrow \infty$ and $x \rightarrow -\infty$

horizontal asymptote

if $\lim_{x \rightarrow \infty} f(x) = L$ or $\lim_{x \rightarrow -\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of f

infinite limit at infinity

a function that becomes arbitrarily large as x becomes large

limit at infinity

the limiting value, if it exists, of a function as $x \rightarrow \infty$ or $x \rightarrow -\infty$

oblique asymptote

the line $y = mx + b$ if $f(x)$ approaches it as $x \rightarrow \infty$ or $x \rightarrow -\infty$

The Precise Definition of a Limit

- Describe the epsilon-delta definition of a limit.
- Apply the epsilon-delta definition to find the limit of a function.
- Describe the epsilon-delta definitions of one-sided limits and infinite limits.
- Use the epsilon-delta definition to prove the limit laws.

By now you have progressed from the very informal definition of a limit in the introduction of this chapter to the intuitive understanding of a limit. At this point, you should have a very strong intuitive sense of what the limit of a function means and how you can find it. In this section, we convert this intuitive idea of a limit into a formal definition using precise mathematical language. The formal definition of a limit is quite possibly one of the most challenging definitions you will encounter early in your study of calculus; however, it is well worth any effort you make to reconcile it with your intuitive notion of a limit. Understanding this definition is the key that opens the door to a better understanding of calculus.

Quantifying Closeness

Before stating the formal definition of a limit, we must introduce a few preliminary ideas. Recall that the distance between two points a and b on a number line is given by $|a - b|$.

- The statement $|f(x) - L| < \varepsilon$ may be interpreted as: *The distance between $f(x)$ and L is less than ε .*
- The statement $0 < |x - a| < \delta$ may be interpreted as: *$x \neq a$ and the distance between x and a is less than δ .*

It is also important to look at the following equivalences for absolute value:

- The statement $|f(x) - L| < \varepsilon$ is equivalent to the statement $L - \varepsilon < f(x) < L + \varepsilon$.
- The statement $0 < |x - a| < \delta$ is equivalent to the statement $a - \delta < x < a + \delta$ and $x \neq a$.

With these clarifications, we can state the formal **epsilon-delta definition of the limit**.

Note:

Definition

Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a . Let L be a real number. Then

Equation:

$$\lim_{x \rightarrow a} f(x) = L$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

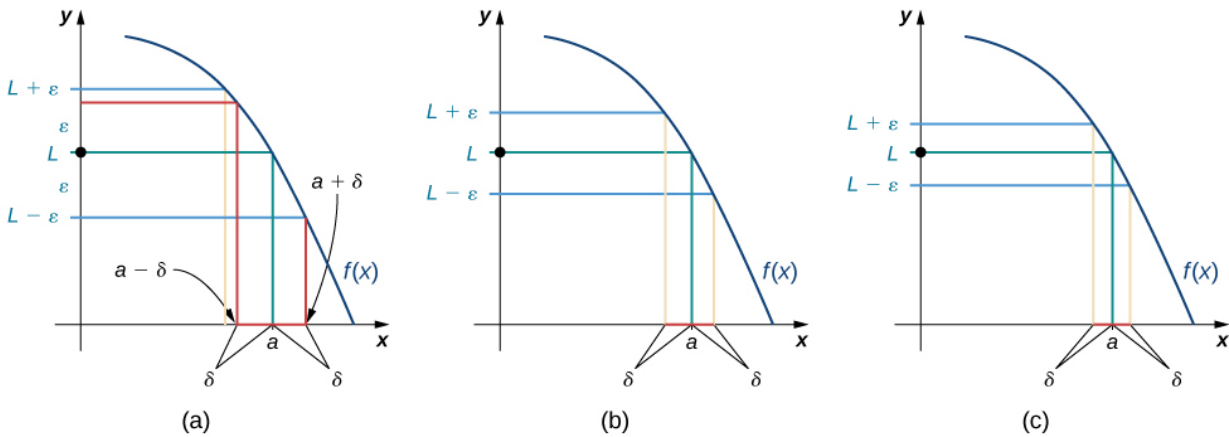
This definition may seem rather complex from a mathematical point of view, but it becomes easier to understand if we break it down phrase by phrase. The statement itself involves something called a *universal quantifier* (for every $\varepsilon > 0$), an *existential quantifier* (there exists a $\delta > 0$), and, last, a

conditional statement (if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$). Let's take a look at [\[link\]](#), which breaks down the definition and translates each part.

Definition	Translation
1. For every $\varepsilon > 0$,	1. For every positive distance ε from L ,
2. there exists a $\delta > 0$,	2. There is a positive distance δ from a ,
3. such that	3. such that
4. if $0 < x - a < \delta$, then $ f(x) - L < \varepsilon$.	4. if x is closer than δ to a and $x \neq a$, then $f(x)$ is closer than ε to L .

Translation of the Epsilon-Delta Definition of the Limit

We can get a better handle on this definition by looking at the definition geometrically. [\[link\]](#) shows possible values of δ for various choices of $\varepsilon > 0$ for a given function $f(x)$, a number a , and a limit L at a . Notice that as we choose smaller values of ε (the distance between the function and the limit), we can always find a δ small enough so that if we have chosen an x value within δ of a , then the value of $f(x)$ is within ε of the limit L .



These graphs show possible values of δ , given successively smaller choices of ε .

Note:

Visit the following applet to experiment with finding values of δ for selected values of ε :

- [The epsilon-delta definition of limit](#)

[\[link\]](#) shows how you can use this definition to prove a statement about the limit of a specific function at a specified value.

Example:

Exercise:

Problem:

Proving a Statement about the Limit of a Specific Function

Prove that $\lim_{x \rightarrow 1} (2x + 1) = 3$.

Solution:

Let $\varepsilon > 0$.

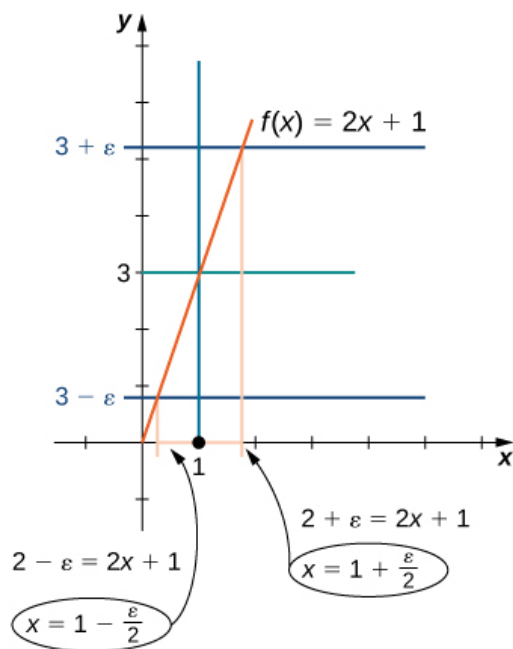
The first part of the definition begins “For every $\varepsilon > 0$.” This means we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating “Let $\varepsilon > 0$,” we signal our intent to do so.

Choose $\delta = \frac{\varepsilon}{2}$.

The definition continues with “there exists a $\delta > 0$.” The phrase “there exists” in a mathematical statement is always a signal for a scavenger hunt. In other words, we must go and find δ . So, where exactly did $\delta = \varepsilon/2$ come from? There are two basic approaches to tracking down δ . One method is purely algebraic and the other is geometric.

We begin by tackling the problem from an algebraic point of view. Since ultimately we want $|(2x + 1) - 3| < \varepsilon$, we begin by manipulating this expression: $|(2x + 1) - 3| < \varepsilon$ is equivalent to $|2x - 2| < \varepsilon$, which in turn is equivalent to $|2||x - 1| < \varepsilon$. Last, this is equivalent to $|x - 1| < \varepsilon/2$. Thus, it would seem that $\delta = \varepsilon/2$ is appropriate.

We may also find δ through geometric methods. [\[link\]](#) demonstrates how this is done.



δ is the length of the smaller of the two distances marked in brown.

$$\begin{aligned}\delta &= \min \left\{ 1 + \frac{\varepsilon}{2} - 1, 1 - \left(1 - \frac{\varepsilon}{2} \right) \right\} \\ &= \min \left\{ \frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right\} \\ &= \frac{\varepsilon}{2}\end{aligned}$$

This graph shows how we find δ geometrically.

Assume $0 < |x - 1| < \delta$. When δ has been chosen, our goal is to show that if $0 < |x - 1| < \delta$, then $|(2x + 1) - 3| < \varepsilon$. To prove any statement of the form “If this, then that,” we begin by assuming “this” and trying to get “that.”

Thus,

$$\begin{aligned}|(2x + 1) - 3| &= |2x - 2| && \text{property of absolute value} \\ &= |2(x - 1)| \\ &= |2| |x - 1| && |2| = 2 \\ &= 2|x - 1| \\ &< 2 \cdot \delta && \text{here's where we use the assumption that } 0 < |x - 1| < \delta \\ &= 2 \cdot \frac{\varepsilon}{2} = \varepsilon && \text{here's where we use our choice of } \delta = \varepsilon/2\end{aligned}$$

Analysis

In this part of the proof, we started with $|(2x + 1) - 3|$ and used our assumption $0 < |x - 1| < \delta$ in a key part of the chain of inequalities to get $|(2x + 1) - 3|$ to be less than ε . We could just as easily have manipulated the assumed inequality $0 < |x - 1| < \delta$ to arrive at $|(2x + 1) - 3| < \varepsilon$ as follows:

$$\begin{aligned}
0 < |x - 1| < \delta &\Rightarrow |x - 1| < \delta \\
&\Rightarrow -\delta < x - 1 < \delta \\
&\Rightarrow -\frac{\varepsilon}{2} < x - 1 < \frac{\varepsilon}{2} \\
&\Rightarrow -\varepsilon < 2x - 2 < \varepsilon \\
&\Rightarrow -\varepsilon < 2x - 2 < \varepsilon \\
&\Rightarrow |2x - 2| < \varepsilon \\
&\Rightarrow |(2x + 1) - 3| < \varepsilon.
\end{aligned}$$

Therefore, $\lim_{x \rightarrow 1} (2x + 1) = 3$. (Having completed the proof, we state what we have accomplished.)

After removing all the remarks, here is a final version of the proof:

Let $\varepsilon > 0$.

Choose $\delta = \varepsilon/2$.

Assume $0 < |x - 1| < \delta$.

Thus,

$$\begin{aligned}
|(2x + 1) - 3| &= |2x - 2| \\
&= |2(x - 1)| \\
&= |2| |x - 1| \\
&= 2 |x - 1| \\
&< 2 \cdot \delta \\
&= 2 \cdot \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, $\lim_{x \rightarrow 1} (2x + 1) = 3$.

The following Problem-Solving Strategy summarizes the type of proof we worked out in [\[link\]](#).

Note:

Problem-Solving Strategy: Proving That $\lim_{x \rightarrow a} f(x) = L$ for a Specific Function $f(x)$

1. Let's begin the proof with the following statement: Let $\varepsilon > 0$.
2. Next, we need to obtain a value for δ . After we have obtained this value, we make the following statement, filling in the blank with our choice of δ : Choose $\delta = \underline{\hspace{2cm}}$.
3. The next statement in the proof should be (at this point, we fill in our given value for a): Assume $0 < |x - a| < \delta$.
4. Next, based on this assumption, we need to show that $|f(x) - L| < \varepsilon$, where $f(x)$ and L are our function $f(x)$ and our limit L . At some point, we need to use $0 < |x - a| < \delta$.

5. We conclude our proof with the statement: Therefore, $\lim_{x \rightarrow a} f(x) = L$.

Example:

Exercise:

Problem:

Proving a Statement about a Limit

Complete the proof that $\lim_{x \rightarrow -1} (4x + 1) = -3$ by filling in the blanks.

Let _____.

Choose $\delta =$ _____.

Assume $0 < |x - \text{_____}| < \delta$.

Thus, $|\text{_____} - \text{_____}| = \text{_____} \varepsilon$.

Solution:

We begin by filling in the blanks where the choices are specified by the definition. Thus, we have

Let $\varepsilon > 0$.

Choose $\delta =$ _____.

Assume $0 < |x - (-1)| < \delta$. (or equivalently, $0 < |x + 1| < \delta$.)

Thus, $|(4x + 1) - (-3)| = |4x + 4| = |4||x + 1| < 4\delta \text{_____} \varepsilon$.

Focusing on the final line of the proof, we see that we should choose $\delta = \frac{\varepsilon}{4}$.

We now complete the final write-up of the proof:

Let $\varepsilon > 0$.

Choose $\delta = \frac{\varepsilon}{4}$.

Assume $0 < |x - (-1)| < \delta$ (or equivalently, $0 < |x + 1| < \delta$.)

Thus, $|(4x + 1) - (-3)| = |4x + 4| = |4||x + 1| < 4\delta = 4(\varepsilon/4) = \varepsilon$.

Note:

Exercise:

Problem: Complete the proof that $\lim_{x \rightarrow 2} (3x - 2) = 4$ by filling in the blanks.

Let _____.

Choose $\delta =$ _____.

Assume $0 < |x - \text{_____}| < \text{_____}$.

Thus,

$|\text{_____} - \text{_____}| = \text{_____} \varepsilon.$

Therefore, $\lim_{x \rightarrow 2} (3x - 2) = 4.$

Solution:

Let $\varepsilon > 0$; choose $\delta = \frac{\varepsilon}{3}$; assume $0 < |x - 2| < \delta$.

Thus, $|(3x - 2) - 4| = |3x - 6| = |3| \cdot |x - 2| < 3 \cdot \delta = 3 \cdot (\varepsilon/3) = \varepsilon.$

Therefore, $\lim_{x \rightarrow 2} 3x - 2 = 4.$

Hint

Follow the outline in the Problem-Solving Strategy that we worked out in full in [\[link\]](#).

In [\[link\]](#) and [\[link\]](#), the proofs were fairly straightforward, since the functions with which we were working were linear. In [\[link\]](#), we see how to modify the proof to accommodate a nonlinear function.

Example:

Exercise:

Problem:

Proving a Statement about the Limit of a Specific Function (Geometric Approach)

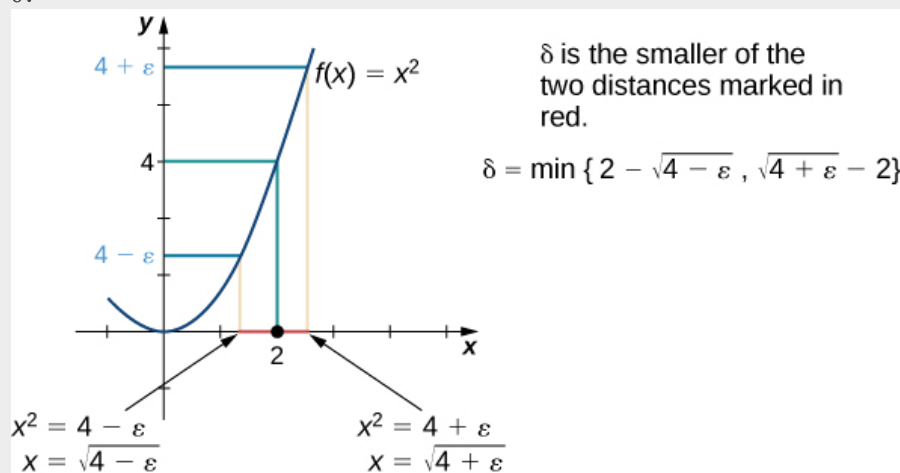
Prove that $\lim_{x \rightarrow 2} x^2 = 4.$

Solution:

1. Let $\varepsilon > 0$. The first part of the definition begins “For every $\varepsilon > 0$,” so we must prove that whatever follows is true no matter what positive value of ε is chosen. By stating “Let $\varepsilon > 0$,” we signal our intent to do so.
2. Without loss of generality, assume $\varepsilon \leq 4$. Two questions present themselves: Why do we want $\varepsilon \leq 4$ and why is it okay to make this assumption? In answer to the first question: Later on, in the process of solving for δ , we will discover that δ involves the quantity

$\sqrt{4 - \varepsilon}$. Consequently, we need $\varepsilon \leq 4$. In answer to the second question: If we can find $\delta > 0$ that “works” for $\varepsilon \leq 4$, then it will “work” for any $\varepsilon > 4$ as well. Keep in mind that, although it is always okay to put an upper bound on ε , it is never okay to put a lower bound (other than zero) on ε .

3. Choose $\delta = \min \left\{ 2 - \sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2 \right\}$. [\[link\]](#) shows how we made this choice of δ .



This graph shows how we find δ geometrically for a given ε for the proof in [\[link\]](#).

4. We must show: If $0 < |x - 2| < \delta$, then $|x^2 - 4| < \varepsilon$, so we must begin by assuming
Equation:

$$0 < |x - 2| < \delta.$$

We don't really need $0 < |x - 2|$ (in other words, $x \neq 2$) for this proof. Since $0 < |x - 2| < \delta \Rightarrow |x - 2| < \delta$, it is okay to drop $0 < |x - 2|$.

Equation:

$$|x - 2| < \delta.$$

Hence,

Equation:

$$-\delta < x - 2 < \delta.$$

Recall that $\delta = \min \left\{ 2 - \sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2 \right\}$. Thus, $\delta \geq 2 - \sqrt{4 - \varepsilon}$ and consequently $-\left(2 - \sqrt{4 - \varepsilon}\right) \leq -\delta$. We also use $\delta \leq \sqrt{4 + \varepsilon} - 2$ here. We might ask at this point: Why did we substitute $2 - \sqrt{4 - \varepsilon}$ for δ on the left-hand side of the inequality and $\sqrt{4 + \varepsilon} - 2$ on the right-hand side of the inequality? If we look at [\[link\]](#),

we see that $2 - \sqrt{4 - \varepsilon}$ corresponds to the distance on the left of 2 on the x -axis and $\sqrt{4 + \varepsilon} - 2$ corresponds to the distance on the right. Thus,

Equation:

$$-\left(2 - \sqrt{4 - \varepsilon}\right) \leq -\delta < x - 2 < \delta \leq \sqrt{4 + \varepsilon} - 2.$$

We simplify the expression on the left:

Equation:

$$-2 + \sqrt{4 - \varepsilon} < x - 2 < \sqrt{4 + \varepsilon} - 2.$$

Then, we add 2 to all parts of the inequality:

Equation:

$$\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon}.$$

We square all parts of the inequality. It is okay to do so, since all parts of the inequality are positive:

Equation:

$$4 - \varepsilon < x^2 < 4 + \varepsilon.$$

We subtract 4 from all parts of the inequality:

Equation:

$$-\varepsilon < x^2 - 4 < \varepsilon.$$

Last,

Equation:

$$|x^2 - 4| < \varepsilon.$$

5. Therefore,

Equation:

$$\lim_{x \rightarrow 2} x^2 = 4.$$

Note:

Exercise:

Problem: Find δ corresponding to $\varepsilon > 0$ for a proof that $\lim_{x \rightarrow 9} \sqrt{x} = 3$.

Solution:

Choose $\delta = \min \left\{ 9 - (3 - \varepsilon)^2, (3 + \varepsilon)^2 - 9 \right\}$.

Hint

Draw a graph similar to the one in [\[link\]](#).

The geometric approach to proving that the limit of a function takes on a specific value works quite well for some functions. Also, the insight into the formal definition of the limit that this method provides is invaluable. However, we may also approach limit proofs from a purely algebraic point of view. In many cases, an algebraic approach may not only provide us with additional insight into the definition, it may prove to be simpler as well. Furthermore, an algebraic approach is the primary tool used in proofs of statements about limits. For [\[link\]](#), we take on a purely algebraic approach.

Example:

Exercise:

Problem:

Proving a Statement about the Limit of a Specific Function (Algebraic Approach)

Prove that $\lim_{x \rightarrow -1} (x^2 - 2x + 3) = 6$.

Solution:

Let's use our outline from the Problem-Solving Strategy:

1. Let $\varepsilon > 0$.
2. Choose $\delta = \min \{1, \varepsilon/5\}$. This choice of δ may appear odd at first glance, but it was obtained by taking a look at our ultimate desired inequality: $|(x^2 - 2x + 3) - 6| < \varepsilon$. This inequality is equivalent to $|x + 1| \cdot |x - 3| < \varepsilon$. At this point, the temptation simply to choose $\delta = \frac{\varepsilon}{x-3}$ is very strong. Unfortunately, our choice of δ must depend on ε only and no other variable. If we can replace $|x - 3|$ by a numerical value, our problem can be resolved. This is the place where assuming $\delta \leq 1$ comes into play. The choice of $\delta \leq 1$ here is arbitrary. We could have just as easily used any other positive number. In some proofs, greater care in this choice may be necessary. Now, since $\delta \leq 1$ and $|x + 1| < \delta \leq 1$, we are able to show that $|x - 3| < 5$. Consequently, $|x + 1| \cdot |x - 3| < |x + 1| \cdot 5$. At this point we realize that we also need $\delta \leq \varepsilon/5$. Thus, we choose $\delta = \min \{1, \varepsilon/5\}$.
3. Assume $0 < |x + 1| < \delta$. Thus,

Equation:

$$|x + 1| < 1 \text{ and } |x + 1| < \frac{\varepsilon}{5}.$$

Since $|x + 1| < 1$, we may conclude that $-1 < x + 1 < 1$. Thus, by subtracting 4 from all parts of the inequality, we obtain $-5 < x - 3 < -1$. Consequently, $|x - 3| < 5$. This gives us

Equation:

$$|(x^2 - 2x + 3) - 6| = |x + 1| \cdot |x - 3| < \frac{\varepsilon}{5} \cdot 5 = \varepsilon.$$

Therefore,

Equation:

$$\lim_{x \rightarrow -1} (x^2 - 2x + 3) = 6.$$

Note:

Exercise:

Problem: Complete the proof that $\lim_{x \rightarrow 1} x^2 = 1$.

Let $\varepsilon > 0$; choose $\delta = \min \{1, \varepsilon/3\}$; assume $0 < |x - 1| < \delta$.

Since $|x - 1| < 1$, we may conclude that $-1 < x - 1 < 1$. Thus, $1 < x + 1 < 3$. Hence, $|x + 1| < 3$.

Solution:

$$|x^2 - 1| = |x - 1| \cdot |x + 1| < \varepsilon/3 \cdot 3 = \varepsilon$$

Hint

Use [\[link\]](#) as a guide.

You will find that, in general, the more complex a function, the more likely it is that the algebraic approach is the easiest to apply. The algebraic approach is also more useful in proving statements about limits.

Proving Limit Laws

We now demonstrate how to use the epsilon-delta definition of a limit to construct a rigorous proof of one of the limit laws. The **triangle inequality** is used at a key point of the proof, so we first review

this key property of absolute value.

Note:

Definition

The **triangle inequality** states that if a and b are any real numbers, then $|a + b| \leq |a| + |b|$.

Proof

We prove the following limit law: If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, then $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M$.

Let $\varepsilon > 0$.

Choose $\delta_1 > 0$ so that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \varepsilon/2$.

Choose $\delta_2 > 0$ so that if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \varepsilon/2$.

Choose $\delta = \min \{\delta_1, \delta_2\}$.

Assume $0 < |x - a| < \delta$.

Thus,

Equation:

$$0 < |x - a| < \delta_1 \text{ and } 0 < |x - a| < \delta_2.$$

Hence,

Equation:

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

We now explore what it means for a limit not to exist. The limit $\lim_{x \rightarrow a} f(x)$ does not exist if there is no real number L for which $\lim_{x \rightarrow a} f(x) = L$. Thus, for all real numbers L , $\lim_{x \rightarrow a} f(x) \neq L$. To understand what this means, we look at each part of the definition of $\lim_{x \rightarrow a} f(x) = L$ together with its opposite. A translation of the definition is given in [\[link\]](#).

Definition	Opposite
1. For every $\varepsilon > 0$,	1. There exists $\varepsilon > 0$ so that
2. there exists a $\delta > 0$, so that	2. for every $\delta > 0$,
3. if $0 < x - a < \delta$, then $ f(x) - L < \varepsilon$.	3. There is an x satisfying $0 < x - a < \delta$ so that $ f(x) - L \geq \varepsilon$.

Translation of the Definition of $\lim_{x \rightarrow a} f(x) = L$ and its Opposite

Finally, we may state what it means for a limit not to exist. The limit $\lim_{x \rightarrow a} f(x)$ does not exist if for every real number L , there exists a real number $\varepsilon > 0$ so that for all $\delta > 0$, there is an x satisfying $0 < |x - a| < \delta$, so that $|f(x) - L| \geq \varepsilon$. Let's apply this in [\[link\]](#) to show that a limit does not exist.

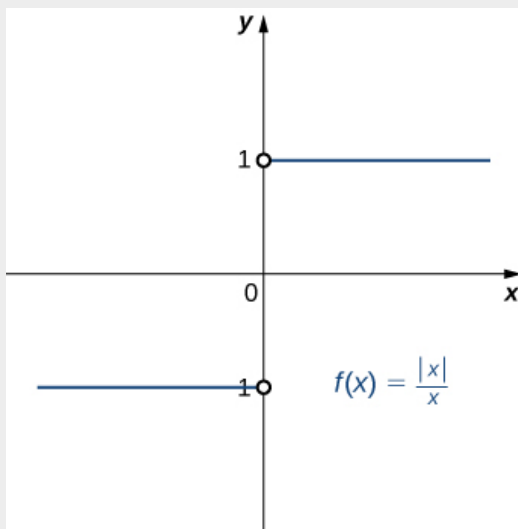
Example:

Exercise:

Problem:

Showing That a Limit Does Not Exist

Show that $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist. The graph of $f(x) = |x|/x$ is shown here:



Solution:

Suppose that L is a candidate for a limit. Choose $\varepsilon = 1/2$.

Let $\delta > 0$. Either $L \geq 0$ or $L < 0$. If $L \geq 0$, then let $x = -\delta/2$. Thus,

Equation:

$$|x - 0| = \left| -\frac{\delta}{2} - 0 \right| = \frac{\delta}{2} < \delta$$

and

Equation:

$$\left| \frac{\left| -\frac{\delta}{2} \right|}{-\frac{\delta}{2}} - L \right| = |-1 - L| = L + 1 \geq 1 > \frac{1}{2} = \varepsilon.$$

On the other hand, if $L < 0$, then let $x = \delta/2$. Thus,

Equation:

$$|x - 0| = \left| \frac{\delta}{2} - 0 \right| = \frac{\delta}{2} < \delta$$

and

Equation:

$$\left| \frac{\left| \frac{\delta}{2} \right|}{\frac{\delta}{2}} - L \right| = |1 - L| = |L| + 1 \geq 1 > \frac{1}{2} = \varepsilon.$$

Thus, for any value of L , $\lim_{x \rightarrow 0} \frac{|x|}{x} \neq L$.

One-Sided and Infinite Limits

Just as we first gained an intuitive understanding of limits and then moved on to a more rigorous definition of a limit, we now revisit one-sided limits. To do this, we modify the epsilon-delta definition of a limit to give formal epsilon-delta definitions for limits from the right and left at a point. These definitions only require slight modifications from the definition of the limit. In the definition of the limit from the right, the inequality $0 < x - a < \delta$ replaces $0 < |x - a| < \delta$, which ensures that we only consider values of x that are greater than (to the right of) a . Similarly, in the definition of the limit from the left, the inequality $-\delta < x - a < 0$ replaces $0 < |x - a| < \delta$, which ensures that we only consider values of x that are less than (to the left of) a .

Note:

Definition

Limit from the Right: Let $f(x)$ be defined over an open interval of the form (a, b) where $a < b$.

Then,

Equation:

$$\lim_{x \rightarrow a^+} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$.

Limit from the Left: Let $f(x)$ be defined over an open interval of the form (b, c) where $b < c$. Then,

Equation:

$$\lim_{x \rightarrow a^-} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $-\delta < x - a < 0$, then $|f(x) - L| < \varepsilon$.

Example:

Exercise:

Problem:

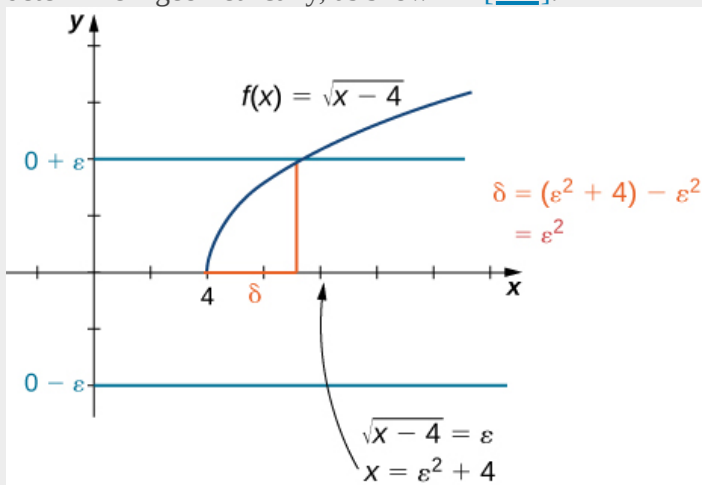
Proving a Statement about a Limit From the Right

Prove that $\lim_{x \rightarrow 4^+} \sqrt{x - 4} = 0$.

Solution:

Let $\varepsilon > 0$.

Choose $\delta = \varepsilon^2$. Since we ultimately want $|\sqrt{x - 4} - 0| < \varepsilon$, we manipulate this inequality to get $\sqrt{x - 4} < \varepsilon$ or, equivalently, $0 < x - 4 < \varepsilon^2$, making $\delta = \varepsilon^2$ a clear choice. We may also determine δ geometrically, as shown in [\[link\]](#).



This graph shows how we find δ for the proof in [\[link\]](#).

Assume $0 < x - 4 < \delta$. Thus, $0 < x - 4 < \varepsilon^2$. Hence, $0 < \sqrt{x - 4} < \varepsilon$. Finally, $|\sqrt{x - 4} - 0| < \varepsilon$.

Therefore, $\lim_{x \rightarrow 4^+} \sqrt{x - 4} = 0$.

Note:

Exercise:

Problem: Find δ corresponding to ε for a proof that $\lim_{x \rightarrow 1^-} \sqrt{1 - x} = 0$.

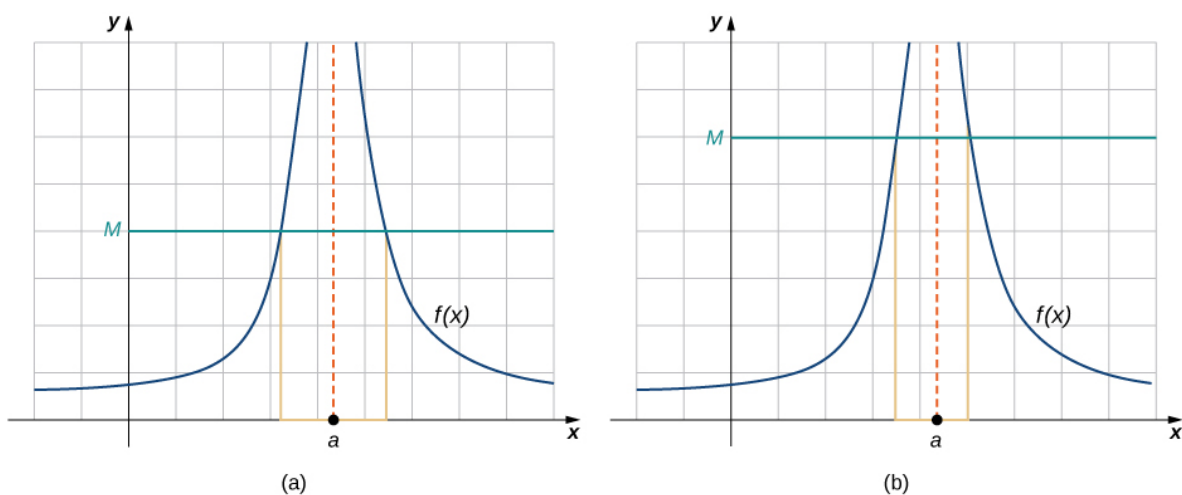
Solution:

$$\delta = \varepsilon^2$$

Hint

Sketch the graph and use [\[link\]](#) as a solving guide.

We conclude the process of converting our intuitive ideas of various types of limits to rigorous formal definitions by pursuing a formal definition of infinite limits. To have $\lim_{x \rightarrow a} f(x) = +\infty$, we want the values of the function $f(x)$ to get larger and larger as x approaches a . Instead of the requirement that $|f(x) - L| < \varepsilon$ for arbitrarily small ε when $0 < |x - a| < \delta$ for small enough δ , we want $f(x) > M$ for arbitrarily large positive M when $0 < |x - a| < \delta$ for small enough δ . [\[link\]](#) illustrates this idea by showing the value of δ for successively larger values of M .



In each graph, δ is the smaller of the lengths of the two brown intervals.

These graphs plot values of δ for M to show that $\lim_{x \rightarrow a} f(x) = +\infty$.

Note:

Definition

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a . Then, we have an infinite limit

Equation:

$$\lim_{x \rightarrow a} f(x) = +\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > M$.

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a . Then, we have a negative infinite limit

Equation:

$$\lim_{x \rightarrow a} f(x) = -\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) < -M$.

Key Concepts

- The intuitive notion of a limit may be converted into a rigorous mathematical definition known as the *epsilon-delta definition of the limit*.
- The epsilon-delta definition may be used to prove statements about limits.
- The epsilon-delta definition of a limit may be modified to define one-sided limits.

In the following exercises, write the appropriate $\varepsilon - \delta$ definition for each of the given statements.

Exercise:

Problem: $\lim_{x \rightarrow a} f(x) = N$

Exercise:

Problem: $\lim_{t \rightarrow b} g(t) = M$

Solution:

For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |t - b| < \delta$, then $|g(t) - M| < \varepsilon$

Exercise:

Problem: $\lim_{x \rightarrow c} h(x) = L$

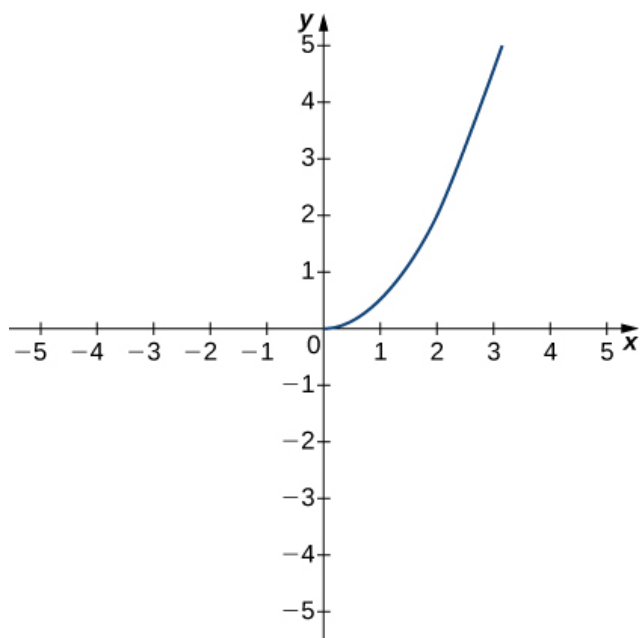
Exercise:

Problem: $\lim_{x \rightarrow a} \varphi(x) = A$

Solution:

For every $\varepsilon > 0$, there exists a $\delta > 0$, so that if $0 < |x - a| < \delta$, then $|\varphi(x) - A| < \varepsilon$

The following graph of the function f satisfies $\lim_{x \rightarrow 2} f(x) = 2$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



Exercise:

Problem: If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 1$.

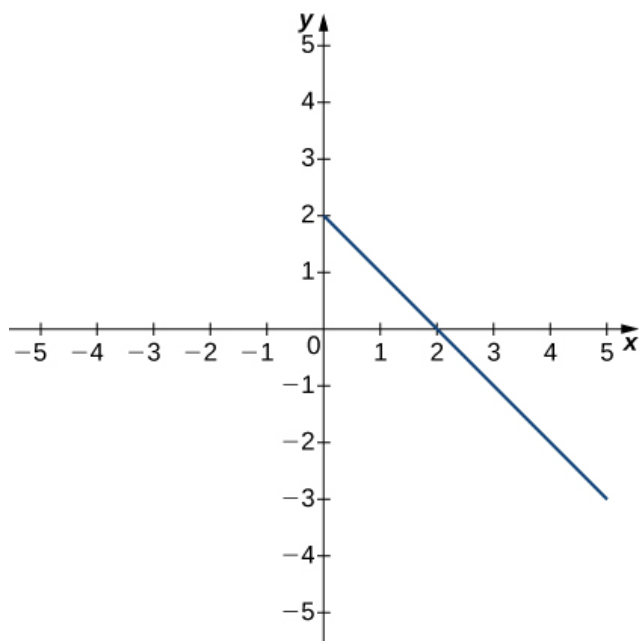
Exercise:

Problem: If $0 < |x - 2| < \delta$, then $|f(x) - 2| < 0.5$.

Solution:

$$\delta \leq 0.25$$

The following graph of the function f satisfies $\lim_{x \rightarrow 3} f(x) = -1$. In the following exercises, determine a value of $\delta > 0$ that satisfies each statement.



Exercise:

Problem: If $0 < |x - 3| < \delta$, then $|f(x) + 1| < 1$.

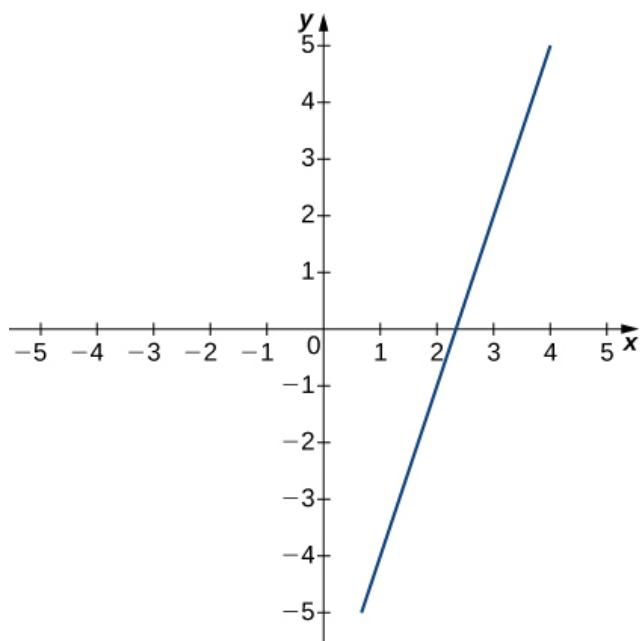
Exercise:

Problem: If $0 < |x - 3| < \delta$, then $|f(x) + 1| < 2$.

Solution:

$$\delta \leq 2$$

The following graph of the function f satisfies $\lim_{x \rightarrow 3} f(x) = 2$. In the following exercises, for each value of ε , find a value of $\delta > 0$ such that the precise definition of limit holds true.



Exercise:

Problem: $\varepsilon = 1.5$

Exercise:

Problem: $\varepsilon = 3$

Solution:

$$\delta \leq 1$$

[T] In the following exercises, use a graphing calculator to find a number δ such that the statements hold true.

Exercise:

Problem: $\left| \sin(2x) - \frac{1}{2} \right| < 0.1$, whenever $\left| x - \frac{\pi}{12} \right| < \delta$

Exercise:

Problem: $\left| \sqrt{x-4} - 2 \right| < 0.1$, whenever $|x - 8| < \delta$

Solution:

$$\delta < 0.3900$$

In the following exercises, use the precise definition of limit to prove the given limits.

Exercise:

Problem: $\lim_{x \rightarrow 2} (5x + 8) = 18$

Exercise:

Problem: $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = 6$

Solution:

Let $\delta = \varepsilon$. If $0 < |x - 3| < \varepsilon$, then $|x + 3 - 6| = |x - 3| < \varepsilon$.

Exercise:

Problem: $\lim_{x \rightarrow 2} \frac{2x^2 - 3x - 2}{x - 2} = 5$

Exercise:

Problem: $\lim_{x \rightarrow 0} x^4 = 0$

Solution:

Let $\delta = \sqrt[4]{\varepsilon}$. If $0 < |x| < \sqrt[4]{\varepsilon}$, then $|x^4| = x^4 < \varepsilon$.

Exercise:

Problem: $\lim_{x \rightarrow 2} (x^2 + 2x) = 8$

In the following exercises, use the precise definition of limit to prove the given one-sided limits.

Exercise:

Problem: $\lim_{x \rightarrow 5^-} \sqrt{5 - x} = 0$

Solution:

Let $\delta = \varepsilon^2$. If $5 - \varepsilon^2 < x < 5$, then $|\sqrt{5 - x}| = \sqrt{5 - x} < \varepsilon$.

Exercise:

Problem: $\lim_{x \rightarrow 0^+} f(x) = -2$, where $f(x) = \begin{cases} 8x - 3, & \text{if } x < 0 \\ 4x - 2, & \text{if } x \geq 0 \end{cases}$

Exercise:

Problem: $\lim_{x \rightarrow 1^-} f(x) = 3$, where $f(x) = \begin{cases} 5x - 2, & \text{if } x < 1 \\ 7x - 1, & \text{if } x \geq 1 \end{cases}$

Solution:

Let $\delta = \varepsilon/5$. If $1 - \varepsilon/5 < x < 1$, then $|f(x) - 3| = 5x - 5 < \varepsilon$.

In the following exercises, use the precise definition of limit to prove the given infinite limits.

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$

Exercise:

Problem: $\lim_{x \rightarrow -1} \frac{3}{(x+1)^2} = \infty$

Solution:

Let $\delta = \sqrt{\frac{3}{N}}$. If $0 < |x + 1| < \sqrt{\frac{3}{N}}$, then $f(x) = \frac{3}{(x+1)^2} > N$.

Exercise:

Problem: $\lim_{x \rightarrow 2} -\frac{1}{(x-2)^2} = -\infty$

Exercise:**Problem:**

An engineer is using a machine to cut a flat square of Aerogel of area 144 cm^2 . If there is a maximum error tolerance in the area of 8 cm^2 , how accurately must the engineer cut on the side, assuming all sides have the same length? How do these numbers relate to δ , ε , a , and L ?

Solution:

0.033 cm , $\varepsilon = 8$, $\delta = 0.33$, $a = 12$, $L = 144$

Exercise:**Problem:**

Use the precise definition of limit to prove that the following limit does not exist: $\lim_{x \rightarrow 1} \frac{|x-1|}{x-1}$.

Exercise:**Problem:**

Using precise definitions of limits, prove that $\lim_{x \rightarrow 0} f(x)$ does not exist, given that $f(x)$ is the ceiling function. (*Hint:* Try any $\delta < 1$.)

Solution:

Answers may vary.

Exercise:**Problem:**

Using precise definitions of limits, prove that $\lim_{x \rightarrow 0} f(x)$ does not exist:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} \quad (\text{Hint: Think about how you can always choose a rational number } 0 < r < d, \text{ but } |f(r) - 0| = 1.)$$

Exercise:**Problem:**

Using precise definitions of limits, determine $\lim_{x \rightarrow 0} f(x)$ for $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$. (Hint: Break into two cases, x rational and x irrational.)

Solution:

0

Exercise:**Problem:**

Using the function from the previous exercise, use the precise definition of limits to show that $\lim_{x \rightarrow a} f(x)$ does not exist for $a \neq 0$.

For the following exercises, suppose that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$ both exist. Use the precise definition of limits to prove the following limit laws:

Exercise:

Problem: $\lim_{x \rightarrow a} (f(x) - g(x)) = L - M$

Solution:

$$f(x) - g(x) = f(x) + (-1)g(x)$$

Exercise:**Problem:**

$$\lim_{x \rightarrow a} [cf(x)] = cL \text{ for any real constant } c \text{ (Hint: Consider two cases: } c = 0 \text{ and } c \neq 0.)$$

Exercise:**Problem:**

$$\lim_{x \rightarrow a} [f(x)g(x)] = LM. \text{ (Hint: } |f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM| \leq |f(x)||g(x) - M| + |M||f(x) - L|.)$$

Solution:

Answers may vary.

Chapter Review Exercises

True or False. In the following exercises, justify your answer with a proof or a counterexample.

Exercise:

Problem: A function has to be continuous at $x = a$ if the $\lim_{x \rightarrow a} f(x)$ exists.

Exercise:

Problem: You can use the quotient rule to evaluate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$.

Solution:

False

Exercise:**Problem:**

If there is a vertical asymptote at $x = a$ for the function $f(x)$, then f is undefined at the point $x = a$.

Exercise:

Problem: If $\lim_{x \rightarrow a} f(x)$ does not exist, then f is undefined at the point $x = a$.

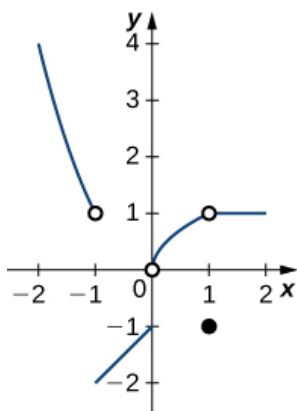
Solution:

False. A removable discontinuity is possible.

Exercise:

Problem: Using the graph, find each limit or explain why the limit does not exist.

- a. $\lim_{x \rightarrow -1} f(x)$
- b. $\lim_{x \rightarrow 1} f(x)$
- c. $\lim_{x \rightarrow 0^+} f(x)$
- d. $\lim_{x \rightarrow 2} f(x)$



In the following exercises, evaluate the limit algebraically or explain why the limit does not exist.

Exercise:

Problem: $\lim_{x \rightarrow 2} \frac{2x^2 - 3x - 2}{x - 2}$

Solution:

5

Exercise:

Problem: $\lim_{x \rightarrow 0} 3x^2 - 2x + 4$

Exercise:

Problem: $\lim_{x \rightarrow 3} \frac{x^3 - 2x^2 - 1}{3x - 2}$

Solution:

8/7

Exercise:

Problem: $\lim_{x \rightarrow \pi/2} \frac{\cot x}{\cos x}$

Exercise:

Problem: $\lim_{x \rightarrow -5} \frac{x^2 + 25}{x + 5}$

Solution:

DNE

Exercise:

Problem: $\lim_{x \rightarrow 2} \frac{3x^2 - 2x - 8}{x^2 - 4}$

Exercise:

Problem: $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x^3 - 1}$

Solution:

$$2/3$$

Exercise:

Problem: $\lim_{x \rightarrow 1} \frac{x^2 - 1}{\sqrt{x} - 1}$

Exercise:

Problem: $\lim_{x \rightarrow 4} \frac{4 - x}{\sqrt{x} - 2}$

Solution:

$$-4;$$

Exercise:

Problem: $\lim_{x \rightarrow 4} \frac{1}{\sqrt{x} - 2}$

In the following exercises, use the squeeze theorem to prove the limit.

Exercise:

Problem: $\lim_{x \rightarrow 0} x^2 \cos(2\pi x) = 0$

Solution:

Since $-1 \leq \cos(2\pi x) \leq 1$, then $-x^2 \leq x^2 \cos(2\pi x) \leq x^2$. Since $\lim_{x \rightarrow 0} x^2 = 0 = \lim_{x \rightarrow 0} -x^2$, it follows that $\lim_{x \rightarrow 0} x^2 \cos(2\pi x) = 0$.

Exercise:

Problem: $\lim_{x \rightarrow 0} x^3 \sin\left(\frac{\pi}{x}\right) = 0$

Exercise:

Problem:

Determine the domain such that the function $f(x) = \sqrt{x-2} + xe^x$ is continuous over its domain.

Solution:

$$[2, \infty]$$

In the following exercises, determine the value of c such that the function remains continuous. Draw your resulting function to ensure it is continuous.

Exercise:

Problem: $f(x) = \begin{cases} x^2 + 1, & x > c \\ 2x, & x \leq c \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} \sqrt{x+1}, & x > -1 \\ x^2 + c, & x \leq -1 \end{cases}$

Solution:

$$c = -1$$

In the following exercises, use the precise definition of limit to prove the limit.

Exercise:

Problem: $\lim_{x \rightarrow 1} (8x + 16) = 24$

Exercise:

Problem: $\lim_{x \rightarrow 0} x^3 = 0$

Solution:

$$\delta = \sqrt[3]{\varepsilon}$$

Exercise:**Problem:**

A ball is thrown into the air and the vertical position is given by $x(t) = -4.9t^2 + 25t + 5$. Use the Intermediate Value Theorem to show that the ball must land on the ground sometime between 5 sec and 6 sec after the throw.

Exercise:

Problem:

A particle moving along a line has a displacement according to the function $x(t) = t^2 - 2t + 4$, where x is measured in meters and t is measured in seconds. Find the average velocity over the time period $t = [0, 2]$.

Solution:

0 m/sec

Exercise:**Problem:**

From the previous exercises, estimate the instantaneous velocity at $t = 2$ by checking the average velocity within $t = 0.01$ sec.

Glossary

epsilon-delta definition of the limit

$\lim_{x \rightarrow a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

triangle inequality

If a and b are any real numbers, then $|a + b| \leq |a| + |b|$

Introduction

class="introduction"

The
Hennessey
Venom GT
can go from
0 to 200
mph in
14.51
seconds.
(credit:
modificatio
n of work
by
Codex41,
Flickr)



The Hennessey Venom GT is one of the fastest cars in the world. In 2014, it reached a record-setting speed of 270.49 mph. It can go from 0 to 200 mph in 14.51 seconds. The techniques in this chapter can be used to calculate the acceleration the Venom achieves in this feat (see [\[link\]](#).)

Calculating velocity and changes in velocity are important uses of calculus, but it is far more widespread than that. Calculus is important in all branches of mathematics, science, and engineering, and it is critical to analysis in business and health as well. In this chapter, we explore one of the main tools of calculus, the derivative, and show convenient ways to calculate derivatives. We apply these rules to a variety of functions in this chapter so that we can then explore applications of these techniques.

Defining the Derivative

- Recognize the meaning of the tangent to a curve at a point.
- Calculate the slope of a tangent line.
- Identify the derivative as the limit of a difference quotient.
- Calculate the derivative of a given function at a point.
- Describe the velocity as a rate of change.
- Explain the difference between average velocity and instantaneous velocity.
- Estimate the derivative from a table of values.

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac Newton (1643–1727) and the German Gottfried Leibniz (1646–1716), whose images appear in [\[link\]](#). When we credit Newton and Leibniz with developing calculus, we are really referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, in later years a bitter controversy erupted over whose work took precedence. Although it seems likely that Newton did, indeed, arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.



Newton and Leibniz are credited with developing calculus independently.

Tangent Lines

We begin our study of calculus by revisiting the notion of secant lines and tangent lines. Recall that we used the slope of a secant line to a function at a point $(a, f(a))$ to estimate the rate of change, or the rate at which one variable changes in relation to another variable. We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points $(a, f(a))$ and $(x, f(x))$, as shown in [\[link\]](#). The slope of this line is given by an equation in the form of a difference quotient:

Equation:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

We can also calculate the slope of a secant line to a function at a value a by using this equation and replacing x with $a + h$, where h is a value close to 0. We can then calculate the slope of the line through the points $(a, f(a))$ and $(a + h, f(a + h))$. In this case, we find the secant line has a slope given by the following difference quotient with increment h :

Equation:

$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{a + h - a} = \frac{f(a + h) - f(a)}{h}.$$

Note:

Definition

Let f be a function defined on an interval I containing a . If $x \neq a$ is in I , then

Equation:

$$Q = \frac{f(x) - f(a)}{x - a}$$

is a **difference quotient**.

Also, if $h \neq 0$ is chosen so that $a + h$ is in I , then

Equation:

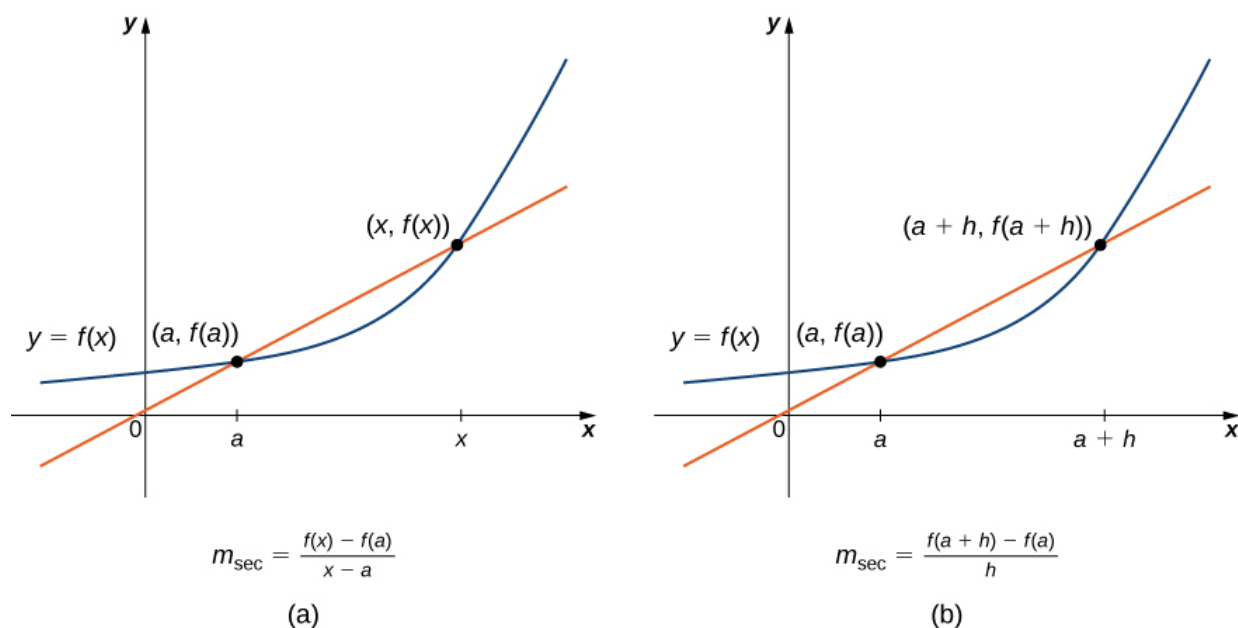
$$Q = \frac{f(a + h) - f(a)}{h}$$

is a difference quotient with increment h .

Note:

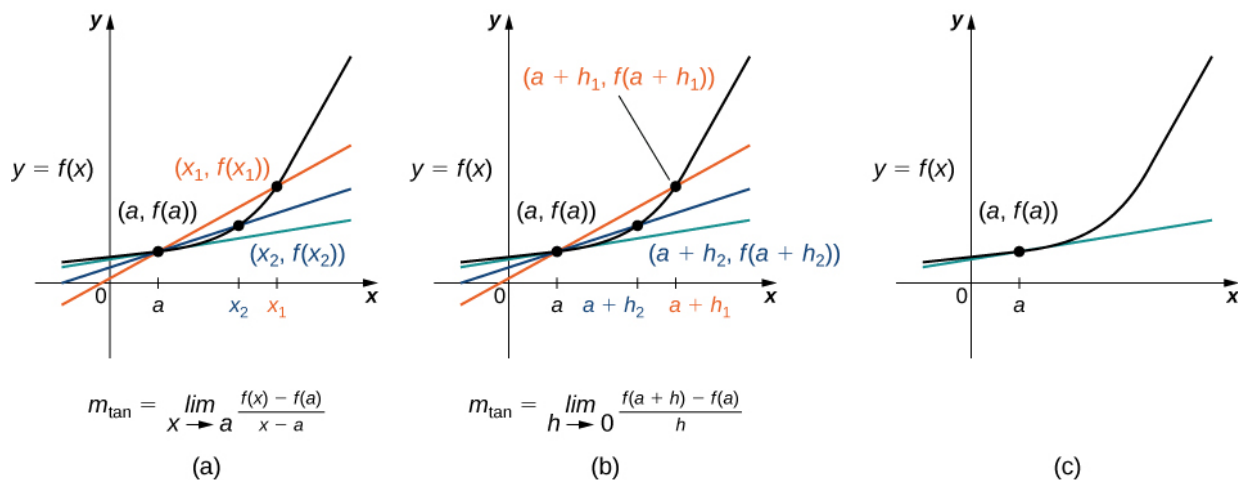
View the development of the [derivative](#) with this applet.

These two expressions for calculating the slope of a secant line are illustrated in [\[link\]](#). We will see that each of these two methods for finding the slope of a secant line is of value. Depending on the setting, we can choose one or the other. The primary consideration in our choice usually depends on ease of calculation.



We can calculate the slope of a secant line in either of two ways.

In [\[link\]](#)(a) we see that, as the values of x approach a , the slopes of the secant lines provide better estimates of the rate of change of the function at a . Furthermore, the secant lines themselves approach the tangent line to the function at a , which represents the limit of the secant lines. Similarly, [\[link\]](#)(b) shows that as the values of h get closer to 0, the secant lines also approach the tangent line. The slope of the tangent line at a is the rate of change of the function at a , as shown in [\[link\]](#)(c).

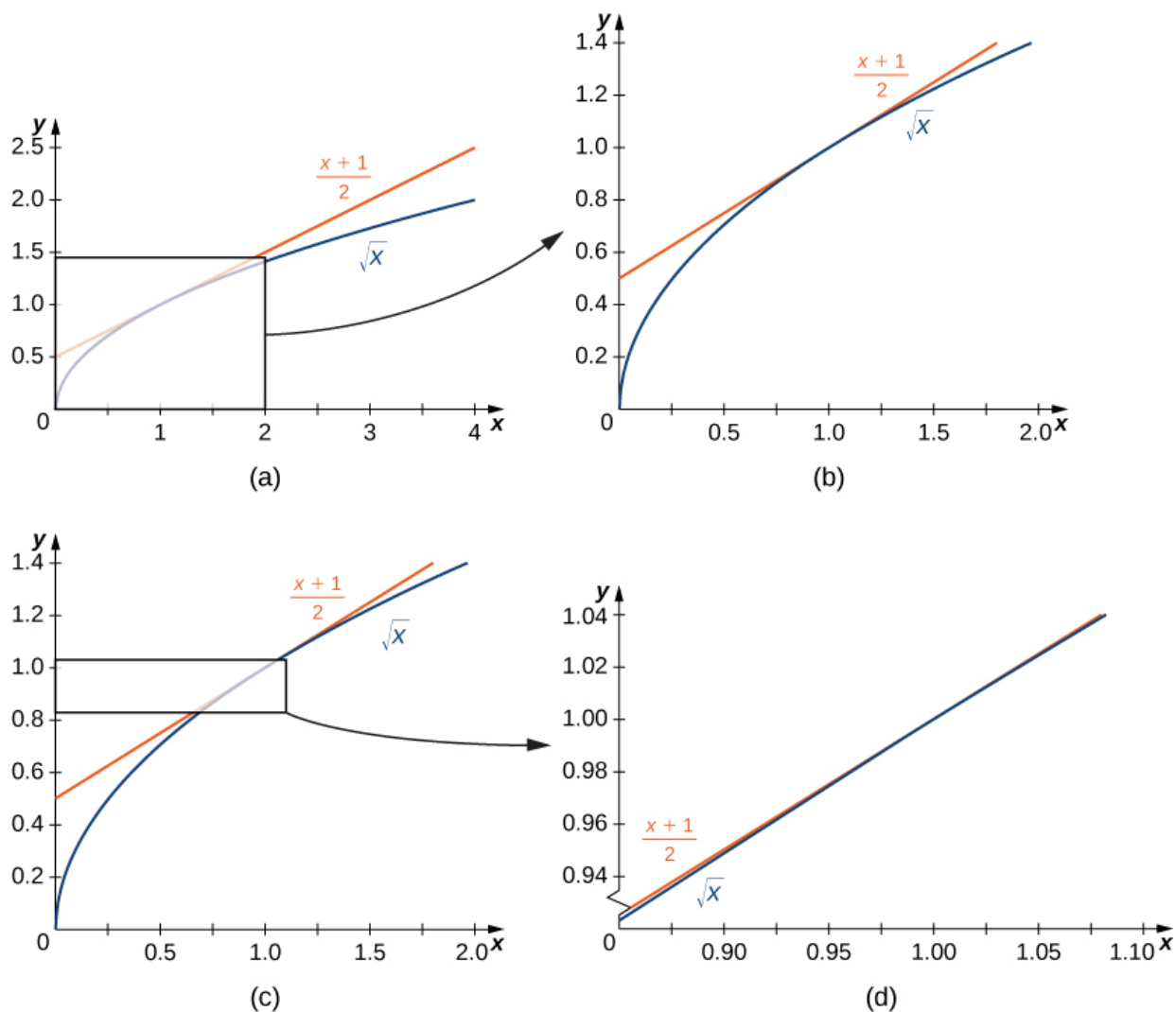


The secant lines approach the tangent line (shown in green) as the second point approaches the first.

Note:

You can use this [site](#) to explore graphs to see if they have a tangent line at a point.

In [\[link\]](#) we show the graph of $f(x) = \sqrt{x}$ and its tangent line at $(1, 1)$ in a series of tighter intervals about $x = 1$. As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line a good approximation to the values of the function for choices of x close to 1. In fact, the graph of $f(x)$ itself appears to be locally linear in the immediate vicinity of $x = 1$.



For values of x close to 1, the graph of $f(x) = \sqrt{x}$ and its tangent line appear to coincide.

Formally we may define the tangent line to the graph of a function as follows.

Note:

Definition

Let $f(x)$ be a function defined in an open interval containing a . The *tangent line* to $f(x)$ at a is the line passing through the point $(a, f(a))$ having slope

Equation:

$$m_{\text{tan}} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists.

Equivalently, we may define the tangent line to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

Equation:

$$m_{\text{tan}} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists.

Just as we have used two different expressions to define the slope of a secant line, we use two different forms to define the slope of the tangent line. In this text we use both forms of the definition. As before, the choice of definition will depend on the setting. Now that we have formally defined a tangent line to a function at a point, we can use this definition to find equations of tangent lines.

Example:

Exercise:

Problem:

Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution:

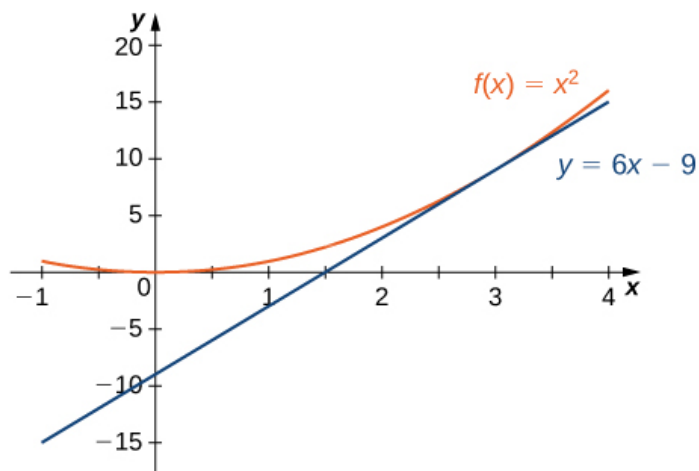
First find the slope of the tangent line. In this example, use [\[link\]](#).

Equation:

$$\begin{aligned} m_{\text{tan}} &= \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} && \text{Apply the definition.} \\ &= \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} && \text{Substitute } f(x) = x^2 \text{ and } f(3) = 9. \\ &= \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \rightarrow 3} (x+3) = 6 && \text{Factor the numerator to evaluate the limit.} \end{aligned}$$

Next, find a point on the tangent line. Since the line is tangent to the graph of $f(x)$ at $x = 3$, it passes through the point $(3, f(3))$. We have $f(3) = 9$, so the tangent line passes through the point $(3, 9)$.

Using the point-slope equation of the line with the slope $m = 6$ and the point $(3, 9)$, we obtain the line $y - 9 = 6(x - 3)$. Simplifying, we have $y = 6x - 9$. The graph of $f(x) = x^2$ and its tangent line at 3 are shown in [\[link\]](#).



The tangent line to $f(x)$ at $x = 3$.

Example:

Exercise:

Problem:

The Slope of a Tangent Line Revisited

Use [\[link\]](#) to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution:

The steps are very similar to [\[link\]](#). See [\[link\]](#) for the definition.

Equation:

$$m_{\text{tan}} = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$$

Apply the definition.

$$= \lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$$

Substitute $f(3+h) = (3+h)^2$ and $f(3) = 9$.

$$= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 - 9}{h}$$

Expand and simplify to evaluate the limit.

$$= \lim_{h \rightarrow 0} \frac{h(6+h)}{h} = \lim_{h \rightarrow 0} (6+h) = 6$$

We obtained the same value for the slope of the tangent line by using the other definition, demonstrating that the formulas can be interchanged.

Example:

Exercise:**Problem:****Finding the Equation of a Tangent Line**

Find the equation of the line tangent to the graph of $f(x) = 1/x$ at $x = 2$.

Solution:

We can use [\[link\]](#), but as we have seen, the results are the same if we use [\[link\]](#).

Equation:

$$\begin{aligned} m_{\text{tan}} &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} \cdot \frac{2x}{2x} \\ &= \lim_{x \rightarrow 2} \frac{(2-x)}{(x-2)(2x)} \\ &= \lim_{x \rightarrow 2} \frac{-1}{2x} \\ &= -\frac{1}{4} \end{aligned}$$

Apply the definition.

Substitute $f(x) = \frac{1}{x}$ and $f(2) = \frac{1}{2}$.

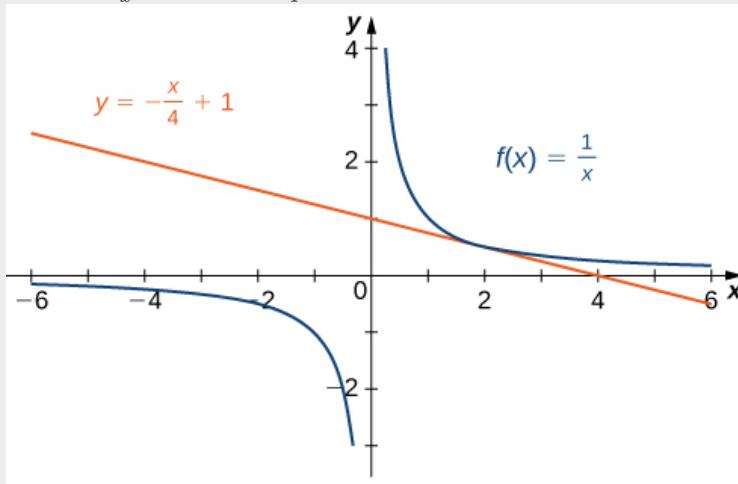
Multiply numerator and denominator by $2x$ to simplify fractions.

Simplify.

Simplify using $\frac{2-x}{x-2} = -1$, for $x \neq 2$.

Evaluate the limit.

We now know that the slope of the tangent line is $-\frac{1}{4}$. To find the equation of the tangent line, we also need a point on the line. We know that $f(2) = \frac{1}{2}$. Since the tangent line passes through the point $(2, \frac{1}{2})$ we can use the point-slope equation of a line to find the equation of the tangent line. Thus the tangent line has the equation $y = -\frac{1}{4}x + 1$. The graphs of $f(x) = \frac{1}{x}$ and $y = -\frac{1}{4}x + 1$ are shown in [\[link\]](#).



The line is tangent to $f(x)$ at $x = 2$.

Note:

Exercise:

Problem: Find the slope of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

Solution:

$$\frac{1}{4}$$

Hint

Use either [\[link\]](#) or [\[link\]](#). Multiply the numerator and the denominator by a conjugate.

The Derivative of a Function at a Point

The type of limit we compute in order to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business, and growth rates in biology. This limit occurs so frequently that we give this value a special name: the **derivative**. The process of finding a derivative is called **differentiation**.

Note:

Definition

Let $f(x)$ be a function defined in an open interval containing a . The derivative of the function $f(x)$ at a , denoted by $f'(a)$, is defined by

Equation:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided this limit exists.

Alternatively, we may also define the derivative of $f(x)$ at a as

Equation:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

Example:

Exercise:

Problem:

Estimating a Derivative

For $f(x) = x^2$, use a table to estimate $f'(3)$ using [\[link\]](#).

Solution:

Create a table using values of x just below 3 and just above 3.

x	$\frac{x^2-9}{x-3}$
2.9	5.9
2.99	5.99
2.999	5.999
3.001	6.001
3.01	6.01
3.1	6.1

After examining the table, we see that a good estimate is $f'(3) = 6$.

Note:**Exercise:**

Problem: For $f(x) = x^2$, use a table to estimate $f'(3)$ using [\[link\]](#).

Solution:

6

Hint

Evaluate $\frac{(x+h)-x^2}{h}$ at $h = -0.1, -0.01, -0.001, 0.001, 0.01, 0.1$

Example:**Exercise:****Problem:****Finding a Derivative**

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using [\[link\]](#).

Solution:

Substitute the given function and value directly into the equation.

Equation:

$$\begin{aligned}
 f'(x) &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \\
 &= \lim_{x \rightarrow 2} \frac{(3x^2 - 4x + 1) - 5}{x - 2} \\
 &= \lim_{x \rightarrow 2} \frac{(x - 2)(3x + 2)}{x - 2} \\
 &= \lim_{x \rightarrow 2} (3x + 2) \\
 &= 8
 \end{aligned}$$

Apply the definition.

Substitute $f(x) = 3x^2 - 4x + 1$ and $f(2) = 5$.

Simplify and factor the numerator.

Cancel the common factor.

Evaluate the limit.

Example:**Exercise:****Problem:****Revisiting the Derivative**

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using [\[link\]](#).

Solution:

Using this equation, we can substitute two values of the function into the equation, and we should get the same value as in [\[link\]](#).

Equation:

$$\begin{aligned}
 f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(3(2+h)^2 - 4(2+h) + 1) - 5}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3h^2 + 8h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(3h + 8)}{h} \\
 &= \lim_{h \rightarrow 0} (3h + 8) \\
 &= 8
 \end{aligned}$$

Apply the definition.

Substitute $f(2) = 5$ and
 $f(2 + h) = 3(2 + h)^2 - 4(2 + h) + 1$.

Simplify the numerator.

Factor the numerator.

Cancel the common factor.

Evaluate the limit.

The results are the same whether we use [\[link\]](#) or [\[link\]](#).

Note:

Exercise:

Problem: For $f(x) = x^2 + 3x + 2$, find $f'(1)$.

Solution:

$$f'(1) = 5$$

Hint

Use either [\[link\]](#), [\[link\]](#), or try both. Use either [\[link\]](#) or [\[link\]](#) as a guide.

Velocities and Rates of Change

Now that we can evaluate a derivative, we can use it in velocity applications. Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the average velocity of the object over a time interval $[a, t]$ if $t > a$ or $[t, a]$ if $t < a$ is given by the difference quotient

Equation:

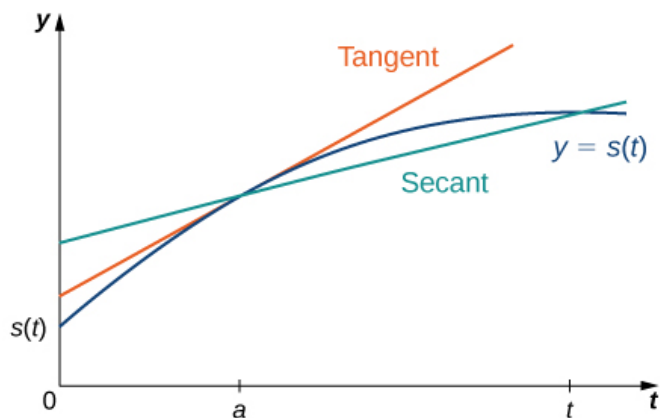
$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}.$$

As the values of t approach a , the values of v_{ave} approach the value we call the instantaneous velocity at a . That is, instantaneous velocity at a , denoted $v(a)$, is given by

Equation:

$$v(a) = s'(a) = \lim_{t \rightarrow a} \frac{s(t) - s(a)}{t - a}.$$

To better understand the relationship between average velocity and instantaneous velocity, see [\[link\]](#). In this figure, the slope of the tangent line (shown in red) is the instantaneous velocity of the object at time $t = a$ whose position at time t is given by the function $s(t)$. The slope of the secant line (shown in green) is the average velocity of the object over the time interval $[a, t]$.



The slope of the secant line is the average velocity over the interval $[a, t]$. The slope of the tangent line is the instantaneous velocity.

We can use [\[link\]](#) to calculate the instantaneous velocity, or we can estimate the velocity of a moving object by using a table of values. We can then confirm the estimate by using [\[link\]](#).

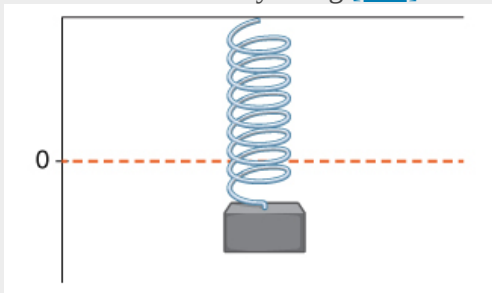
Example:

Exercise:

Problem:

Estimating Velocity

A lead weight on a spring is oscillating up and down. Its position at time t with respect to a fixed horizontal line is given by $s(t) = \sin t$ ([\[link\]](#)). Use a table of values to estimate $v(0)$. Check the estimate by using [\[link\]](#).



A lead weight suspended from a spring in vertical oscillatory motion.

Solution:

We can estimate the instantaneous velocity at $t = 0$ by computing a table of average velocities using values of t approaching 0, as shown in [\[link\]](#).

t	$\frac{\sin t - \sin 0}{t - 0} = \frac{\sin t}{t}$
-0.1	0.998334166
-0.01	0.9999833333
-0.001	0.999999833
0.001	0.999999833
0.01	0.9999833333
0.1	0.998334166

Average velocities using values of t approaching 0

From the table we see that the average velocity over the time interval $[-0.1, 0]$ is 0.998334166, the average velocity over the time interval $[-0.01, 0]$ is 0.9999833333, and so forth. Using this table of values, it appears that a good estimate is $v(0) = 1$.

By using [\[link\]](#), we can see that

Equation:

$$v(0) = s'(0) = \lim_{t \rightarrow 0} \frac{\sin t - \sin 0}{t - 0} = \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1.$$

Thus, in fact, $v(0) = 1$.

Note:

Exercise:

Problem:

A rock is dropped from a height of 64 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Find its instantaneous velocity 1 second after it is dropped, using [\[link\]](#).

Solution:

−32 ft/s

Hint

$v(t) = s'(t)$. Follow the earlier examples of the derivative using [\[link\]](#).

As we have seen throughout this section, the slope of a tangent line to a function and instantaneous velocity are related concepts. Each is calculated by computing a derivative and each measures the instantaneous rate of change of a function, or the rate of change of a function at any point along the function.

Note:

Definition

The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

Example:

Exercise:

Problem:

Chapter Opener: Estimating Rate of Change of Velocity



(credit: modification of work by
Codex41, Flickr)

Reaching a top speed of 270.49 mph, the Hennessey Venom GT is one of the fastest cars in the world. In tests it went from 0 to 60 mph in 3.05 seconds, from 0 to 100 mph in 5.88 seconds, from 0 to 200 mph in 14.51 seconds, and from 0 to 229.9 mph in 19.96 seconds. Use this data to draw a conclusion about the rate of change of velocity (that is, its acceleration) as it approaches 229.9 mph. Does the rate at which the car is accelerating appear to be increasing, decreasing, or constant?

Solution:

First observe that 60 mph = 88 ft/s, 100 mph \approx 146.67 ft/s, 200 mph \approx 293.33 ft/s, and 229.9 mph \approx 337.19 ft/s. We can summarize the information in a table.

t	$v(t)$
0	0
3.05	88
5.88	147.67
14.51	293.33
19.96	337.19

$v(t)$ at different values of t

Now compute the average acceleration of the car in feet per second on intervals of the form $[t, 19.96]$ as t approaches 19.96, as shown in the following table.

t	$\frac{v(t)-v(19.96)}{t-19.96} = \frac{v(t)-337.19}{t-19.96}$
0.0	16.89
3.05	14.74
5.88	13.46
14.51	8.05

Average acceleration

The rate at which the car is accelerating is decreasing as its velocity approaches 229.9 mph (337.19 ft/s).

Example:

Exercise:

Problem:**Rate of Change of Temperature**

A homeowner sets the thermostat so that the temperature in the house begins to drop from 70°F at 9 p.m., reaches a low of 60° during the night, and rises back to 70° by 7 a.m. the next morning. Suppose that the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \leq t \leq 10$, where t is the number of hours past 9 p.m. Find the instantaneous rate of change of the temperature at midnight.

Solution:

Since midnight is 3 hours past 9 p.m., we want to compute $T'(3)$. Refer to [\[link\]](#).

Equation:

$$T'(3) = \lim_{t \rightarrow 3} \frac{T(t) - T(3)}{t - 3}$$

$$= \lim_{t \rightarrow 3} \frac{0.4t^2 - 4t + 70 - 61.6}{t - 3}$$

$$= \lim_{t \rightarrow 3} \frac{0.4t^2 - 4t + 8.4}{t - 3}$$

$$= \lim_{t \rightarrow 3} \frac{0.4(t-3)(t-7)}{t-3}$$

$$= \lim_{t \rightarrow 3} 0.4(t-7)$$

$$= -1.6$$

Apply the definition.

Substitute $T(t) = 0.4t^2 - 4t + 70$ and $T(3) = 61.6$.

Simplify.

$$= \lim_{t \rightarrow 3} \frac{0.4(t-3)(t-7)}{t-3}$$

Cancel.

Evaluate the limit.

The instantaneous rate of change of the temperature at midnight is -1.6°F per hour.

Example:**Exercise:****Problem:****Rate of Change of Profit**

A toy company can sell x electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x) = 100x + 10,000$ dollars. Find the rate of change of profit when 10,000 games are produced. Should the toy company increase or decrease production?

Solution:

The profit $P(x)$ earned by producing x gaming systems is $R(x) - C(x)$, where $R(x)$ is the revenue obtained from the sale of x games. Since the company can sell x games at $p = -0.01x + 400$ per game,

Equation:

$$R(x) = xp = x(-0.01x + 400) = -0.01x^2 + 400x.$$

Consequently,

Equation:

$$P(x) = -0.01x^2 + 300x - 10,000.$$

Therefore, evaluating the rate of change of profit gives

Equation:

$$\begin{aligned} P'(10000) &= \lim_{x \rightarrow 10000} \frac{P(x) - P(10000)}{x - 10000} \\ &= \lim_{x \rightarrow 10000} \frac{-0.01x^2 + 300x - 10000 - 1990000}{x - 10000} \\ &= \lim_{x \rightarrow 10000} \frac{-0.01x^2 + 300x - 2000000}{x - 10000} \\ &= 100. \end{aligned}$$

Since the rate of change of profit $P'(10,000) > 0$ and $P(10,000) > 0$, the company should increase production.

Note:

Exercise:

Problem:

A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P(s) = -20s^2 + 150s - 10$. The coffee shop currently charges \$3.25 per scone. Find $P'(3.25)$, the rate of change of profit when the price is \$3.25 and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

Solution:

$$P'(3.25) = 20 > 0; \text{ raise prices}$$

Hint

Use [\[link\]](#) for a guide.

Key Concepts

- The slope of the tangent line to a curve measures the instantaneous rate of change of a curve. We can calculate it by finding the limit of the difference quotient or the difference quotient with increment h .
- The derivative of a function $f(x)$ at a value a is found using either of the definitions for the slope of the tangent line.

- Velocity is the rate of change of position. As such, the velocity $v(t)$ at time t is the derivative of the position $s(t)$ at time t . Average velocity is given by

Equation:

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}.$$

Instantaneous velocity is given by

Equation:

$$v(a) = s'(a) = \lim_{t \rightarrow a} \frac{s(t) - s(a)}{t - a}.$$

- We may estimate a derivative by using a table of values.

Key Equations

- **Difference quotient**

$$Q = \frac{f(x) - f(a)}{x - a}$$

- **Difference quotient with increment h**

$$Q = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}$$

- **Slope of tangent line**

$$m_{\text{tan}} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$m_{\text{tan}} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

- **Derivative of $f(x)$ at a**

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

- **Average velocity**

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}$$

- **Instantaneous velocity**

$$v(a) = s'(a) = \lim_{t \rightarrow a} \frac{s(t) - s(a)}{t - a}$$

For the following exercises, use [\[link\]](#) to find the slope of the secant line between the values x_1 and x_2 for each function $y = f(x)$.

Exercise:

Problem: $f(x) = 4x + 7$; $x_1 = 2$, $x_2 = 5$

Solution:

4

Exercise:

Problem: $f(x) = 8x - 3; x_1 = -1, x_2 = 3$

Exercise:

Problem: $f(x) = x^2 + 2x + 1; x_1 = 3, x_2 = 3.5$

Solution:

8.5

Exercise:

Problem: $f(x) = -x^2 + x + 2; x_1 = 0.5, x_2 = 1.5$

Exercise:

Problem: $f(x) = \frac{4}{3x-1}; x_1 = 1, x_2 = 3$

Solution:

$-\frac{3}{4}$

Exercise:

Problem: $f(x) = \frac{x-7}{2x+1}; x_1 = -2, x_2 = 0$

Exercise:

Problem: $f(x) = \sqrt{x}; x_1 = 1, x_2 = 16$

Solution:

0.2

Exercise:

Problem: $f(x) = \sqrt{x-9}; x_1 = 10, x_2 = 13$

Exercise:

Problem: $f(x) = x^{1/3} + 1; x_1 = 0, x_2 = 8$

Solution:

0.25

Exercise:

Problem: $f(x) = 6x^{2/3} + 2x^{1/3}; x_1 = 1, x_2 = 27$

For the following functions,

- use [\[link\]](#) to find the slope of the tangent line $m_{\text{tan}} = f'(a)$, and
- find the equation of the tangent line to f at $x = a$.

Exercise:

Problem: $f(x) = 3 - 4x, a = 2$

Solution:

a. -4 b. $y = 3 - 4x$

Exercise:

Problem: $f(x) = \frac{x}{5} + 6, a = -1$

Exercise:

Problem: $f(x) = x^2 + x, a = 1$

Solution:

a. 3 b. $y = 3x - 1$

Exercise:

Problem: $f(x) = 1 - x - x^2, a = 0$

Exercise:

Problem: $f(x) = \frac{7}{x}, a = 3$

Solution:

a. $-\frac{7}{9}$ b. $y = -\frac{7}{9}x + \frac{14}{3}$

Exercise:

Problem: $f(x) = \sqrt{x+8}, a = 1$

Exercise:

Problem: $f(x) = 2 - 3x^2, a = -2$

Solution:

a. 12 b. $y = 12x + 14$

Exercise:

Problem: $f(x) = \frac{-3}{x-1}, a = 4$

Exercise:

Problem: $f(x) = \frac{2}{x+3}, a = -4$

Solution:

a. -2 b. $y = -2x - 10$

Exercise:

Problem: $f(x) = \frac{3}{x^2}, a = 3$

For the following functions $y = f(x)$, find $f'(a)$ using [\[link\]](#).

Exercise:

Problem: $f(x) = 5x + 4, a = -1$

Solution:

5

Exercise:

Problem: $f(x) = -7x + 1, a = 3$

Exercise:

Problem: $f(x) = x^2 + 9x, a = 2$

Solution:

13

Exercise:

Problem: $f(x) = 3x^2 - x + 2, a = 1$

Exercise:

Problem: $f(x) = \sqrt{x}, a = 4$

Solution:

$\frac{1}{4}$

Exercise:

Problem: $f(x) = \sqrt{x-2}$, $a = 6$

Exercise:

Problem: $f(x) = \frac{1}{x}$, $a = 2$

Solution:

$$-\frac{1}{4}$$

Exercise:

Problem: $f(x) = \frac{1}{x-3}$, $a = -1$

Exercise:

Problem: $f(x) = \frac{1}{x^3}$, $a = 1$

Solution:

$$-3$$

Exercise:

Problem: $f(x) = \frac{1}{\sqrt{x}}$, $a = 4$

For the following exercises, given the function $y = f(x)$,

- find the slope of the secant line PQ for each point $Q(x, f(x))$ with x value given in the table.
- Use the answers from a. to estimate the value of the slope of the tangent line at P .
- Use the answer from b. to find the equation of the tangent line to f at point P .

Exercise:

Problem: [T] $f(x) = x^2 + 3x + 4$, $P(1, 8)$ (Round to 6 decimal places.)

x	Slope m_{PQ}	x	Slope m_{PQ}
1.1	(i)	0.9	(vii)
1.01	(ii)	0.99	(viii)

x	Slope m_{PQ}	x	Slope m_{PQ}
1.001	(iii)	0.999	(ix)
1.0001	(iv)	0.9999	(x)
1.00001	(v)	0.99999	(xi)
1.000001	(vi)	0.999999	(xii)

Solution:

a. (i) 5.100000, (ii) 5.010000, (iii) 5.001000, (iv) 5.000100, (v) 5.000010, (vi) 5.000001, (vii) 4.900000, (viii) 4.990000, (ix) 4.999000, (x) 4.999900, (xi) 4.999990, (x) 4.999999 b. $m_{\tan} = 5$ c. $y = 5x + 3$

Exercise:

Problem: [T] $f(x) = \frac{x+1}{x^2-1}, P(0, -1)$

x	Slope m_{PQ}	x	Slope m_{PQ}
0.1	(i)	-0.1	(vii)
0.01	(ii)	-0.01	(viii)
0.001	(iii)	-0.001	(ix)
0.0001	(iv)	-0.0001	(x)
0.00001	(v)	-0.00001	(xi)
0.000001	(vi)	-0.000001	(xii)

Exercise:

Problem: [T] $f(x) = 10e^{0.5x}, P(0, 10)$ (Round to 4 decimal places.)

x	Slope m_{PQ}
-0.1	(i)
-0.01	(ii)
-0.001	(iii)
-0.0001	(iv)
-0.00001	(v)
-0.000001	(vi)

Solution:

a. (i) 4.8771, (ii) 4.9875 (iii) 4.9988, (iv) 4.9999, (v) 4.9999, (vi) 4.9999 b. $m_{\tan} = 5$ c. $y = 5x + 10$

Exercise:

Problem: [T] $f(x) = \tan(x), P(\pi, 0)$

x	Slope m_{PQ}
3.1	(i)
3.14	(ii)
3.141	(iii)
3.1415	(iv)
3.14159	(v)
3.141592	(vi)

[T] For the following position functions $y = s(t)$, an object is moving along a straight line, where t is in seconds and s is in meters. Find

a. the simplified expression for the average velocity from $t = 2$ to $t = 2 + h$;

- b. the average velocity between $t = 2$ and $t = 2 + h$, where (i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$, and (iv) $h = 0.0001$; and
 c. use the answer from a. to estimate the instantaneous velocity at $t = 2$ second.

Exercise:

Problem: $s(t) = \frac{1}{3}t + 5$

Solution:

a. $\frac{1}{3}$; b. (i) $0.\bar{3}$ m/s, (ii) $0.\bar{3}$ m/s, (iii) $0.\bar{3}$ m/s, (iv) $0.\bar{3}$ m/s; c. $0.\bar{3} = \frac{1}{3}$ m/s

Exercise:

Problem: $s(t) = t^2 - 2t$

Exercise:

Problem: $s(t) = 2t^3 + 3$

Solution:

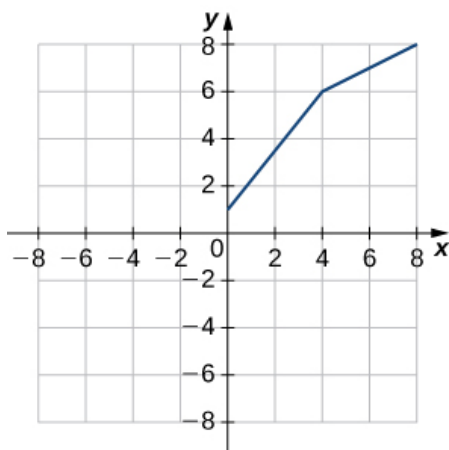
a. $2(h^2 + 6h + 12)$; b. (i) 25.22 m/s, (ii) 24.12 m/s, (iii) 24.01 m/s, (iv) 24 m/s; c. 24 m/s

Exercise:

Problem: $s(t) = \frac{16}{t^2} - \frac{4}{t}$

Exercise:

Problem: Use the following graph to evaluate a. $f'(1)$ and b. $f'(6)$.

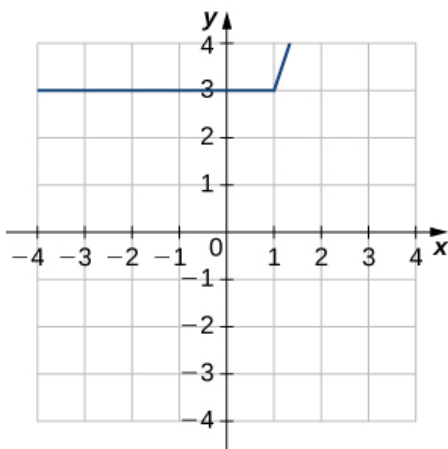


Solution:

a. 1.25; b. 0.5

Exercise:

Problem: Use the following graph to evaluate a. $f'(-3)$ and b. $f'(1.5)$.



For the following exercises, use the limit definition of derivative to show that the derivative does not exist at $x = a$ for each of the given functions.

Exercise:

Problem: $f(x) = x^{1/3}, x = 0$

Solution:

$$\lim_{x \rightarrow 0^-} \frac{x^{1/3} - 0}{x - 0} = \lim_{x \rightarrow 0^-} \frac{1}{x^{2/3}} = \infty$$

Exercise:

Problem: $f(x) = x^{2/3}, x = 0$

Exercise:

Problem: $f(x) = \begin{cases} 1, & x < 1 \\ x, & x \geq 1 \end{cases}, x = 1$

Solution:

$$\lim_{x \rightarrow 1^-} \frac{1-1}{x-1} = 0 \neq 1 = \lim_{x \rightarrow 1^+} \frac{x-1}{x-1}$$

Exercise:

Problem: $f(x) = \frac{|x|}{x}, x = 0$

Exercise:

Problem:

[T] The position in feet of a race car along a straight track after t seconds is modeled by the function $s(t) = 8t^2 - \frac{1}{16}t^3$.

- a. Find the average velocity of the vehicle over the following time intervals to four decimal places:
 - i. $[4, 4.1]$
 - ii. $[4, 4.01]$
 - iii. $[4, 4.001]$
 - iv. $[4, 4.0001]$
 - b. Use a. to draw a conclusion about the instantaneous velocity of the vehicle at $t = 4$ seconds.
-

Solution:

a. (i) 61.7244 ft/s, (ii) 61.0725 ft/s (iii) 61.0072 ft/s (iv) 61.0007 ft/s b. At 4 seconds the race car is traveling at a rate/velocity of 61 ft/s.

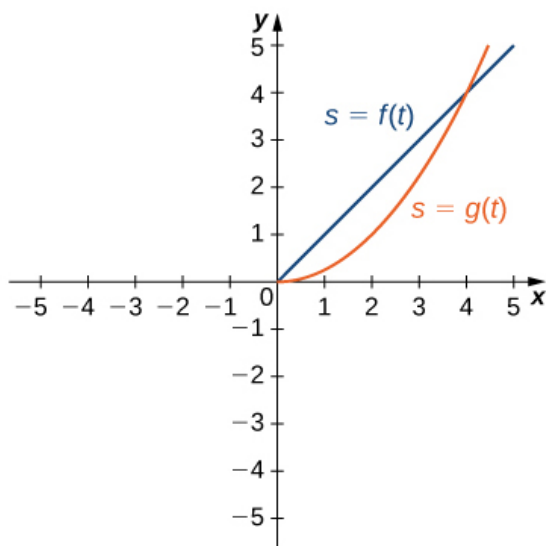
Exercise:**Problem:**

[T] The distance in feet that a ball rolls down an incline is modeled by the function $s(t) = 14t^2$, where t is seconds after the ball begins rolling.

- a. Find the average velocity of the ball over the following time intervals:
 - i. $[5, 5.1]$
 - ii. $[5, 5.01]$
 - iii. $[5, 5.001]$
 - iv. $[5, 5.0001]$
- b. Use the answers from a. to draw a conclusion about the instantaneous velocity of the ball at $t = 5$ seconds.

Exercise:**Problem:**

Two vehicles start out traveling side by side along a straight road. Their position functions, shown in the following graph, are given by $s = f(t)$ and $s = g(t)$, where s is measured in feet and t is measured in seconds.



- Which vehicle has traveled farther at $t = 2$ seconds?
- What is the approximate velocity of each vehicle at $t = 3$ seconds?
- Which vehicle is traveling faster at $t = 4$ seconds?
- What is true about the positions of the vehicles at $t = 4$ seconds?

Solution:

a. The vehicle represented by $f(t)$, because it has traveled 2 feet, whereas $g(t)$ has traveled 1 foot. b. The velocity of $f(t)$ is constant at 1 ft/s, while the velocity of $g(t)$ is approximately 2 ft/s. c. The vehicle represented by $g(t)$, with a velocity of approximately 4 ft/s. d. Both have traveled 4 feet in 4 seconds.

Exercise:

Problem:

[T] The total cost $C(x)$, in hundreds of dollars, to produce x jars of mayonnaise is given by $C(x) = 0.000003x^3 + 4x + 300$.

- Calculate the average cost per jar over the following intervals:
 - $[100, 100.1]$
 - $[100, 100.01]$
 - $[100, 100.001]$
 - $[100, 100.0001]$
- Use the answers from a. to estimate the average cost to produce 100 jars of mayonnaise.

Exercise:

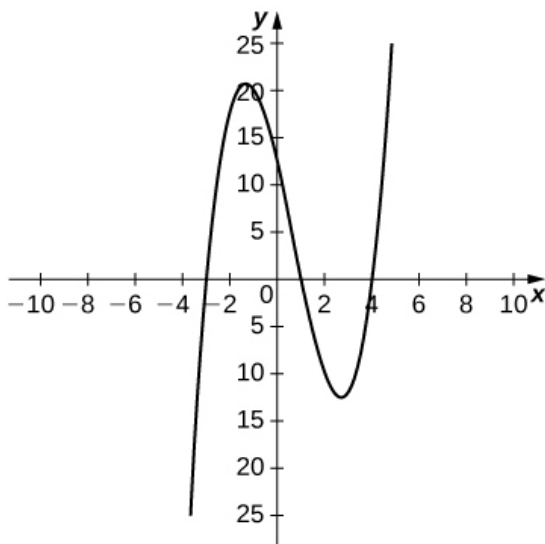
Problem: [T] For the function $f(x) = x^3 - 2x^2 - 11x + 12$, do the following.

- Use a graphing calculator to graph f in an appropriate viewing window.

- b. Use the ZOOM feature on the calculator to approximate the two values of $x = a$ for which $m_{\text{tan}} = f'(a) = 0$.

Solution:

a.



- b. $a \approx -1.361, 2.694$

Exercise:

Problem: [T] For the function $f(x) = \frac{x}{1+x^2}$, do the following.

- Use a graphing calculator to graph f in an appropriate viewing window.
- Use the ZOOM feature on the calculator to approximate the values of $x = a$ for which $m_{\text{tan}} = f'(a) = 0$.

Exercise:

Problem:

Suppose that $N(x)$ computes the number of gallons of gas used by a vehicle traveling x miles. Suppose the vehicle gets 30 mpg.

- Find a mathematical expression for $N(x)$.
- What is $N(100)$? Explain the physical meaning.
- What is $N'(100)$? Explain the physical meaning.

Solution:

- a. $N(x) = \frac{x}{30}$ b. ~ 3.3 gallons. When the vehicle travels 100 miles, it has used 3.3 gallons of gas. c. $\frac{1}{30}$. The rate of gas consumption in gallons per mile that the vehicle is achieving after having traveled 100 miles.

Exercise:

Problem: [T] For the function $f(x) = x^4 - 5x^2 + 4$, do the following.

- Use a graphing calculator to graph f in an appropriate viewing window.
- Use the nDeriv function, which numerically finds the derivative, on a graphing calculator to estimate $f'(-2)$, $f'(-0.5)$, $f'(1.7)$, and $f'(2.718)$.

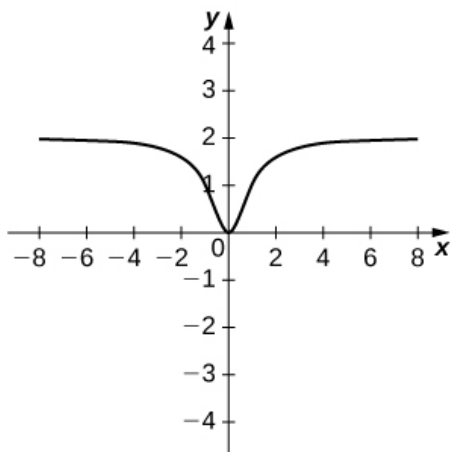
Exercise:

Problem: [T] For the function $f(x) = \frac{x^2}{x^2+1}$, do the following.

- Use a graphing calculator to graph f in an appropriate viewing window.
- Use the nDeriv function on a graphing calculator to find $f'(-4)$, $f'(-2)$, $f'(2)$, and $f'(4)$.

Solution:

a.



b. $-0.028, -0.16, 0.16, 0.028$

Glossary

derivative

the slope of the tangent line to a function at a point, calculated by taking the limit of the difference quotient, is the derivative

difference quotient

of a function $f(x)$ at a is given by

Equation:

$$\frac{f(a+h) - f(a)}{h} \text{ or } \frac{f(x) - f(a)}{x - a}$$

differentiation

the process of taking a derivative

instantaneous rate of change

the rate of change of a function at any point along the function a , also called $f'(a)$, or the derivative of the function at a

The Derivative as a Function

- Define the derivative function of a given function.
- Graph a derivative function from the graph of a given function.
- State the connection between derivatives and continuity.
- Describe three conditions for when a function does not have a derivative.
- Explain the meaning of a higher-order derivative.

As we have seen, the derivative of a function at a given point gives us the rate of change or slope of the tangent line to the function at that point. If we differentiate a position function at a given time, we obtain the velocity at that time. It seems reasonable to conclude that knowing the derivative of the function at every point would produce valuable information about the behavior of the function. However, the process of finding the derivative at even a handful of values using the techniques of the preceding section would quickly become quite tedious. In this section we define the derivative function and learn a process for finding it.

Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

Note:

Definition

Let f be a function. The **derivative function**, denoted by f' , is the function whose domain consists of those values of x such that the following limit exists:

Equation:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

A function $f(x)$ is said to be **differentiable at a** if $f'(a)$ exists. More generally, a function is said to be **differentiable on S** if it is differentiable at every point in an open set S , and a **differentiable function** is one in which $f'(x)$ exists on its domain.

In the next few examples we use [\[link\]](#) to find the derivative of a function.

Example:

Exercise:

Problem:

Finding the Derivative of a Square-Root Function

Find the derivative of $f(x) = \sqrt{x}$.

Solution:

Start directly with the definition of the derivative function. Use [\[link\]](#).

Equation:

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
 &= \lim_{h \rightarrow 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})} \\
 &= \frac{1}{2\sqrt{x}}
 \end{aligned}$$

Substitute $f(x+h) = \sqrt{x+h}$ and $f(x) = \sqrt{x}$ into $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$.

Multiply numerator and denominator by $\sqrt{x+h} + \sqrt{x}$ without distributing in the denominator.

Multiply the numerators and simplify.

Cancel the h .

Evaluate the limit.

Example:

Exercise:

Problem:

Finding the Derivative of a Quadratic Function

Find the derivative of the function $f(x) = x^2 - 2x$.

Solution:

Follow the same procedure here, but without having to multiply by the conjugate.

Equation:

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{((x+h)^2 - 2(x+h)) - (x^2 - 2x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh - 2h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(2x - 2 + h)}{h} \\
 &= \lim_{h \rightarrow 0} (2x - 2 + h) \\
 &= 2x - 2
 \end{aligned}$$

Substitute $f(x+h) = (x+h)^2 - 2(x+h)$ and $f(x) = x^2 - 2x$ into

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Expand $(x+h)^2 - 2(x+h)$.

Simplify.

Factor out h from the numerator.

Cancel the common factor of h .

Evaluate the limit.

Note:

Exercise:

Problem: Find the derivative of $f(x) = x^2$.

Solution:

$$f'(x) = 2x$$

Hint

Use [\[link\]](#) and follow the example.

We use a variety of different notations to express the derivative of a function. In [\[link\]](#) we showed that if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$. If we had expressed this function in the form $y = x^2 - 2x$, we could have expressed the derivative as $y' = 2x - 2$ or $\frac{dy}{dx} = 2x - 2$. We could have conveyed the same information by writing $\frac{d}{dx}(x^2 - 2x) = 2x - 2$. Thus, for the function $y = f(x)$, each of the following notations represents the derivative of $f(x)$:

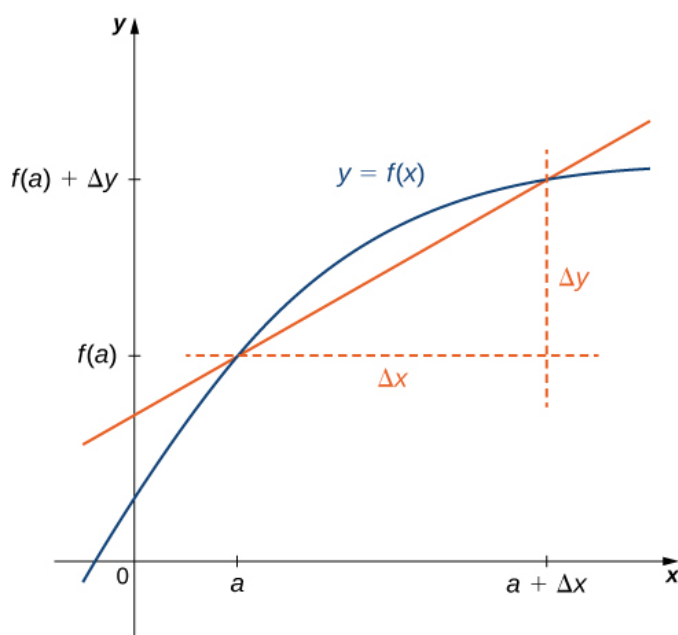
Equation:

$$f'(x), \frac{dy}{dx}, y', \frac{d}{dx}(f(x)).$$

In place of $f'(a)$ we may also use $\left. \frac{dy}{dx} \right|_{x=a}$. Use of the $\frac{dy}{dx}$ notation (called Leibniz notation) is quite common in engineering and physics. To understand this notation better, recall that the derivative of a function at a point is the limit of the slopes of secant lines as the secant lines approach the tangent line. The slopes of these secant lines are often expressed in the form $\frac{\Delta y}{\Delta x}$ where Δy is the difference in the y values corresponding to the difference in the x values, which are expressed as Δx ([\[link\]](#)). Thus the derivative, which can be thought of as the instantaneous rate of change of y with respect to x , is expressed as

Equation:

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}.$$

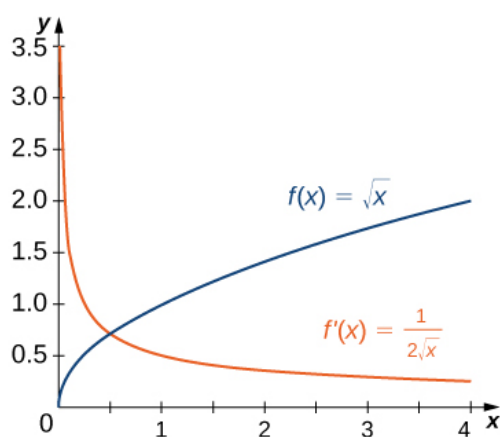


The derivative is expressed as $\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$.

Graphing a Derivative

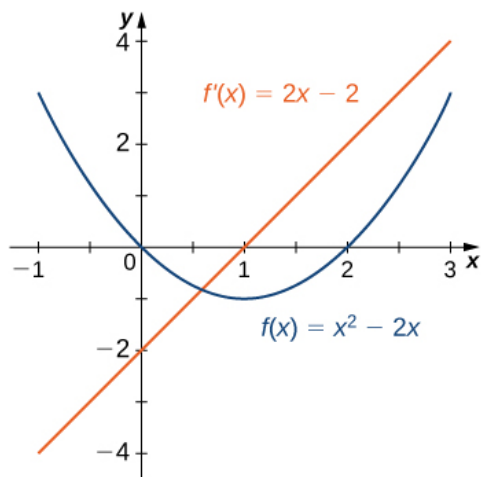
We have already discussed how to graph a function, so given the equation of a function or the equation of a derivative function, we could graph it. Given both, we would expect to see a correspondence between the graphs of these two functions, since $f'(x)$ gives the rate of change of a function $f(x)$ (or slope of the tangent line to $f(x)$).

In [\[link\]](#) we found that for $f(x) = \sqrt{x}$, $f'(x) = 1/2\sqrt{x}$. If we graph these functions on the same axes, as in [\[link\]](#), we can use the graphs to understand the relationship between these two functions. First, we notice that $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive. Consequently, we expect $f'(x) > 0$ for all values of x in its domain. Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing and we expect to see a corresponding decrease in $f'(x)$. We also observe that $f(0)$ is undefined and that $\lim_{x \rightarrow 0^+} f'(x) = +\infty$, corresponding to a vertical tangent to $f(x)$ at 0.



The derivative $f'(x)$ is positive everywhere because the function $f(x)$ is increasing.

In [\[link\]](#) we found that for $f(x) = x^2 - 2x$, $f'(x) = 2x - 2$. The graphs of these functions are shown in [\[link\]](#). Observe that $f(x)$ is decreasing for $x < 1$. For these same values of x , $f'(x) < 0$. For values of $x > 1$, $f(x)$ is increasing and $f'(x) > 0$. Also, $f(x)$ has a horizontal tangent at $x = 1$ and $f'(1) = 0$.



The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing.

The derivative is zero where the function has a horizontal tangent.

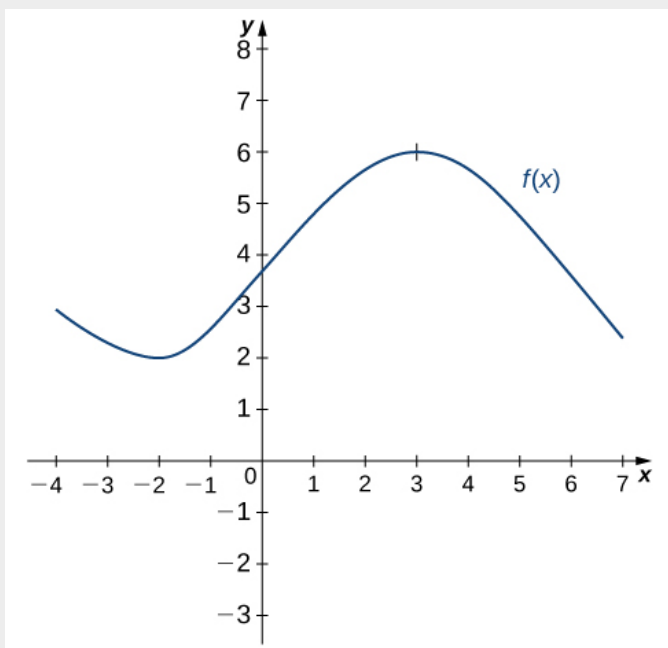
Example:

Exercise:

Problem:

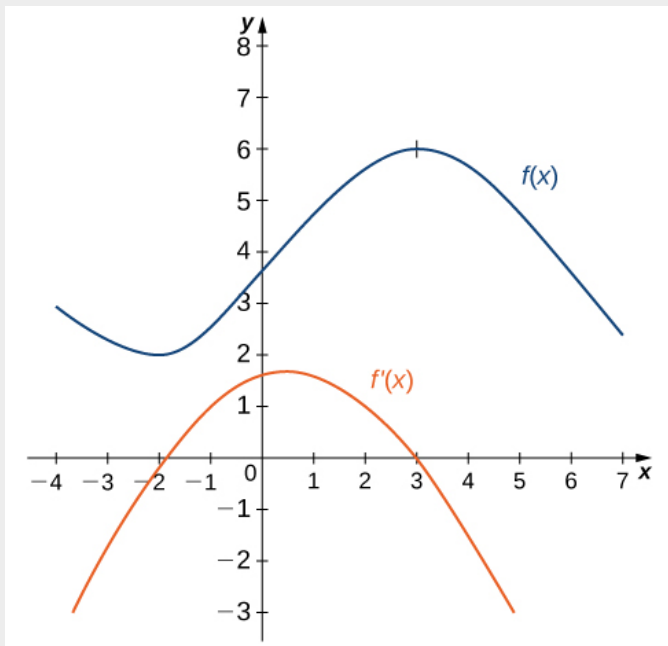
Sketching a Derivative Using a Function

Use the following graph of $f(x)$ to sketch a graph of $f'(x)$.



Solution:

The solution is shown in the following graph. Observe that $f(x)$ is increasing and $f'(x) > 0$ on $(-2, 3)$. Also, $f(x)$ is decreasing and $f'(x) < 0$ on $(-\infty, -2)$ and on $(3, +\infty)$. Also note that $f(x)$ has horizontal tangents at -2 and 3 , and $f'(-2) = 0$ and $f'(3) = 0$.

**Note:****Exercise:**

Problem: Sketch the graph of $f(x) = x^2 - 4$. On what interval is the graph of $f'(x)$ above the x -axis?

Solution:

$(0, +\infty)$

Hint

The graph of $f'(x)$ is positive where $f(x)$ is increasing.

Derivatives and Continuity

Now that we can graph a derivative, let's examine the behavior of the graphs. First, we consider the relationship between differentiability and continuity. We will see that if a function is differentiable at a point, it must be continuous there; however, a function that is continuous at a point need not be differentiable at that point. In fact, a function may be continuous at a point and fail to be differentiable at the point for one of several reasons.

Note:**Differentiability Implies Continuity**

Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a , then f is continuous at a .

Proof

If $f(x)$ is differentiable at a , then $f'(a)$ exists and

Equation:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

We want to show that $f(x)$ is continuous at a by showing that $\lim_{x \rightarrow a} f(x) = f(a)$. Thus,

Equation:

$$\begin{aligned} \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} (f(x) - f(a) + f(a)) \\ &= \lim_{x \rightarrow a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a) \right) && \text{Multiply and divide } f(x) - f(a) \text{ by } x - a. \\ &= \left(\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \right) \cdot \left(\lim_{x \rightarrow a} (x - a) \right) + \lim_{x \rightarrow a} f(a) \\ &= f'(a) \cdot 0 + f(a) \\ &= f(a). \end{aligned}$$

Therefore, since $f(a)$ is defined and $\lim_{x \rightarrow a} f(x) = f(a)$, we conclude that f is continuous at a .

□

We have just proven that differentiability implies continuity, but now we consider whether continuity implies differentiability. To determine an answer to this question, we examine the function $f(x) = |x|$. This function is continuous everywhere; however, $f'(0)$ is undefined. This observation leads us to believe that continuity does not imply differentiability. Let's explore further. For $f(x) = |x|$,

Equation:

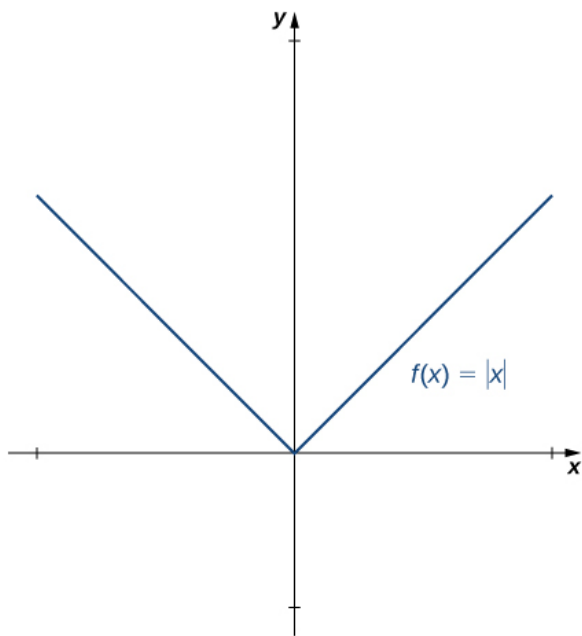
$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{|x| - |0|}{x - 0} = \lim_{x \rightarrow 0} \frac{|x|}{x}.$$

This limit does not exist because

Equation:

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1 \text{ and } \lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1.$$

See [\[link\]](#).



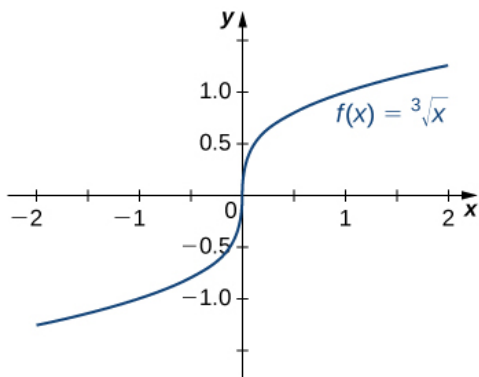
The function $f(x) = |x|$ is continuous at 0 but is not differentiable at 0.

Let's consider some additional situations in which a continuous function fails to be differentiable. Consider the function $f(x) = \sqrt[3]{x}$:

Equation:

$$f'(0) = \lim_{x \rightarrow 0} \frac{\sqrt[3]{x} - 0}{x - 0} = \lim_{x \rightarrow 0} \frac{1}{\sqrt[3]{x^2}} = +\infty.$$

Thus $f'(0)$ does not exist. A quick look at the graph of $f(x) = \sqrt[3]{x}$ clarifies the situation. The function has a vertical tangent line at 0 ([link](#)).



The function $f(x) = \sqrt[3]{x}$ has a vertical tangent at $x = 0$. It is continuous at 0 but is not differentiable at

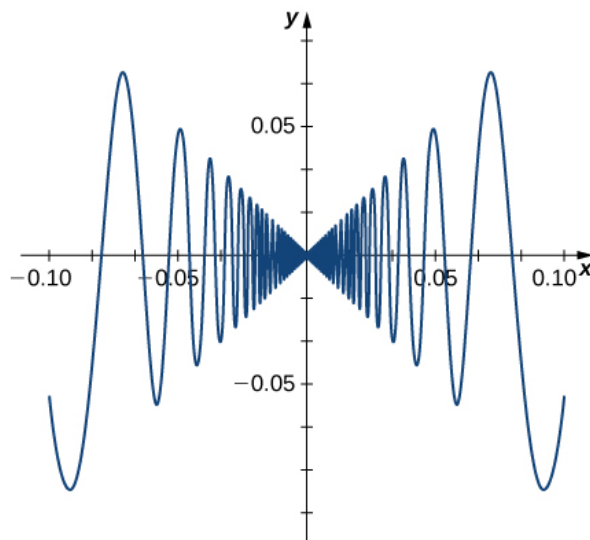
0.

The function $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ also has a derivative that exhibits interesting behavior at 0. We see that

Equation:

$$f'(0) = \lim_{x \rightarrow 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right).$$

This limit does not exist, essentially because the slopes of the secant lines continuously change direction as they approach zero ([link](#)).



The function $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is not differentiable at 0.

In summary:

1. We observe that if a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.
2. We saw that $f(x) = |x|$ failed to be differentiable at 0 because the limit of the slopes of the tangent lines on the left and right were not the same. Visually, this resulted in a sharp corner on the graph of the function at 0. From this we conclude that in order to be differentiable at a point, a function must be “smooth” at that point.
3. As we saw in the example of $f(x) = \sqrt[3]{x}$, a function fails to be differentiable at a point where there is a vertical tangent line.

4. As we saw with $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ a function may fail to be differentiable at a point in more complicated ways as well.

Example:

Exercise:

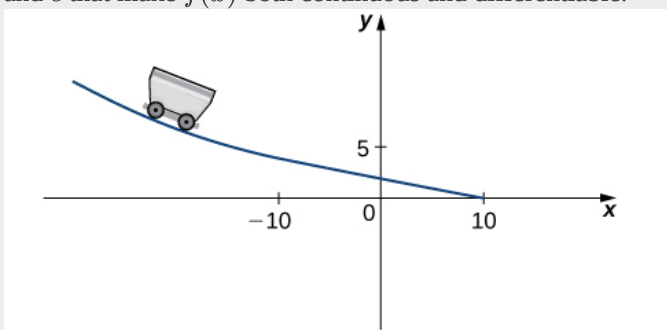
Problem:

A Piecewise Function that is Continuous and Differentiable

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line ([link](#)). The function that describes the track is to have the form

$$f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c & \text{if } x < -10 \\ -\frac{1}{4}x + \frac{5}{2} & \text{if } x \geq -10 \end{cases} \quad \text{where } x \text{ and } f(x) \text{ are in inches. For the car to move smoothly}$$

along the track, the function $f(x)$ must be both continuous and differentiable at -10 . Find values of b and c that make $f(x)$ both continuous and differentiable.



For the car to move smoothly along the track, the function must be both continuous and differentiable.

Solution:

For the function to be continuous at $x = -10$, $\lim_{x \rightarrow -10^-} f(x) = f(-10)$. Thus, since

Equation:

$$\lim_{x \rightarrow -10^-} f(x) = \frac{1}{10}(-10)^2 - 10b + c = 10 - 10b + c$$

and $f(-10) = 5$, we must have $10 - 10b + c = 5$. Equivalently, we have $c = 10b - 5$.

For the function to be differentiable at -10 ,

Equation:

$$f'(10) = \lim_{x \rightarrow -10} \frac{f(x) - f(-10)}{x + 10}$$

must exist. Since $f(x)$ is defined using different rules on the right and the left, we must evaluate this limit from the right and the left and then set them equal to each other:

Equation:

$$\begin{aligned}\lim_{x \rightarrow -10^-} \frac{f(x) - f(-10)}{x + 10} &= \lim_{x \rightarrow -10^-} \frac{\frac{1}{10}x^2 + bx + c - 5}{x + 10} \\ &= \lim_{x \rightarrow -10^-} \frac{\frac{1}{10}x^2 + bx + (10b - 5) - 5}{x + 10} && \text{Substitute } c = 10b - 5. \\ &= \lim_{x \rightarrow -10^-} \frac{x^2 - 100 + 10bx + 100b}{10(x + 10)} \\ &= \lim_{x \rightarrow -10^-} \frac{(x + 10)(x - 10 + 10b)}{10(x + 10)} && \text{Factor by grouping.} \\ &= b - 2.\end{aligned}$$

We also have

Equation:

$$\begin{aligned}\lim_{x \rightarrow -10^+} \frac{f(x) - f(-10)}{x + 10} &= \lim_{x \rightarrow -10^+} \frac{-\frac{1}{4}x + \frac{5}{2} - 5}{x + 10} \\ &= \lim_{x \rightarrow -10^+} \frac{-(x + 10)}{4(x + 10)} \\ &= -\frac{1}{4}.\end{aligned}$$

This gives us $b - 2 = -\frac{1}{4}$. Thus $b = \frac{7}{4}$ and $c = 10\left(\frac{7}{4}\right) - 5 = \frac{25}{2}$.

Note:

Exercise:

Problem:

Find values of a and b that make $f(x) = \begin{cases} ax + b & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases}$ both continuous and differentiable at 3.

Solution:

$$a = 6 \text{ and } b = -9$$

Hint

Use [\[link\]](#) as a guide.

Higher-Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position, or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue to take derivatives to obtain the third derivative, fourth derivative, and so on. Collectively, these are referred to as **higher-order derivatives**. The notation for the higher-order derivatives of $y = f(x)$ can be expressed in any of the following forms:

Equation:

$$f''(x), f'''(x), f^{(4)}(x), \dots, f^{(n)}(x)$$

Equation:

$$y''(x), y'''(x), y^{(4)}(x), \dots, y^{(n)}(x)$$

Equation:

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \dots, \frac{d^ny}{dx^n}.$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly. Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$.

Example:

Exercise:

Problem:

Finding a Second Derivative

For $f(x) = 2x^2 - 3x + 1$, find $f''(x)$.

Solution:

First find $f'(x)$.

Equation:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{(2(x+h)^2 - 3(x+h) + 1) - (2x^2 - 3x + 1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4xh + h^2 - 3h}{h} \\ &= \lim_{h \rightarrow 0} (4x + h - 3) \\ &= 4x - 3 \end{aligned}$$

Substitute $f(x) = 2x^2 - 3x + 1$

and

$$f(x+h) = 2(x+h)^2 - 3(x+h) + 1$$

$$\text{into } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Simplify the numerator.

Factor out the h in the numerator

and cancel with the h in the

denominator.

Take the limit.

Next, find $f''(x)$ by taking the derivative of $f'(x) = 4x - 3$.

Equation:

$$\begin{aligned}
 f''(x) &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(4(x+h) - 3) - (4x - 3)}{h} \\
 &= \lim_{h \rightarrow 0} 4 \\
 &= 4
 \end{aligned}$$

Use $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ with $f'(x)$ in place of $f(x)$.

Substitute $f'(x+h) = 4(x+h) - 3$ and $f'(x) = 4x - 3$.

Simplify.

Take the limit.

Note:

Exercise:

Problem: Find $f''(x)$ for $f(x) = x^2$.

Solution:

$$f''(x) = 2$$

Hint

We found $f'(x) = 2x$ in a previous checkpoint. Use [\[link\]](#) to find the derivative of $f'(x)$

Example:

Exercise:

Problem:
Finding Acceleration

The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t) = 3t^2 - 4t + 1$ (in meters). Find the function that describes its acceleration at time t .

Solution:

Since $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$, we begin by finding the derivative of $s(t)$:

Equation:

$$\begin{aligned}
 s'(t) &= \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3(t+h)^2 - 4(t+h) + 1 - (3t^2 - 4t + 1)}{h} \\
 &= 6t - 4.
 \end{aligned}$$

Next,

Equation:

$$\begin{aligned}
 s''(t) &= \lim_{h \rightarrow 0} \frac{s'(t+h) - s'(t)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{6(t+h) - 4 - (6t - 4)}{h} \\
 &= 6.
 \end{aligned}$$

Thus, $a = 6 \text{ m/s}^2$.

Note:

Exercise:

Problem: For $s(t) = t^3$, find $a(t)$.

Solution:

$$a(t) = 6t$$

Hint

Use [\[link\]](#) as a guide.

Key Concepts

- The derivative of a function $f(x)$ is the function whose value at x is $f'(x)$.
- The graph of a derivative of a function $f(x)$ is related to the graph of $f(x)$. Where $f(x)$ has a tangent line with positive slope, $f'(x) > 0$. Where $f(x)$ has a tangent line with negative slope, $f'(x) < 0$. Where $f(x)$ has a horizontal tangent line, $f'(x) = 0$.
- If a function is differentiable at a point, then it is continuous at that point. A function is not differentiable at a point if it is not continuous at the point, if it has a vertical tangent line at the point, or if the graph has a sharp corner or cusp.
- Higher-order derivatives are derivatives of derivatives, from the second derivative to the n th derivative.

Key Equations

- **The derivative function**

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

For the following exercises, use the definition of a derivative to find $f'(x)$.

Exercise:

Problem: $f(x) = 6$

Exercise:

Problem: $f(x) = 2 - 3x$

Solution:

−3

Exercise:

Problem: $f(x) = \frac{2x}{7} + 1$

Exercise:

Problem: $f(x) = 4x^2$

Solution:

$$8x$$

Exercise:

Problem: $f(x) = 5x - x^2$

Exercise:

Problem: $f(x) = \sqrt{2x}$

Solution:

$$\frac{1}{\sqrt{2x}}$$

Exercise:

Problem: $f(x) = \sqrt{x-6}$

Exercise:

Problem: $f(x) = \frac{9}{x}$

Solution:

$$\frac{-9}{x^2}$$

Exercise:

Problem: $f(x) = x + \frac{1}{x}$

Exercise:

Problem: $f(x) = \frac{1}{\sqrt{x}}$

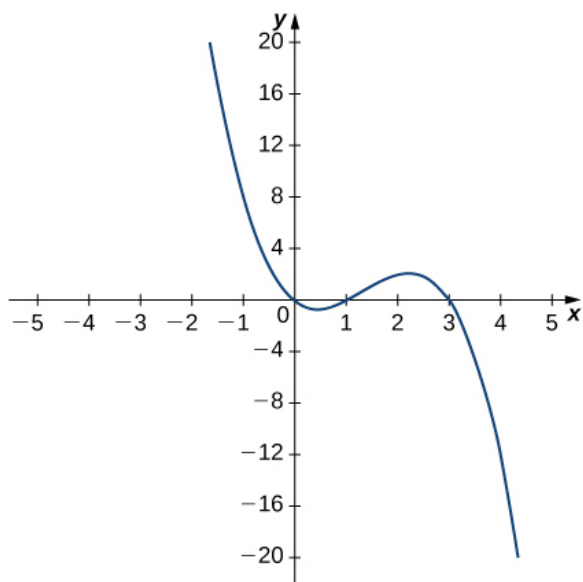
Solution:

$$\frac{-1}{2x^{3/2}}$$

For the following exercises, use the graph of $y = f(x)$ to sketch the graph of its derivative $f'(x)$.

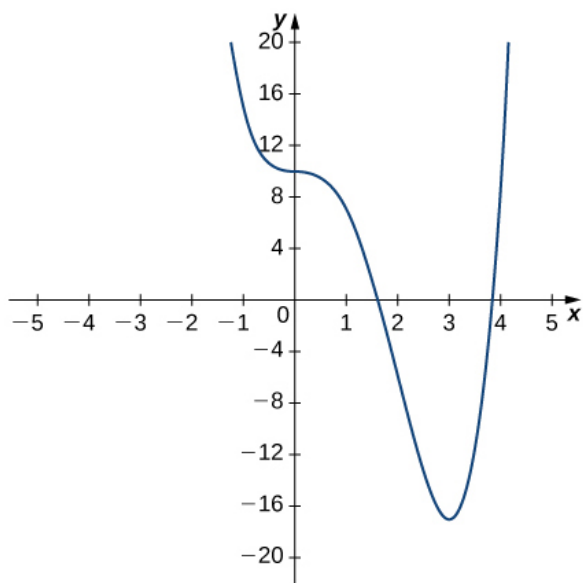
Exercise:

Problem:

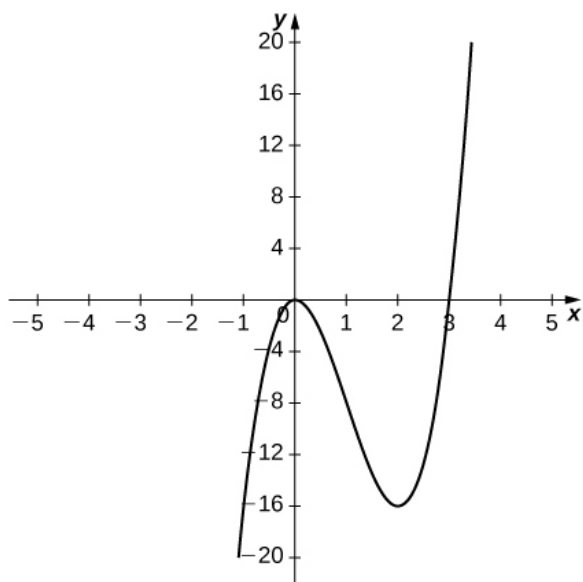


Exercise:

Problem:

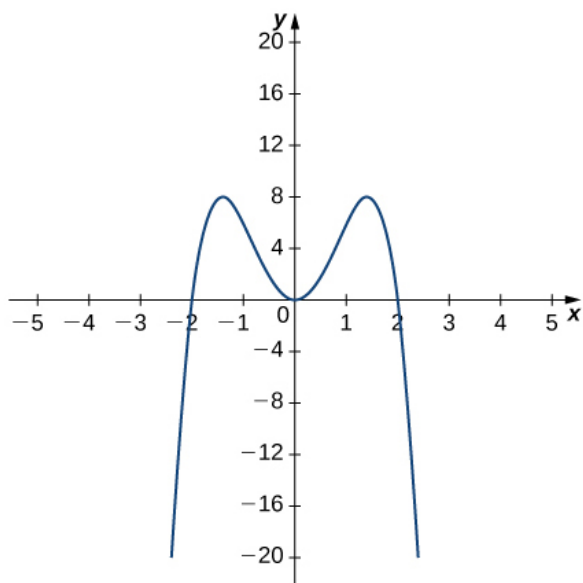


Solution:



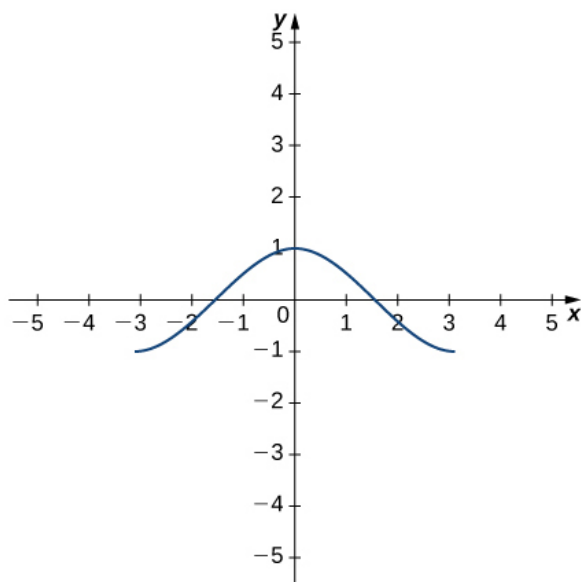
Exercise:

Problem:

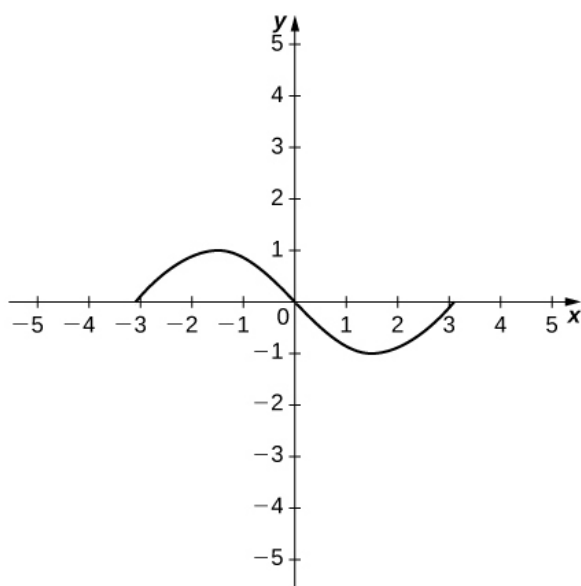


Exercise:

Problem:



Solution:



For the following exercises, the given limit represents the derivative of a function $y = f(x)$ at $x = a$. Find $f(x)$ and a .

Exercise:

Problem: $\lim_{h \rightarrow 0} \frac{(1+h)^{2/3} - 1}{h}$

Exercise:

Problem: $\lim_{h \rightarrow 0} \frac{[3(2+h)^2 + 2] - 14}{h}$

Solution:

$$f(x) = 3x^2 + 2, a = 2$$

Exercise:

Problem: $\lim_{h \rightarrow 0} \frac{\cos(\pi+h)+1}{h}$

Exercise:

Problem: $\lim_{h \rightarrow 0} \frac{(2+h)^4 - 16}{h}$

Solution:

$$f(x) = x^4, a = 2$$

Exercise:

Problem: $\lim_{h \rightarrow 0} \frac{[2(3+h)^2 - (3+h)] - 15}{h}$

Exercise:

Problem: $\lim_{h \rightarrow 0} \frac{e^h - 1}{h}$

Solution:

$$f(x) = e^x, a = 0$$

For the following functions,

- sketch the graph and
- use the definition of a derivative to show that the function is not differentiable at $x = 1$.

Exercise:

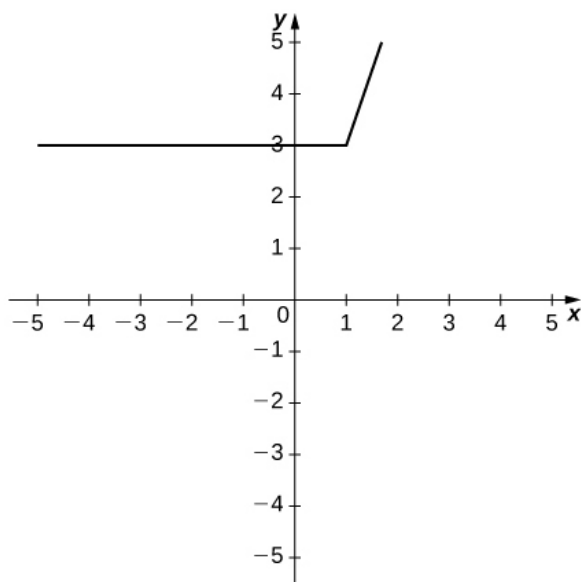
Problem: $f(x) = \begin{cases} 2\sqrt{x}, & 0 \leq x \leq 1 \\ 3x - 1, & x > 1 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} 3, & x < 1 \\ 3x, & x \geq 1 \end{cases}$

Solution:

a.



b. $\lim_{h \rightarrow 1^-} \frac{3-3}{h} \neq \lim_{h \rightarrow 1^+} \frac{3h}{h}$

Exercise:

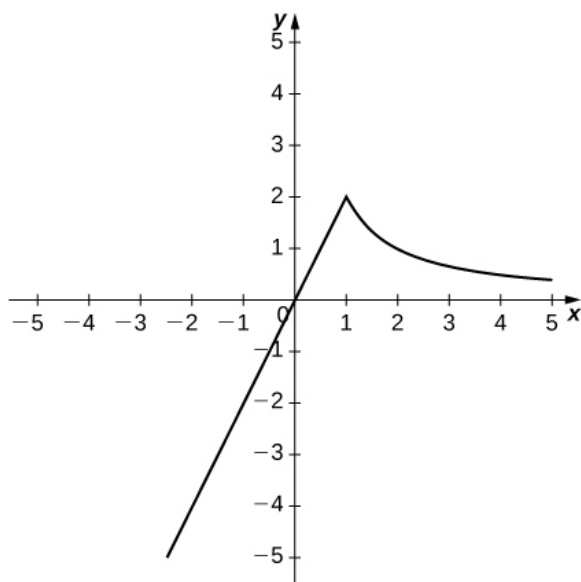
Problem: $f(x) = \begin{cases} -x^2 + 2, & x \leq 1 \\ x, & x > 1 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} 2x, & x \leq 1 \\ \frac{2}{x}, & x > 1 \end{cases}$

Solution:

a.



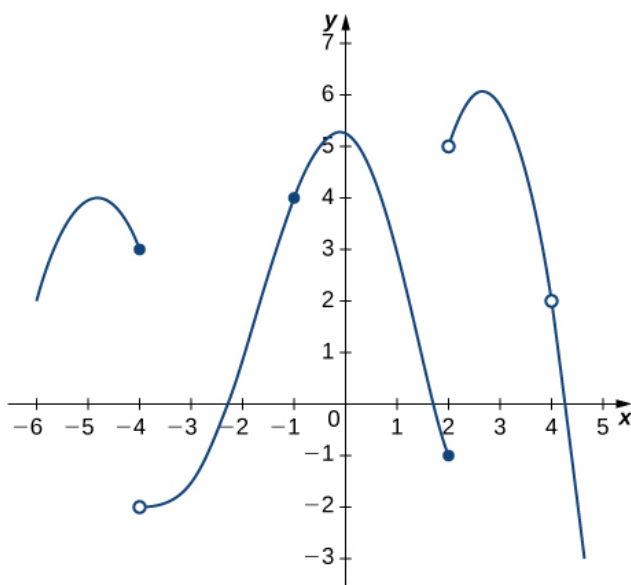
b. $\lim_{h \rightarrow 1^-} \frac{2h}{h} \neq \lim_{h \rightarrow 1^+} \frac{\frac{2}{x+h} - \frac{2}{x}}{h}.$

For the following graphs,

- determine for which values of $x = a$ the $\lim_{x \rightarrow a} f(x)$ exists but f is not continuous at $x = a$, and
- determine for which values of $x = a$ the function is continuous but not differentiable at $x = a$.

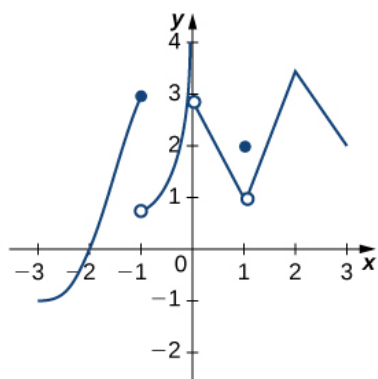
Exercise:

Problem:



Exercise:

Problem:

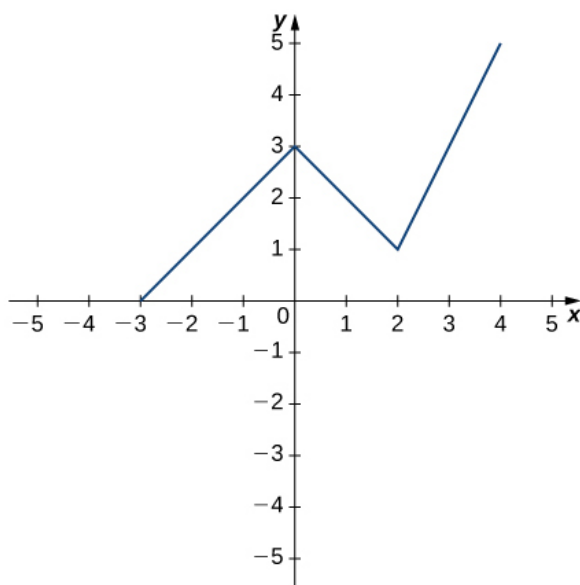


Solution:

a. $x = 1$, b. $x = 2$

Exercise:

Problem: Use the graph to evaluate a. $f'(-0.5)$, b. $f'(0)$, c. $f'(1)$, d. $f'(2)$, and e. $f'(3)$, if it exists.



For the following functions, use $f''(x) = \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h}$ to find $f''(x)$.

Exercise:

Problem: $f(x) = 2 - 3x$

Solution:

0

Exercise:

Problem: $f(x) = 4x^2$

Exercise:

Problem: $f(x) = x + \frac{1}{x}$

Solution:

$$\frac{2}{x^3}$$

For the following exercises, use a calculator to graph $f(x)$. Determine the function $f'(x)$, then use a calculator to graph $f'(x)$.

Exercise:

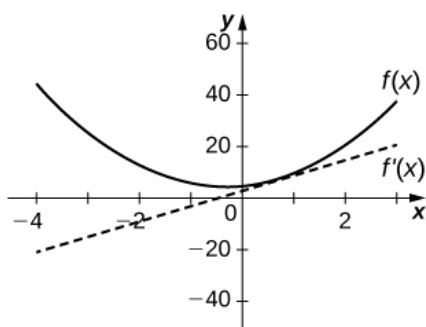
Problem: [T] $f(x) = -\frac{5}{x}$

Exercise:

Problem: [T] $f(x) = 3x^2 + 2x + 4$.

Solution:

$$f'(x) = 6x + 2$$

**Exercise:**

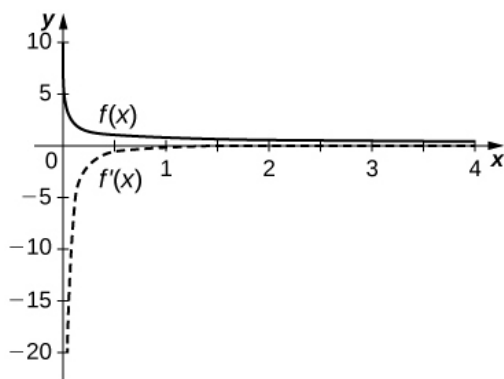
Problem: [T] $f(x) = \sqrt{x} + 3x$

Exercise:

Problem: [T] $f(x) = \frac{1}{\sqrt{2x}}$

Solution:

$$f'(x) = -\frac{1}{(2x)^{3/2}}$$



Exercise:

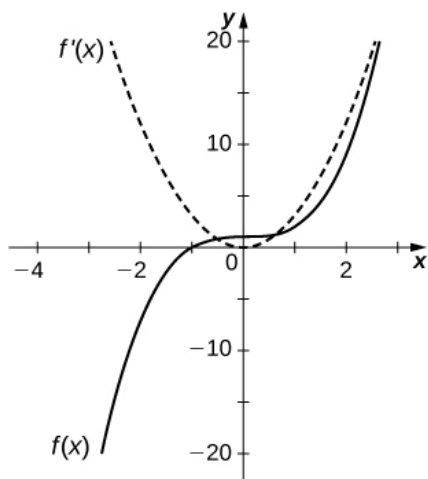
Problem: [T] $f(x) = 1 + x + \frac{1}{x}$

Exercise:

Problem: [T] $f(x) = x^3 + 1$

Solution:

$$f'(x) = 3x^2$$



For the following exercises, describe what the two expressions represent in terms of each of the given situations. Be sure to include units.

a. $\frac{f(x+h)-f(x)}{h}$

b. $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$

Exercise:

Problem: $P(x)$ denotes the population of a city at time x in years.

Exercise:

Problem:

$C(x)$ denotes the total amount of money (in thousands of dollars) spent on concessions by x customers at an amusement park.

Solution:

a. Average rate at which customers spent on concessions in thousands per customer. b. Rate (in thousands per customer) at which x customers spent money on concessions in thousands per customer.

Exercise:

Problem: $R(x)$ denotes the total cost (in thousands of dollars) of manufacturing x clock radios.

Exercise:

Problem: $g(x)$ denotes the grade (in percentage points) received on a test, given x hours of studying.

Solution:

a. Average grade received on the test with an average study time between two values. b. Rate (in percentage points per hour) at which the grade on the test increased or decreased for a given average study time of x hours.

Exercise:**Problem:**

$B(x)$ denotes the cost (in dollars) of a sociology textbook at university bookstores in the United States in x years since 1990.

Exercise:

Problem: $p(x)$ denotes atmospheric pressure at an altitude of x feet.

Solution:

a. Average change of atmospheric pressure between two different altitudes. b. Rate (torr per foot) at which atmospheric pressure is increasing or decreasing at x feet.

Exercise:

Problem: Sketch the graph of a function $y = f(x)$ with all of the following properties:

- a. $f'(x) > 0$ for $-2 \leq x < 1$
- b. $f'(2) = 0$
- c. $f'(x) > 0$ for $x > 2$
- d. $f(2) = 2$ and $f(0) = 1$
- e. $\lim_{x \rightarrow -\infty} f(x) = 0$ and $\lim_{x \rightarrow \infty} f(x) = \infty$
- f. $f'(1)$ does not exist.

Exercise:**Problem:**

Suppose temperature T in degrees Fahrenheit at a height x in feet above the ground is given by $y = T(x)$.

- a. Give a physical interpretation, with units, of $T'(x)$.
- b. If we know that $T'(1000) = -0.1$, explain the physical meaning.

Solution:

- a. The rate (in degrees per foot) at which temperature is increasing or decreasing for a given height x .
- b. The rate of change of temperature as altitude changes at 1000 feet is -0.1 degrees per foot.

Exercise:

Problem:

Suppose the total profit of a company is $y = P(x)$ thousand dollars when x units of an item are sold.

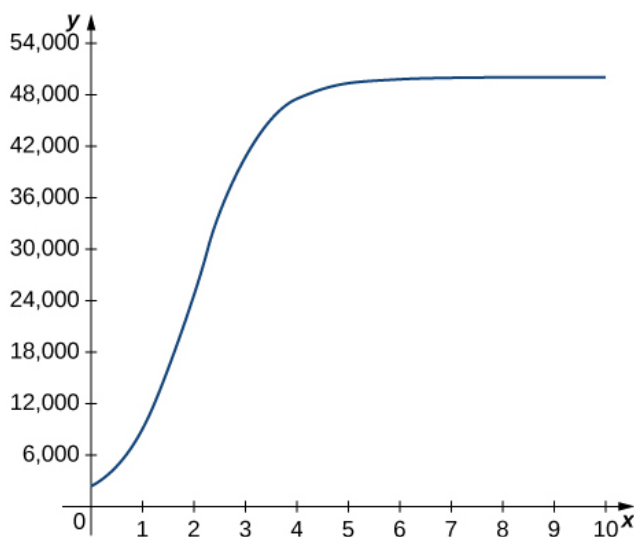
- a. What does $\frac{P(b)-P(a)}{b-a}$ for $0 < a < b$ measure, and what are the units?
- b. What does $P'(x)$ measure, and what are the units?
- c. Suppose that $P'(30) = 5$, what is the approximate change in profit if the number of items sold increases from 30 to 31?

Exercise:

Problem:

The graph in the following figure models the number of people $N(t)$ who have come down with the flu t weeks after its initial outbreak in a town with a population of 50,000 citizens.

- a. Describe what $N'(t)$ represents and how it behaves as t increases.
- b. What does the derivative tell us about how this town is affected by the flu outbreak?



Solution:

- a. The rate at which the number of people who have come down with the flu is changing t weeks after the initial outbreak.
- b. The rate is increasing sharply up to the third week, at which point it slows down and then becomes constant.

For the following exercises, use the following table, which shows the height h of the Saturn V rocket for the Apollo 11 mission t seconds after launch.

Time (seconds)	Height (meters)
0	0
1	2
2	4
3	13
4	25
5	32

Exercise:

Problem: What is the physical meaning of $h'(t)$? What are the units?

Exercise:

Problem:

[T] Construct a table of values for $h'(t)$ and graph both $h(t)$ and $h'(t)$ on the same graph. (*Hint:* for interior points, estimate both the left limit and right limit and average them.)

Solution:

Time (seconds)	$h'(t)$ (m/s)
0	2
1	2
2	5.5
3	10.5
4	9.5
5	7

Exercise:**Problem:**

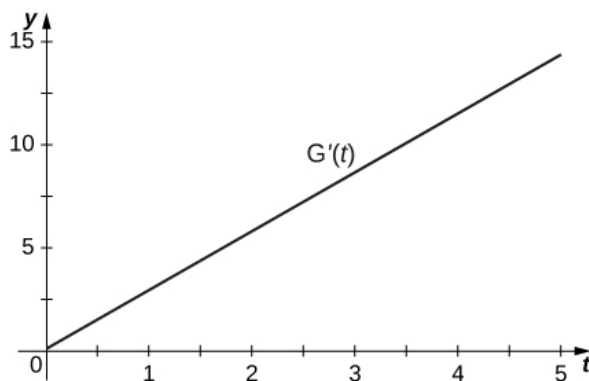
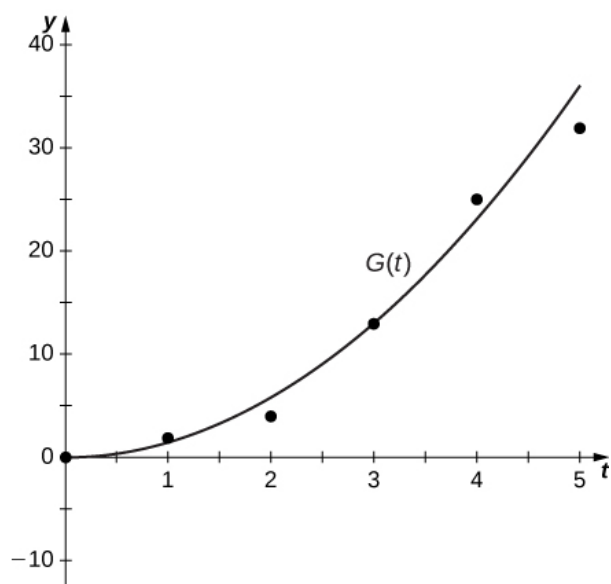
[T] The best linear fit to the data is given by $H(t) = 7.229t - 4.905$, where H is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $H'(t)$. Graph $H(t)$ with the given data and, on a separate coordinate plane, graph $H'(t)$.

Exercise:**Problem:**

[T] The best quadratic fit to the data is given by $G(t) = 1.429t^2 + 0.0857t - 0.1429$, where G is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $G'(t)$. Graph $G(t)$ with the given data and, on a separate coordinate plane, graph $G'(t)$.

Solution:

$$G'(t) = 2.858t + 0.0857$$

**Exercise:**

Problem:

[T] The best cubic fit to the data is given by $F(t) = 0.2037t^3 + 2.956t^2 - 2.705t + 0.4683$, where F is the height of the rocket (in m) and t is the time elapsed since take off. From this equation, determine $F'(t)$. Graph $F(t)$ with the given data and, on a separate coordinate plane, graph $F'(t)$. Does the linear, quadratic, or cubic function fit the data best?

Exercise:**Problem:**

Using the best linear, quadratic, and cubic fits to the data, determine what $H''(t)$, $G''(t)$ and $F''(t)$ are. What are the physical meanings of $H''(t)$, $G''(t)$ and $F''(t)$, and what are their units?

Solution:

$H''(t) = 0$, $G''(t) = 2.858$ and $f''(t) = 1.222t + 5.912$ represent the acceleration of the rocket, with units of meters per second squared (m/s^2).

Glossary

derivative function

gives the derivative of a function at each point in the domain of the original function for which the derivative is defined

differentiable at a

a function for which $f'(a)$ exists is differentiable at a

differentiable on S

a function for which $f'(x)$ exists for each x in the open set S is differentiable on S

differentiable function

a function for which $f'(x)$ exists is a differentiable function

higher-order derivative

a derivative of a derivative, from the second derivative to the n th derivative, is called a higher-order derivative

Differentiation Rules

- State the constant, constant multiple, and power rules.
- Apply the sum and difference rules to combine derivatives.
- Use the product rule for finding the derivative of a product of functions.
- Use the quotient rule for finding the derivative of a quotient of functions.
- Extend the power rule to functions with negative exponents.
- Combine the differentiation rules to find the derivative of a polynomial or rational function.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ by using a process that involved multiplying an expression by a conjugate prior to evaluating a limit. The process that we could use to evaluate $\frac{d}{dx}(\sqrt[3]{x})$ using the definition, while similar, is more complicated. In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

The Basic Rules

The functions $f(x) = c$ and $g(x) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f(x) = c$. For this function, both $f(x) = c$ and $f(x+h) = c$, so we obtain the following result:

Equation:

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{c - c}{h} \\&= \lim_{h \rightarrow 0} \frac{0}{h} \\&= \lim_{h \rightarrow 0} 0 = 0.\end{aligned}$$

The rule for differentiating constant functions is called the **constant rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0. We restate this rule in the following theorem.

Note:

The Constant Rule

Let c be a constant.

If $f(x) = c$, then $f'(c) = 0$.

Alternatively, we may express this rule as

Equation:

$$\frac{d}{dx}(c) = 0.$$

Example:**Exercise:****Problem:****Applying the Constant Rule**

Find the derivative of $f(x) = 8$.

Solution:

This is just a one-step application of the rule:

Equation:

$$f'(8) = 0.$$

Note:**Exercise:**

Problem: Find the derivative of $g(x) = -3$.

Solution:

0

Hint

Use the preceding example as a guide.

The Power Rule

We have shown that

Equation:

$$\frac{d}{dx}(x^2) = 2x \text{ and } \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}(x^n)$. We continue our examination of derivative formulas by differentiating power functions of the form $f(x) = x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}(x^3)$. As we go through this derivation, pay special attention to the portion of the expression in boldface, as the technique used in this case is essentially the same as the technique used to prove the general case.

Example:**Exercise:****Problem:****Differentiating x^3**

Find $\frac{d}{dx}(x^3)$.

Solution:

Equation:

$$\begin{aligned}\frac{d}{dx}(x^3) &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\&= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\&= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\&= \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\&= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) \\&= 3x^2\end{aligned}$$

Notice that the first term in the expansion of $(x+h)^3$ is x^3 and the second term is $3x^2h$. All other terms contain powers of h that are two or greater.

In this step the x^3 terms have been cancelled, leaving only terms containing h .

Factor out the common factor of h .

After cancelling the common factor of h , the only term not containing h is $3x^2$.

Let h go to 0.

Note:

Exercise:

Problem: Find $\frac{d}{dx}(x^4)$.

Solution:

$$4x^3$$

Hint

Use $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$ and follow the procedure outlined in the preceding example.

As we shall see, the procedure for finding the derivative of the general form $f(x) = x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x) = x^3$, the power on x becomes the coefficient of x^2 in the derivative and the power on x in the derivative decreases by 1. The following theorem states that the **power rule** holds for all positive integer powers of x . We will eventually extend this result to negative integer powers. Later, we will see that this rule may also be extended first to rational powers of x and then to arbitrary powers of x . Be aware, however, that this rule does not apply to functions in which a constant is raised to a variable power, such as $f(x) = 3^x$.

Note:

The Power Rule

Let n be a positive integer. If $f(x) = x^n$, then

Equation:

$$f'(x) = nx^{n-1}.$$

Alternatively, we may express this rule as

Equation:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof

For $f(x) = x^n$ where n is a positive integer, we have

Equation:

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}.$$

Equation:

$$\text{Since } (x+h)^n = x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \dots + nxh^{n-1} + h^n,$$

we see that

Equation:

$$(x+h)^n - x^n = nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \dots + nxh^{n-1} + h^n.$$

Next, divide both sides by h :

Equation:

$$\frac{(x+h)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \dots + nxh^{n-1} + h^n}{h}.$$

Thus,

Equation:

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \dots + nxh^{n-2} + h^{n-1}.$$

Finally,

Equation:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \binom{n}{3}x^{n-3}h^2 + \dots + nxh^{n-2} + h^{n-1} \right) \\ &= nx^{n-1}. \end{aligned}$$

□

Example:**Exercise:****Problem:****Applying the Power Rule**

Find the derivative of the function $f(x) = x^{10}$ by applying the power rule.

Solution:

Using the power rule with $n = 10$, we obtain

Equation:

$$f'(x) = 10x^{10-1} = 10x^9.$$

Note:**Exercise:**

Problem: Find the derivative of $f(x) = x^7$.

Solution:

$$f'(x) = 7x^6$$

Hint

Use the power rule with $n = 7$.

The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Note:**Sum, Difference, and Constant Multiple Rules**

Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.

Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g .

Equation:

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x));$$

that is,

Equation:

$$\text{for } j(x) = f(x) + g(x), j'(x) = f'(x) + g'(x).$$

Difference Rule. The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g :

Equation:

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x));$$

that is,

Equation:

$$\text{for } j(x) = f(x) - g(x), j'(x) = f'(x) - g'(x).$$

Constant Multiple Rule. The derivative of a constant k multiplied by a function f is the same as the constant multiplied by the derivative:

Equation:

$$\frac{d}{dx}(kf(x)) = k \frac{d}{dx}(f(x));$$

that is,

Equation:

$$\text{for } j(x) = kf(x), j'(x) = kf'(x).$$

Proof

We provide only the proof of the sum rule here. The rest follow in a similar manner.

For differentiable functions $f(x)$ and $g(x)$, we set $j(x) = f(x) + g(x)$. Using the limit definition of the derivative we have

Equation:

$$j'(x) = \lim_{h \rightarrow 0} \frac{j(x+h) - j(x)}{h}.$$

By substituting $j(x+h) = f(x+h) + g(x+h)$ and $j(x) = f(x) + g(x)$, we obtain

Equation:

$$j'(x) = \lim_{h \rightarrow 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}.$$

Rearranging and regrouping the terms, we have

Equation:

$$j'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right).$$

We now apply the sum law for limits and the definition of the derivative to obtain

Equation:

$$j'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right) + \lim_{h \rightarrow 0} \left(\frac{g(x+h) - g(x)}{h} \right) = f'(x) + g'(x).$$

□

Example:

Exercise:

Problem:

Applying the Constant Multiple Rule

Find the derivative of $g(x) = 3x^2$ and compare it to the derivative of $f(x) = x^2$.

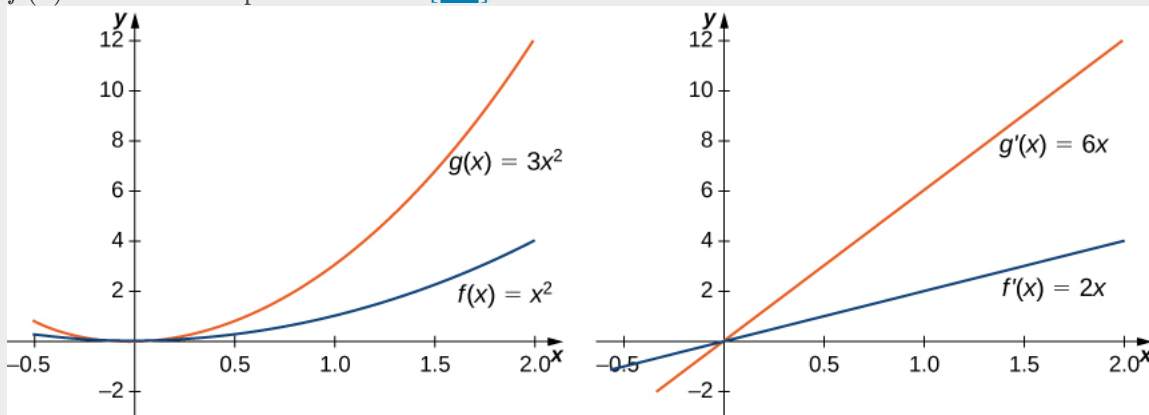
Solution:

We use the power rule directly:

Equation:

$$g'(x) = \frac{d}{dx}(3x^2) = 3 \frac{d}{dx}(x^2) = 3(2x) = 6x.$$

Since $f(x) = x^2$ has derivative $f'(x) = 2x$, we see that the derivative of $g(x)$ is 3 times the derivative of $f(x)$. This relationship is illustrated in [\[link\]](#).



The derivative of $g(x)$ is 3 times the derivative of $f(x)$.

Example:

Exercise:

Problem:

Applying Basic Derivative Rules

Find the derivative of $f(x) = 2x^5 + 7$.

Solution:

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the sequence in which the differentiation rules are applied, we use Leibniz notation throughout the solution:

Equation:

$$\begin{aligned}
 f'(x) &= \frac{d}{dx}(2x^5 + 7) \\
 &= \frac{d}{dx}(2x^5) + \frac{d}{dx}(7) && \text{Apply the sum rule.} \\
 &= 2 \frac{d}{dx}(x^5) + \frac{d}{dx}(7) && \text{Apply the constant multiple rule.} \\
 &= 2(5x^4) + 0 && \text{Apply the power rule and the constant rule.} \\
 &= 10x^4. && \text{Simplify.}
 \end{aligned}$$

Note:

Exercise:

Problem: Find the derivative of $f(x) = 2x^3 - 6x^2 + 3$.

Solution:

$$f'(x) = 6x^2 - 12x.$$

Hint

Use the preceding example as a guide.

Example:

Exercise:

Problem:
Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$.

Solution:

To find the equation of the tangent line, we need a point and a slope. To find the point, compute

Equation:

$$f(1) = 1^2 - 4(1) + 6 = 3.$$

This gives us the point $(1, 3)$. Since the slope of the tangent line at 1 is $f'(1)$, we must first find $f'(x)$.

Using the definition of a derivative, we have

Equation:

$$f'(x) = 2x - 4$$

so the slope of the tangent line is $f'(1) = -2$. Using the point-slope formula, we see that the equation of the tangent line is

Equation:

$$y - 3 = -2(x - 1).$$

Putting the equation of the line in slope-intercept form, we obtain

Equation:

$$y = -2x + 5.$$

Note:

Exercise:

Problem:

Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 11$ at $x = 2$. Use the point-slope form.

Solution:

$$y = 12x - 23$$

Hint

Use the preceding example as a guide.

The Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the sum and difference rules, the **product rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x) = x^2$, whose derivative is $f'(x) = 2x$ and not $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$.

Note:

Product Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

Equation:

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x).$$

That is,

Equation:

$$\text{if } j(x) = f(x)g(x), \text{ then } j'(x) = f'(x)g(x) + g'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Proof

We begin by assuming that $f(x)$ and $g(x)$ are differentiable functions. At a key point in this proof we need to use the fact that, since $g(x)$ is differentiable, it is also continuous. In particular, we use the fact that since $g(x)$ is continuous, $\lim_{h \rightarrow 0} g(x+h) = g(x)$.

By applying the limit definition of the derivative to $j(x) = f(x)g(x)$, we obtain

Equation:

$$j'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x+h)$ in the numerator, we have

Equation:

$$j'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

After breaking apart this quotient and applying the sum law for limits, the derivative becomes

Equation:

$$j'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} \right) + \lim_{h \rightarrow 0} \left(\frac{f(x)g(x+h) - f(x)g(x)}{h} \right).$$

Rearranging, we obtain

Equation:

$$j'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) \right) + \lim_{h \rightarrow 0} \left(\frac{g(x+h) - g(x)}{h} \cdot f(x) \right).$$

By using the continuity of $g(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the limit laws, we arrive at the product rule,

Equation:

$$j'(x) = f'(x)g(x) + g'(x)f(x).$$

□

Example:

Exercise:

Problem:

Applying the Product Rule to Functions at a Point

For $j(x) = f(x)g(x)$, use the product rule to find $j'(2)$ if $f(2) = 3$, $f'(2) = -4$, $g(2) = 1$, and $g'(2) = 6$.

Solution:

Since $j(x) = f(x)g(x)$, $j'(x) = f'(x)g(x) + g'(x)f(x)$, and hence

Equation:

$$j'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = 14.$$

Example:

Exercise:

Problem:
Applying the Product Rule to Binomials

For $j(x) = (x^2 + 2)(3x^3 - 5x)$, find $j'(x)$ by applying the product rule. Check the result by first finding the product and then differentiating.

Solution:

If we set $f(x) = x^2 + 2$ and $g(x) = 3x^3 - 5x$, then $f'(x) = 2x$ and $g'(x) = 9x^2 - 5$. Thus,

Equation:

$$j'(x) = f'(x)g(x) + g'(x)f(x) = (2x)(3x^3 - 5x) + (9x^2 - 5)(x^2 + 2).$$

Simplifying, we have

Equation:

$$j'(x) = 15x^4 + 3x^2 - 10.$$

To check, we see that $j(x) = 3x^5 + x^3 - 10x$ and, consequently, $j'(x) = 15x^4 + 3x^2 - 10$.

Note:

Exercise:

Problem: Use the product rule to obtain the derivative of $j(x) = 2x^5(4x^2 + x)$.

Solution:

$$j'(x) = 10x^4(4x^2 + x) + (8x + 1)(2x^5) = 56x^6 + 12x^5.$$

Hint

Set $f(x) = 2x^5$ and $g(x) = 4x^2 + x$ and use the preceding example as a guide.

The Quotient Rule

Having developed and practiced the product rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that

Equation:

$$\frac{d}{dx}(x^2) = 2x, \text{ not } \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x)} = \frac{3x^2}{1} = 3x^2.$$

Note:

The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

Equation:

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - \frac{d}{dx}(g(x)) \cdot f(x)}{(g(x))^2}.$$

That is,

Equation:

$$\text{if } j(x) = \frac{f(x)}{g(x)}, \text{ then } j'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}.$$

The proof of the **quotient rule** is very similar to the proof of the product rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

Example:

Exercise:

Problem:

Applying the Quotient Rule

Use the quotient rule to find the derivative of $k(x) = \frac{5x^2}{4x+3}$.

Solution:

Let $f(x) = 5x^2$ and $g(x) = 4x + 3$. Thus, $f'(x) = 10x$ and $g'(x) = 4$. Substituting into the quotient rule, we have

Equation:

$$k'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} = \frac{10x(4x+3) - 4(5x^2)}{(4x+3)^2}.$$

Simplifying, we obtain

Equation:

$$k'(x) = \frac{20x^2 + 30x}{(4x+3)^2}.$$

Note:

Exercise:

Problem: Find the derivative of $h(x) = \frac{3x+1}{4x-3}$.

Solution:

$$h'(x) = -\frac{13}{(4x-3)^2}.$$

Hint

Apply the quotient rule with $f(x) = 3x + 1$ and $g(x) = 4x - 3$.

It is now possible to use the quotient rule to extend the power rule to find derivatives of functions of the form x^k where k is a negative integer.

Note:**Extended Power Rule**

If k is a negative integer, then

Equation:

$$\frac{d}{dx}(x^k) = kx^{k-1}.$$

Proof

If k is a negative integer, we may set $n = -k$, so that n is a positive integer with $k = -n$. Since for each positive integer n , $x^{-n} = \frac{1}{x^n}$, we may now apply the quotient rule by setting $f(x) = 1$ and $g(x) = x^n$. In this case, $f'(x) = 0$ and $g'(x) = nx^{n-1}$. Thus,

Equation:

$$\frac{d}{dx}(x^{-n}) = \frac{0(x^n) - 1(nx^{n-1})}{(x^n)^2}.$$

Simplifying, we see that

Equation:

$$\frac{d}{dx}(x^{-n}) = \frac{-nx^{n-1}}{x^{2n}} = -nx^{(n-1)-2n} = -nx^{-n-1}.$$

Finally, observe that since $k = -n$, by substituting we have

Equation:

$$\frac{d}{dx}(x^k) = kx^{k-1}.$$

□

Example:**Exercise:****Problem:****Using the Extended Power Rule**

Find $\frac{d}{dx}(x^{-4})$.

Solution:

By applying the extended power rule with $k = -4$, we obtain

Equation:

$$\frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5}.$$

Example:**Exercise:****Problem:****Using the Extended Power Rule and the Constant Multiple Rule**

Use the extended power rule and the constant multiple rule to find $f'(x) = \frac{6}{x^2}$.

Solution:

It may seem tempting to use the quotient rule to find this derivative, and it would certainly not be incorrect to do so. However, it is far easier to differentiate this function by first rewriting it as $f(x) = 6x^{-2}$.

Equation:

$f'(x) = \frac{d}{dx}\left(\frac{6}{x^2}\right) = \frac{d}{dx}(6x^{-2})$	Rewrite $\frac{6}{x^2}$ as $6x^{-2}$.
$= 6 \frac{d}{dx}(x^{-2})$	Apply the constant multiple rule.
$= 6(-2x^{-3})$	Use the extended power rule to differentiate x^{-2} .
$= -12x^{-3}$	Simplify.

Note:**Exercise:**

Problem: Find the derivative of $g(x) = \frac{1}{x^7}$ using the extended power rule.

Solution:

$$g'(x) = -7x^{-8}.$$

Hint

Rewrite $g(x) = \frac{1}{x^7} = x^{-7}$. Use the extended power rule with $k = -7$.

Combining Differentiation Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one differentiation rule to find the derivative of a given function. At this point, by combining the differentiation rules, we may find the derivatives of any polynomial or rational function. Later on we will encounter more complex combinations of differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

Example:**Exercise:****Problem:****Combining Differentiation Rules**

For $k(x) = 3h(x) + x^2g(x)$, find $k'(x)$.

Solution:

Finding this derivative requires the sum rule, the constant multiple rule, and the product rule.

Equation:

$$\begin{aligned}
 k'(x) &= \frac{d}{dx}(3h(x) + x^2g(x)) = \frac{d}{dx}(3h(x)) + \frac{d}{dx}(x^2g(x)) && \text{Apply the sum rule.} \\
 &= 3\frac{d}{dx}(h(x)) + \left(\frac{d}{dx}(x^2)g(x) + \frac{d}{dx}(g(x))x^2\right) && \text{Apply the constant multiple rule to} \\
 &= 3h'(x) + 2xg(x) + g'(x)x^2 && \text{differentiate } 3h(x) \text{ and the product} \\
 & && \text{rule to differentiate } x^2g(x).
 \end{aligned}$$

Example:**Exercise:****Problem:****Extending the Product Rule**

For $k(x) = f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x)$, $g(x)$, $h(x)$, and their derivatives.

Solution:

We can think of the function $k(x)$ as the product of the function $f(x)g(x)$ and the function $h(x)$. That is, $k(x) = (f(x)g(x)) \cdot h(x)$. Thus,

Equation:

$$\begin{aligned}
 k'(x) &= \frac{d}{dx}(f(x)g(x)) \cdot h(x) + \frac{d}{dx}(h(x)) \cdot (f(x)g(x)) && \text{Apply the product rule to the product} \\
 &= (f'(x)g(x) + g'(x)f(x))h(x) + h'(x)f(x)g(x) && \text{of } f(x)g(x) \text{ and } h(x). \\
 &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x). && \text{Apply the product rule to } f(x)g(x). \\
 & && \text{Simplify.}
 \end{aligned}$$

Example:**Exercise:****Problem:****Combining the Quotient Rule and the Product Rule**

For $h(x) = \frac{2x^3k(x)}{3x+2}$, find $h'(x)$.

Solution:

This procedure is typical for finding the derivative of a rational function.

Equation:

$$\begin{aligned}
 h'(x) &= \frac{\frac{d}{dx}(2x^3k(x)) \cdot (3x+2) - \frac{d}{dx}(3x+2) \cdot (2x^3k(x))}{(3x+2)^2} \\
 &= \frac{(6x^2k(x) + k'(x) \cdot 2x^3)(3x+2) - 3(2x^3k(x))}{(3x+2)^2} \\
 &= \frac{-6x^3k(x) + 18x^3k(x) + 12x^2k(x) + 6x^4k'(x) + 4x^3k'(x)}{(3x+2)^2}
 \end{aligned}$$

Apply the quotient rule.

Apply the product rule to find

$\frac{d}{dx}(2x^3k(x))$. Use $\frac{d}{dx}(3x+2) = 3$.

Simplify.

Note:**Exercise:**

Problem: Find $\frac{d}{dx}(3f(x) - 2g(x))$.

Solution:

$$3f'(x) - 2g'(x).$$

Hint

Apply the difference rule and the constant multiple rule.

Example:**Exercise:****Problem:****Determining Where a Function Has a Horizontal Tangent**

Determine the values of x for which $f(x) = x^3 - 7x^2 + 8x + 1$ has a horizontal tangent line.

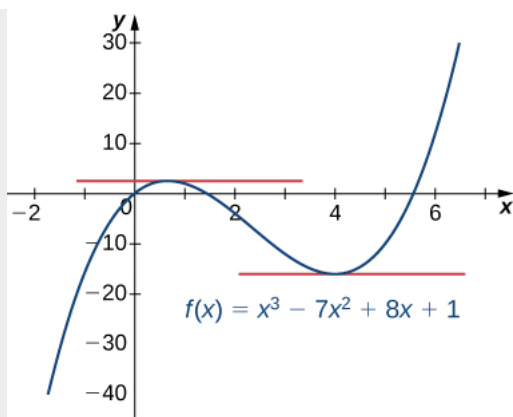
Solution:

To find the values of x for which $f(x)$ has a horizontal tangent line, we must solve $f'(x) = 0$. Since

Equation:

$$f'(x) = 3x^2 - 14x + 8 = (3x - 2)(x - 4),$$

we must solve $(3x - 2)(x - 4) = 0$. Thus we see that the function has horizontal tangent lines at $x = \frac{2}{3}$ and $x = 4$ as shown in the following graph.



This function has horizontal tangent lines at $x = 2/3$ and $x = 4$.

Example:

Exercise:

Problem: Finding a Velocity

The position of an object on a coordinate axis at time t is given by $s(t) = \frac{t}{t^2+1}$. What is the initial velocity of the object?

Solution:

Since the initial velocity is $v(0) = s'(0)$, begin by finding $s'(t)$ by applying the quotient rule:

Equation:

$$s'(t) = \frac{1(t^2+1) - 2t(t)}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2}.$$

After evaluating, we see that $v(0) = 1$.

Note:

Exercise:

Problem:

Find the values of x for which the line tangent to the graph of $f(x) = 4x^2 - 3x + 2$ has a tangent line parallel to the line $y = 2x + 3$.

Solution:

$$\frac{5}{8}$$

Hint

Solve $f'(x) = 2$.

Note:

Formula One Grandstands

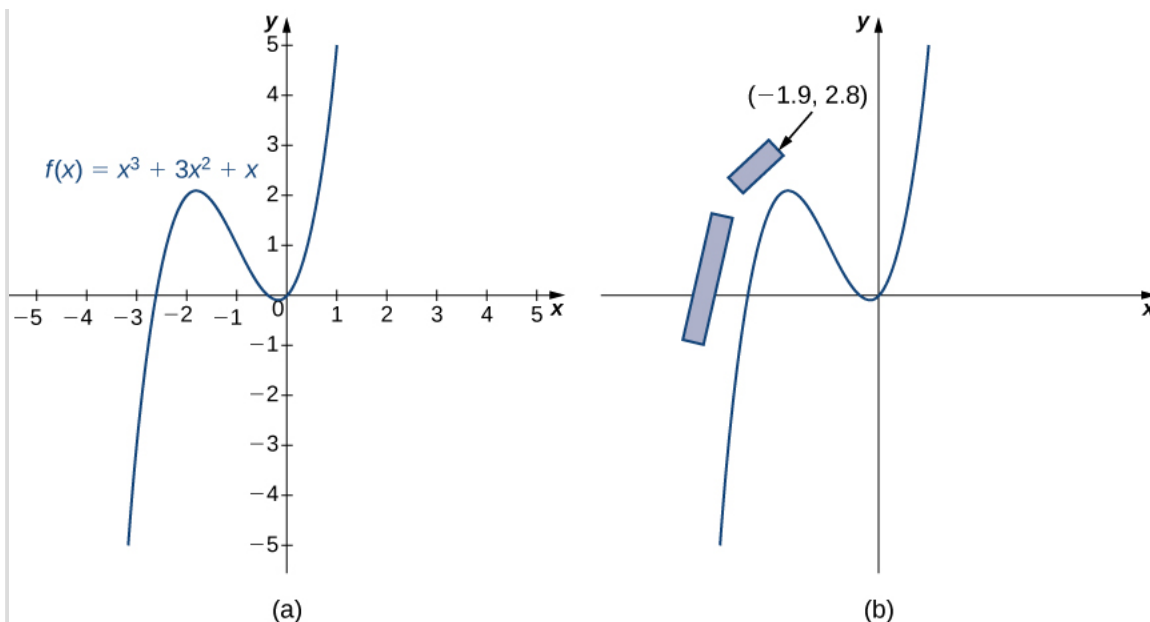
Formula One car races can be very exciting to watch and attract a lot of spectators. Formula One track designers have to ensure sufficient grandstand space is available around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car ([link](#)).



The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, located where the spectators are not in danger.

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this just results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path tangent to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the function $f(x) = x^3 + 3x^2 + x$ ([link](#)). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9, 2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.



(a) One section of the racetrack can be modeled by the function $f(x) = x^3 + 3x^2 + x$. (b) The front corner of the grandstand is located at $(-1.9, 2.8)$.

1. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn, at the point where the slope of the tangent line is 1. Find the (x, y) coordinates of this point near the turn.
2. Find the equation of the tangent line to the curve at this point.
3. To determine whether the spectators are in danger in this scenario, find the x -coordinate of the point where the tangent line crosses the line $y = 2.8$. Is this point safely to the right of the grandstand? Or are the spectators in danger?
4. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5, 0.625)$. What is the slope of the tangent line at this point?
5. If a driver loses control as described in part 4, are the spectators safe?
6. Should you proceed with the current design for the grandstand, or should the grandstands be moved?

Key Concepts

- The derivative of a constant function is zero.
- The derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1.
- The derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative.
- The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g .
- The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g .
- The derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.
- The derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second

function.

- We used the limit definition of the derivative to develop formulas that allow us to find derivatives without resorting to the definition of the derivative. These formulas can be used singly or in combination with each other.

For the following exercises, find $f'(x)$ for each function.

Exercise:

Problem: $f(x) = x^7 + 10$

Exercise:

Problem: $f(x) = 5x^3 - x + 1$

Solution:

$$f'(x) = 15x^2 - 1$$

Exercise:

Problem: $f(x) = 4x^2 - 7x$

Exercise:

Problem: $f(x) = 8x^4 + 9x^2 - 1$

Solution:

$$f'(x) = 32x^3 + 18x$$

Exercise:

Problem: $f(x) = x^4 + \frac{2}{x}$

Exercise:

Problem: $f(x) = 3x \left(18x^4 + \frac{13}{x+1} \right)$

Solution:

$$f'(x) = 270x^4 + \frac{39}{(x+1)^2}$$

Exercise:

Problem: $f(x) = (x+2)(2x^2-3)$

Exercise:

Problem: $f(x) = x^2 \left(\frac{2}{x^2} + \frac{5}{x^3} \right)$

Solution:

$$f'(x) = \frac{-5}{x^2}$$

Exercise:

Problem: $f(x) = \frac{x^3 + 2x^2 - 4}{3}$

Exercise:

Problem: $f(x) = \frac{4x^3 - 2x + 1}{x^2}$

Solution:

$$f'(x) = \frac{4x^4 + 2x^2 - 2x}{x^4}$$

Exercise:

Problem: $f(x) = \frac{x^2 + 4}{x^2 - 4}$

Exercise:

Problem: $f(x) = \frac{x + 9}{x^2 - 7x + 1}$

Solution:

$$f'(x) = \frac{-x^2 - 18x + 64}{(x^2 - 7x + 1)^2}$$

For the following exercises, find the equation of the tangent line $T(x)$ to the graph of the given function at the indicated point. Use a graphing calculator to graph the function and the tangent line.

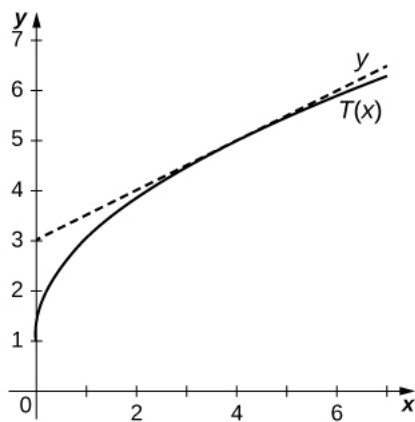
Exercise:

Problem: [T] $y = 3x^2 + 4x + 1$ at $(0, 1)$

Exercise:

Problem: [T] $y = 2\sqrt{x} + 1$ at $(4, 5)$

Solution:



$$T(x) = \frac{1}{2}x + 3$$

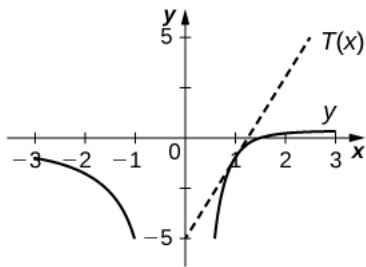
Exercise:

Problem: [T] $y = \frac{2x}{x-1}$ at $(-1, 1)$

Exercise:

Problem: [T] $y = \frac{2}{x} - \frac{3}{x^2}$ at $(1, -1)$

Solution:



$$T(x) = 4x - 5$$

For the following exercises, assume that $f(x)$ and $g(x)$ are both differentiable functions for all x . Find the derivative of each of the functions $h(x)$.

Exercise:

Problem: $h(x) = 4f(x) + \frac{g(x)}{7}$

Exercise:

Problem: $h(x) = x^3 f(x)$

Solution:

$$h'(x) = 3x^2 f(x) + x^3 f'(x)$$

Exercise:

Problem: $h(x) = \frac{f(x)g(x)}{2}$

Exercise:

Problem: $h(x) = \frac{3f(x)}{g(x)+2}$

Solution:

$$h'(x) = \frac{3f'(x)(g(x)+2) - 3f(x)g'(x)}{(g(x)+2)^2}$$

For the following exercises, assume that $f(x)$ and $g(x)$ are both differentiable functions with values as given in the following table. Use the following table to calculate the following derivatives.

x	1	2	3	4
$f(x)$	3	5	-2	0
$g(x)$	2	3	-4	6
$f'(x)$	-1	7	8	-3
$g'(x)$	4	1	2	9

Exercise:

Problem: Find $h'(1)$ if $h(x) = xf(x) + 4g(x)$.

Exercise:

Problem: Find $h'(2)$ if $h(x) = \frac{f(x)}{g(x)}$.

Solution:

$$\frac{16}{9}$$

Exercise:

Problem: Find $h'(3)$ if $h(x) = 2x + f(x)g(x)$.

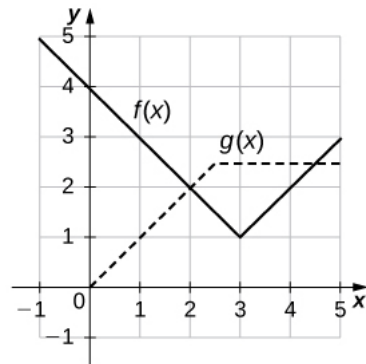
Exercise:

Problem: Find $h'(4)$ if $h(x) = \frac{1}{x} + \frac{g(x)}{f(x)}$.

Solution:

Undefined

For the following exercises, use the following figure to find the indicated derivatives, if they exist.



Exercise:

Problem: Let $h(x) = f(x) + g(x)$. Find

a. $h'(1)$,

- b. $h'(3)$, and
- c. $h'(4)$.

Exercise:

Problem: Let $h(x) = f(x)g(x)$. Find

- a. $h'(1)$,
- b. $h'(3)$, and
- c. $h'(4)$.

Solution:

- a. 2, b. does not exist, c. 2.5

Exercise:

Problem: Let $h(x) = \frac{f(x)}{g(x)}$. Find

- a. $h'(1)$,
- b. $h'(3)$, and
- c. $h'(4)$.

For the following exercises,

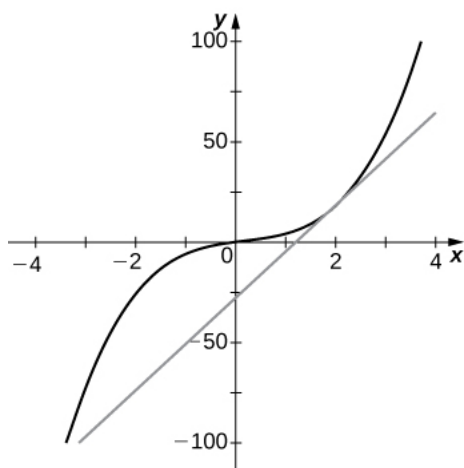
- a. evaluate $f'(a)$, and
- b. graph the function $f(x)$ and the tangent line at $x = a$.

Exercise:

Problem: [T] $f(x) = 2x^3 + 3x - x^2$, $a = 2$

Solution:

- a. 23, b. $y = 23x - 28$



Exercise:

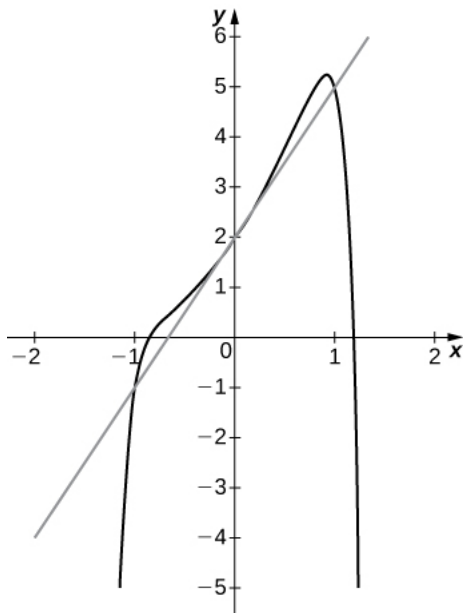
Problem: [T] $f(x) = \frac{1}{x} - x^2, a = 1$

Exercise:

Problem: [T] $f(x) = x^2 - x^{12} + 3x + 2, a = 0$

Solution:

a. 3, b. $y = 3x + 2$



Exercise:

Problem: [T] $f(x) = \frac{1}{x} - x^{2/3}, a = -1$

Exercise:

Problem: Find the equation of the tangent line to the graph of $f(x) = 2x^3 + 4x^2 - 5x - 3$ at $x = -1$.

Solution:

$$y = -7x - 3$$

Exercise:

Problem: Find the equation of the tangent line to the graph of $f(x) = x^2 + \frac{4}{x} - 10$ at $x = 8$.

Exercise:

Problem: Find the equation of the tangent line to the graph of $f(x) = (3x - x^2)(3 - x - x^2)$ at $x = 1$.

Solution:

$$y = -5x + 7$$

Exercise:**Problem:**

Find the point on the graph of $f(x) = x^3$ such that the tangent line at that point has an x intercept of 6.

Exercise:**Problem:**

Find the equation of the line passing through the point $P(3, 3)$ and tangent to the graph of $f(x) = \frac{6}{x-1}$.

Solution:

$$y = -\frac{3}{2}x + \frac{15}{2}$$

Exercise:**Problem:**

Determine all points on the graph of $f(x) = x^3 + x^2 - x - 1$ for which the slope of the tangent line is

- a. horizontal
- b. -1 .

Exercise:

Problem: Find a quadratic polynomial such that $f(1) = 5$, $f'(1) = 3$ and $f''(1) = -6$.

Solution:

$$y = -3x^2 + 9x - 1$$

Exercise:

Problem: A car driving along a freeway with traffic has traveled $s(t) = t^3 - 6t^2 + 9t$ meters in t seconds.

- a. Determine the time in seconds when the velocity of the car is 0.
- b. Determine the acceleration of the car when the velocity is 0.

Exercise:

Problem: [T] A herring swimming along a straight line has traveled $s(t) = \frac{t^2}{t^2+2}$ feet in t seconds.

Determine the velocity of the herring when it has traveled 3 seconds.

Solution:

$$\frac{12}{121} \text{ or } 0.0992 \text{ ft/s}$$

Exercise:**Problem:**

The population in millions of arctic flounder in the Atlantic Ocean is modeled by the function $P(t) = \frac{8t+3}{0.2t^2+1}$, where t is measured in years.

- a. Determine the initial flounder population.

- b. Determine $P'(10)$ and briefly interpret the result.

Exercise:

Problem:

[T] The concentration of antibiotic in the bloodstream t hours after being injected is given by the function $C(t) = \frac{2t^2+t}{t^3+50}$, where C is measured in milligrams per liter of blood.

- Find the rate of change of $C(t)$.
- Determine the rate of change for $t = 8, 12, 24$, and 36 .
- Briefly describe what seems to be occurring as the number of hours increases.

Solution:

- a. $\frac{-2t^4-2t^3+200t+50}{(t^3+50)^2}$ b. -0.02395 mg/L-hr, -0.01344 mg/L-hr, -0.003566 mg/L-hr, -0.001579 mg/L-hr c. The rate at which the concentration of drug in the bloodstream decreases is slowing to 0 as time increases.

Exercise:

Problem:

A book publisher has a cost function given by $C(x) = \frac{x^3+2x+3}{x^2}$, where x is the number of copies of a book in thousands and C is the cost, per book, measured in dollars. Evaluate $C'(2)$ and explain its meaning.

Exercise:

Problem:

[T] According to Newton's law of universal gravitation, the force F between two bodies of constant mass m_1 and m_2 is given by the formula $F = \frac{Gm_1m_2}{d^2}$, where G is the gravitational constant and d is the distance between the bodies.

- Suppose that G , m_1 , and m_2 are constants. Find the rate of change of force F with respect to distance d .
- Find the rate of change of force F with gravitational constant $G = 6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2$, on two bodies 10 meters apart, each with a mass of 1000 kilograms.

Solution:

- a. $F'(d) = \frac{-2Gm_1m_2}{d^3}$ b. $-1.33 \times 10^{-7} \text{ N/m}$

Glossary

constant multiple rule

the derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative: $\frac{d}{dx}(cf(x)) = cf'(x)$

constant rule

the derivative of a constant function is zero: $\frac{d}{dx}(c) = 0$, where c is a constant

difference rule

the derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g : $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$

power rule

the derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1: If n is an integer, then $\frac{d}{dx}x^n = nx^{n-1}$

product rule

the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x)$

quotient rule

the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

sum rule

the derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g : $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

Derivatives as Rates of Change

- Determine a new value of a quantity from the old value and the amount of change.
- Calculate the average rate of change and explain how it differs from the instantaneous rate of change.
- Apply rates of change to displacement, velocity, and acceleration of an object moving along a straight line.
- Predict the future population from the present value and the population growth rate.
- Use derivatives to calculate marginal cost and revenue in a business situation.

In this section we look at some applications of the derivative by focusing on the interpretation of the derivative as the rate of change of a function. These applications include **acceleration** and velocity in physics, **population growth rates** in biology, and marginal functions in economics.

Amount of Change Formula

One application for derivatives is to estimate an unknown value of a function at a point by using a known value of a function at some given point together with its rate of change at the given point. If $f(x)$ is a function defined on an interval $[a, a + h]$, then the **amount of change** of $f(x)$ over the interval is the change in the y values of the function over that interval and is given by

Equation:

$$f(a + h) - f(a).$$

The **average rate of change** of the function f over that same interval is the ratio of the amount of change over that interval to the corresponding change in the x values. It is given by

Equation:

$$\frac{f(a+h) - f(a)}{h}.$$

As we already know, the instantaneous rate of change of $f(x)$ at a is its derivative

Equation:

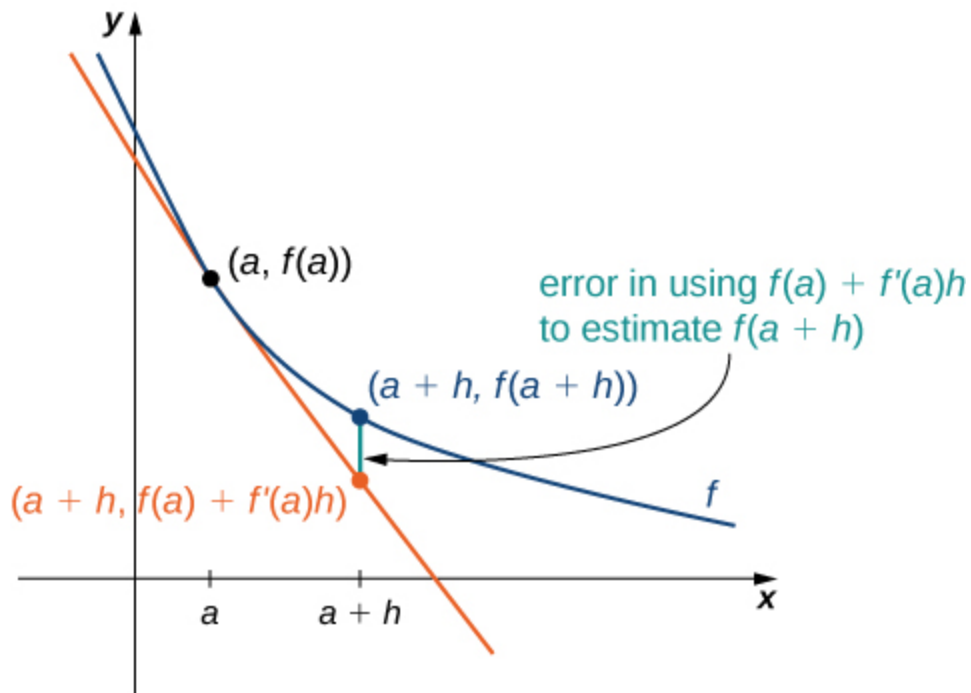
$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

For small enough values of h , $f'(a) \approx \frac{f(a+h) - f(a)}{h}$. We can then solve for $f(a+h)$ to get the amount of change formula:

Equation:

$$f(a+h) \approx f(a) + f'(a)h.$$

We can use this formula if we know only $f(a)$ and $f'(a)$ and wish to estimate the value of $f(a+h)$. For example, we may use the current population of a city and the rate at which it is growing to estimate its population in the near future. As we can see in [\[link\]](#), we are approximating $f(a+h)$ by the y coordinate at $a+h$ on the line tangent to $f(x)$ at $x=a$. Observe that the accuracy of this estimate depends on the value of h as well as the value of $f'(a)$.



The new value of a changed quantity equals the original value plus the rate of change times the interval of change: $f(a+h) \approx f(a) + f'(a)h$.

Note:

Here is an interesting [demonstration](#) of rate of change.

Example:

Exercise:

Problem:

Estimating the Value of a Function

If $f(3) = 2$ and $f'(3) = 5$, estimate $f(3.2)$.

Solution:

Begin by finding h . We have $h = 3.2 - 3 = 0.2$. Thus,

Equation:

$$f(3.2) = f(3 + 0.2) \approx f(3) + (0.2)f'(3) = 2 + 0.2(5) = 3.$$

Note:

Exercise:

Problem: Given $f(10) = -5$ and $f'(10) = 6$, estimate $f(10.1)$.

Solution:

-4.4

Hint

Use the same process as in the preceding example.

Motion along a Line

Another use for the derivative is to analyze motion along a line. We have described velocity as the rate of change of position. If we take the derivative of the velocity, we can find the acceleration, or the rate of change of velocity. It is also important to introduce the idea of **speed**, which is the magnitude of velocity. Thus, we can state the following mathematical definitions.

Note:

Definition

Let $s(t)$ be a function giving the position of an object at time t .

The velocity of the object at time t is given by $v(t) = s'(t)$.

The speed of the object at time t is given by $|v(t)|$.

The acceleration of the object at t is given by $a(t) = v'(t) = s''(t)$.

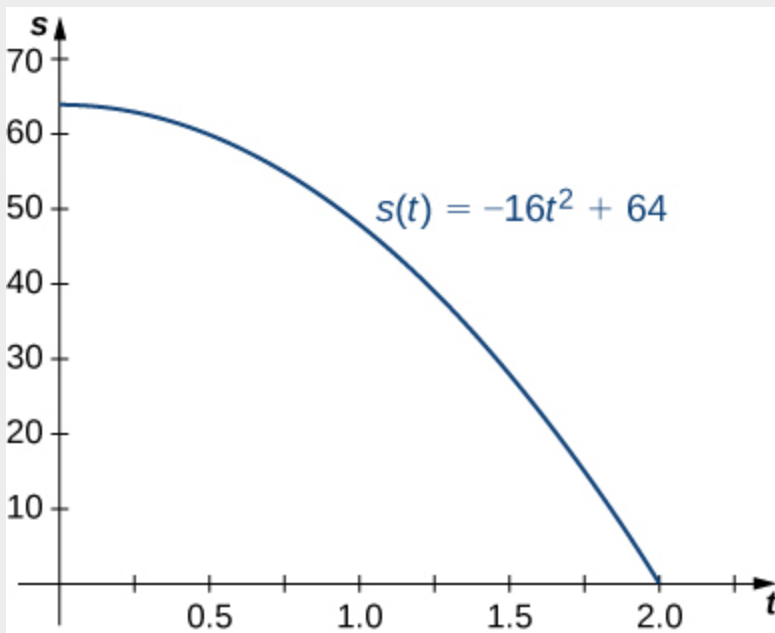
Example:

Exercise:

Problem:

Comparing Instantaneous Velocity and Average Velocity

A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64$.



- What is the instantaneous velocity of the ball when it hits the ground?
- What is the average velocity during its fall?

Solution:

The first thing to do is determine how long it takes the ball to reach the ground. To do this, set $s(t) = 0$. Solving $-16t^2 + 64 = 0$, we

get $t = 2$, so it takes 2 seconds for the ball to reach the ground.

- The instantaneous velocity of the ball as it strikes the ground is $v(2)$. Since $v(t) = s'(t) = -32t$, we obtain $v(t) = -64$ ft/s.
- The average velocity of the ball during its fall is

Equation:

$$v_{ave} = \frac{s(2) - s(0)}{2 - 0} = \frac{0 - 64}{2} = -32 \text{ ft/s.}$$

Example:

Exercise:

Problem:

Interpreting the Relationship between $v(t)$ and $a(t)$

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$ and use these values to answer the following questions.

- Is the particle moving from left to right or from right to left at time $t = 1$?
- Is the particle speeding up or slowing down at time $t = 1$?

Solution:

Begin by finding $v(t)$ and $a(t)$.

and $a(t) = v'(t) = s''(t) = 6t$.

Evaluating these functions at $t = 1$, we obtain $v(1) = -1$ and $a(1) = 6$.

- a. Because $v(1) < 0$, the particle is moving from right to left.
- b. Because $v(1) < 0$ and $a(1) > 0$, velocity and acceleration are acting in opposite directions. In other words, the particle is being accelerated in the direction opposite the direction in which it is traveling, causing $|v(t)|$ to decrease. The particle is slowing down.

Example:

Exercise:

Problem:

Position and Velocity

The position of a particle moving along a coordinate axis is given by $s(t) = t^3 - 9t^2 + 24t + 4, t \geq 0$.

- a. Find $v(t)$.
- b. At what time(s) is the particle at rest?
- c. On what time intervals is the particle moving from left to right?
From right to left?
- d. Use the information obtained to sketch the path of the particle along a coordinate axis.

Solution:

- a. The velocity is the derivative of the position function:

Equation:

$$v(t) = s'(t) = 3t^2 - 18t + 24.$$

- b. The particle is at rest when $v(t) = 0$, so set $3t^2 - 18t + 24 = 0$. Factoring the left-hand side of the equation produces $3(t - 2)(t - 4) = 0$. Solving, we find that the particle is at rest at $t = 2$ and $t = 4$.

- c. The particle is moving from left to right when $v(t) > 0$ and from right to left when $v(t) < 0$. [\[link\]](#) gives the analysis of the sign of $v(t)$ for $t \geq 0$, but it does not represent the axis along which the particle is moving.

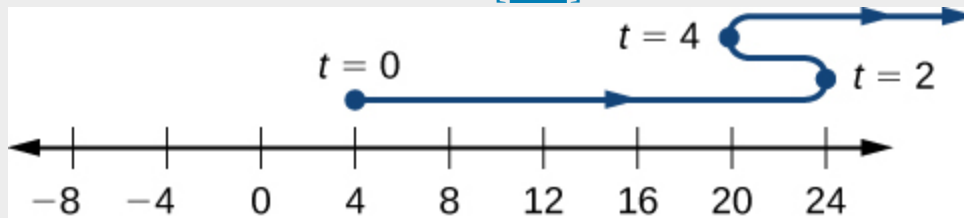


The sign of $v(t)$ determines the direction of the particle.

Since $3t^2 - 18t + 24 > 0$ on $[0, 2) \cup (2, +\infty)$, the particle is moving from left to right on these intervals.

Since $3t^2 - 18t + 24 < 0$ on $(2, 4)$, the particle is moving from right to left on this interval.

- d. Before we can sketch the graph of the particle, we need to know its position at the time it starts moving ($t = 0$) and at the times that it changes direction ($t = 2, 4$). We have $s(0) = 4$, $s(2) = 24$, and $s(4) = 20$. This means that the particle begins on the coordinate axis at 4 and changes direction at 0 and 20 on the coordinate axis. The path of the particle is shown on a coordinate axis in [\[link\]](#).



The path of the particle can be determined by analyzing $v(t)$.

Note:**Exercise:****Problem:**

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^2 - 5t + 1$. Is the particle moving from right to left or from left to right at time $t = 3$?

Solution:

left to right

Hint

Find $v(3)$ and look at the sign.

Population Change

In addition to analyzing velocity, speed, acceleration, and position, we can use derivatives to analyze various types of populations, including those as diverse as bacteria colonies and cities. We can use a current population, together with a growth rate, to estimate the size of a population in the future. The population growth rate is the rate of change of a population and consequently can be represented by the derivative of the size of the population.

Note:**Definition**

If $P(t)$ is the number of entities present in a population, then the population growth rate of $P(t)$ is defined to be $P'(t)$.

Example:

Exercise:**Problem:****Estimating a Population**

The population of a city is tripling every 5 years. If its current population is 10,000, what will be its approximate population 2 years from now?

Solution:

Let $P(t)$ be the population (in thousands) t years from now. Thus, we know that $P(0) = 10$ and based on the information, we anticipate $P(5) = 30$. Now estimate $P'(0)$, the current growth rate, using

Equation:

$$P'(0) \approx \frac{P(5) - P(0)}{5 - 0} = \frac{30 - 10}{5} = 4.$$

By applying [\[link\]](#) to $P(t)$, we can estimate the population 2 years from now by writing

Equation:

$$P(2) \approx P(0) + (2)P'(0) \approx 10 + 2(4) = 18;$$

thus, in 2 years the population will be 18,000.

Note:**Exercise:****Problem:**

The current population of a mosquito colony is known to be 3,000; that is, $P(0) = 3,000$. If $P'(0) = 100$, estimate the size of the population in 3 days, where t is measured in days.

Solution:

3,300

Hint

Use $P(3) \approx P(0) + 3P'(0)$.

Changes in Cost and Revenue

In addition to analyzing motion along a line and population growth, derivatives are useful in analyzing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

Note:**Definition**

If $C(x)$ is the cost of producing x items, then the **marginal cost** $MC(x)$ is $MC(x) = C'(x)$.

If $R(x)$ is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.

If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the **marginal profit** $MP(x)$ is defined to be

$$MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x).$$

We can roughly approximate

Equation:

$$MC(x) = C'(x) = \lim_{h \rightarrow 0} \frac{C(x+h) - C(x)}{h}$$

by choosing an appropriate value for h . Since x represents objects, a reasonable and small value for h is 1. Thus, by substituting $h = 1$, we get the approximation $MC(x) = C'(x) \approx C(x+1) - C(x)$. Consequently, $C'(x)$ for a given value of x can be thought of as the change in cost associated with producing one additional item. In a similar way, $MR(x) = R'(x)$ approximates the revenue obtained by selling one additional item, and $MP(x) = P'(x)$ approximates the profit obtained by producing and selling one additional item.

Example:

Exercise:

Problem:

Applying Marginal Revenue

Assume that the number of barbeque dinners that can be sold, x , can be related to the price charged, p , by the equation $p(x) = 9 - 0.03x$, $0 \leq x \leq 300$.

In this case, the revenue in dollars obtained by selling x barbeque dinners is given by

Equation:

$$R(x) = xp(x) = x(9 - 0.03x) = -0.03x^2 + 9x \text{ for } 0 \leq x \leq 300.$$

Use the marginal revenue function to estimate the revenue obtained from selling the 101st barbeque dinner. Compare this to the actual revenue obtained from the sale of this dinner.

Solution:

First, find the marginal revenue function:

$$MR(x) = R'(x) = -0.06x + 9.$$

Next, use $R'(100)$ to approximate $R(101) - R(100)$, the revenue obtained from the sale of the 101st dinner. Since $R'(100) = 3$, the revenue obtained from the sale of the 101st dinner is approximately \$3.

The actual revenue obtained from the sale of the 101st dinner is

Equation:

$$R(101) - R(100) = 602.97 - 600 = 2.97, \text{ or } \$2.97.$$

The marginal revenue is a fairly good estimate in this case and has the advantage of being easy to compute.

Note:

Exercise:

Problem:

Suppose that the profit obtained from the sale of x fish-fry dinners is given by $P(x) = -0.03x^2 + 8x - 50$. Use the marginal profit function to estimate the profit from the sale of the 101st fish-fry dinner.

Solution:

\$2

Hint

Use $P'(100)$ to approximate $P(101) - P(100)$.

Key Concepts

- Using $f(a + h) \approx f(a) + f'(a)h$, it is possible to estimate $f(a + h)$ given $f'(a)$ and $f(a)$.
- The rate of change of position is velocity, and the rate of change of velocity is acceleration. Speed is the absolute value, or magnitude, of velocity.
- The population growth rate and the present population can be used to predict the size of a future population.
- Marginal cost, marginal revenue, and marginal profit functions can be used to predict, respectively, the cost of producing one more item, the revenue obtained by selling one more item, and the profit obtained by producing and selling one more item.

For the following exercises, the given functions represent the position of a particle traveling along a horizontal line.

- Find the velocity and acceleration functions.
- Determine the time intervals when the object is slowing down or speeding up.

Exercise:

Problem: $s(t) = 2t^3 - 3t^2 - 12t + 8$

Exercise:

Problem: $s(t) = 2t^3 - 15t^2 + 36t - 10$

Solution:

a. $v(t) = 6t^2 - 30t + 36$, $a(t) = 12t - 30$; b. speeds up $(2, 2.5) \cup (3, \infty)$, slows down $(0, 2) \cup (2.5, 3)$

Exercise:

Problem: $s(t) = \frac{t}{1+t^2}$

Exercise:**Problem:**

A rocket is fired vertically upward from the ground. The distance s in feet that the rocket travels from the ground after t seconds is given by $s(t) = -16t^2 + 560t$.

- Find the velocity of the rocket 3 seconds after being fired.
- Find the acceleration of the rocket 3 seconds after being fired.

Solution:

a. 464 ft/s^2 b. -32 ft/s^2

Exercise:**Problem:**

A ball is thrown downward with a speed of 8 ft/s from the top of a 64-foot-tall building. After t seconds, its height above the ground is given by $s(t) = -16t^2 - 8t + 64$.

- Determine how long it takes for the ball to hit the ground.
- Determine the velocity of the ball when it hits the ground.

Exercise:**Problem:**

The position function $s(t) = t^2 - 3t - 4$ represents the position of the back of a car backing out of a driveway and then driving in a straight line, where s is in feet and t is in seconds. In this case, $s(t) = 0$ represents the time at which the back of the car is at the garage door, so $s(0) = -4$ is the starting position of the car, 4 feet inside the garage.

- Determine the velocity of the car when $s(t) = 0$.
- Determine the velocity of the car when $s(t) = 14$.

Solution:

a. 5 ft/s b. 9 ft/s

Exercise:**Problem:**

The position of a hummingbird flying along a straight line in t seconds is given by $s(t) = 3t^3 - 7t$ meters.

- Determine the velocity of the bird at $t = 1$ sec.
- Determine the acceleration of the bird at $t = 1$ sec.
- Determine the acceleration of the bird when the velocity equals 0.

Exercise:**Problem:**

A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot-tall building. The distance in feet that the potato travels from the ground after t seconds is given by $s(t) = -16t^2 + 100t + 85$.

- Find the velocity of the potato after 0.5 s and 5.75 s.
- Find the speed of the potato at 0.5 s and 5.75 s.
- Determine when the potato reaches its maximum height.
- Find the acceleration of the potato at 0.5 s and 1.5 s.
- Determine how long the potato is in the air.
- Determine the velocity of the potato upon hitting the ground.

Solution:

a. 84 ft/s, -84 ft/s b. 84 ft/s c. $\frac{25}{8}$ s d. -32 ft/s^2 in both cases e. $\frac{1}{8} (25 + \sqrt{965})$ s f. $-4\sqrt{965} \text{ ft/s}$

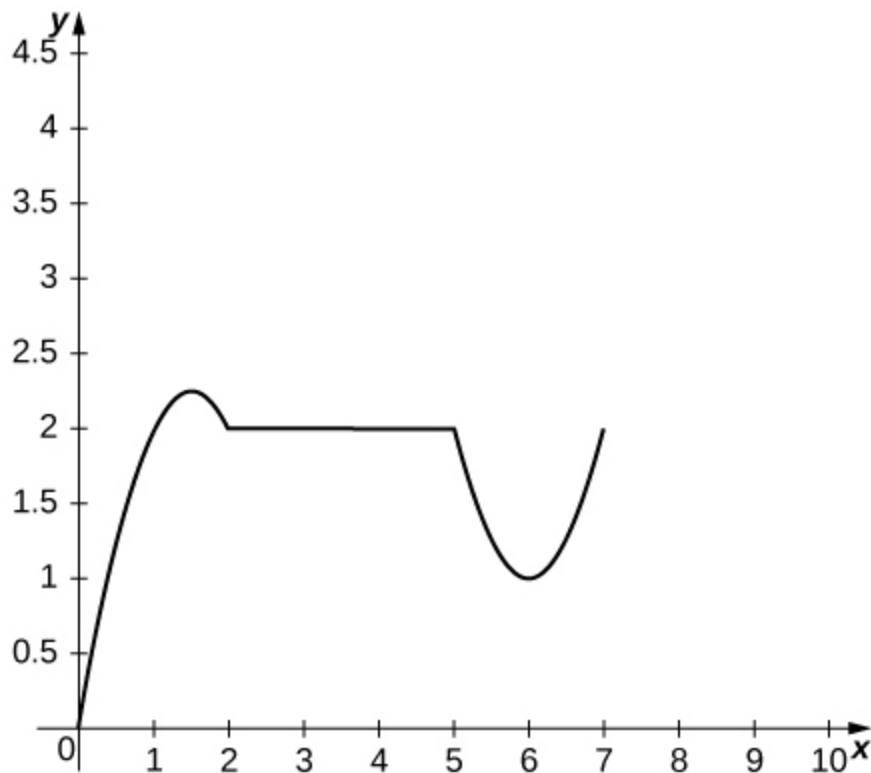
Exercise:**Problem:**

The position function $s(t) = t^3 - 8t$ gives the position in miles of a freight train where east is the positive direction and t is measured in hours.

- Determine the direction the train is traveling when $s(t) = 0$.
- Determine the direction the train is traveling when $a(t) = 0$.
- Determine the time intervals when the train is slowing down or speeding up.

Exercise:**Problem:**

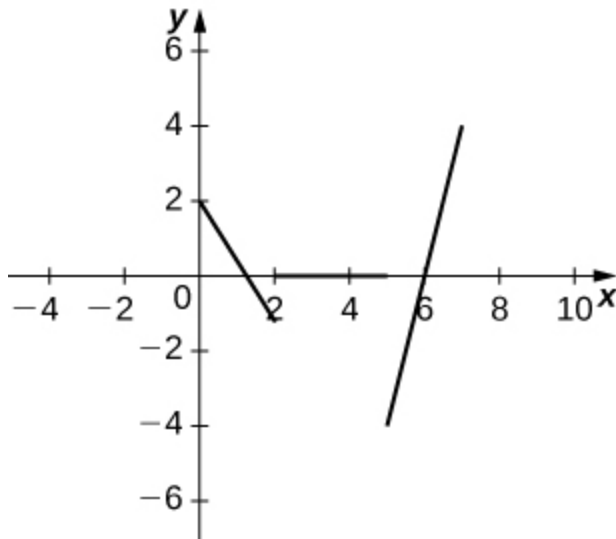
The following graph shows the position $y = s(t)$ of an object moving along a straight line.



- Use the graph of the position function to determine the time intervals when the velocity is positive, negative, or zero.
- Sketch the graph of the velocity function.
- Use the graph of the velocity function to determine the time intervals when the acceleration is positive, negative, or zero.
- Determine the time intervals when the object is speeding up or slowing down.

Solution:

- a. Velocity is positive on $(0, 1.5) \cup (6, 7)$, negative on $(1.5, 2) \cup (5, 6)$, and zero on $(2, 5)$. b.



- c. Acceleration is positive on $(5, 7)$, negative on $(0, 2)$, and zero on $(2, 5)$. d. The object is speeding up on $(6, 7) \cup (1.5, 2)$ and slowing down on $(0, 1.5) \cup (5, 6)$.

Exercise:

Problem:

The cost function, in dollars, of a company that manufactures food processors is given by $C(x) = 200 + \frac{7}{x} + \frac{x^2}{7}$, where x is the number of food processors manufactured.

- Find the marginal cost function.
- Find the marginal cost of manufacturing 12 food processors.
- Find the actual cost of manufacturing the thirteenth food processor.

Exercise:**Problem:**

The price p (in dollars) and the demand x for a certain digital clock radio is given by the price–demand function $p = 10 - 0.001x$.

- Find the revenue function $R(x)$.
- Find the marginal revenue function.
- Find the marginal revenue at $x = 2000$ and 5000.

Solution:

- a. $R(x) = 10x - 0.001x^2$ b. $R'(x) = 10 - 0.002x$ c. \$6 per item, \$0 per item

Exercise:**Problem:**

[T] A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by $P(x) = 30x - 0.3x^2 - 250$, where x is the number of skateboards sold.

- Find the exact profit from the sale of the thirtieth skateboard.

- b. Find the marginal profit function and use it to estimate the profit from the sale of the thirtieth skateboard.

Exercise:

Problem:

[T] In general, the profit function is the difference between the revenue and cost functions: $P(x) = R(x) - C(x)$.

Suppose the price-demand and cost functions for the production of cordless drills is given respectively by $p = 143 - 0.03x$ and $C(x) = 75,000 + 65x$, where x is the number of cordless drills that are sold at a price of p dollars per drill and $C(x)$ is the cost of producing x cordless drills.

- Find the marginal cost function.
- Find the revenue and marginal revenue functions.
- Find $R'(1000)$ and $R'(4000)$. Interpret the results.
- Find the profit and marginal profit functions.
- Find $P'(1000)$ and $P'(4000)$. Interpret the results.

Solution:

a. $C'(x) = 65$ b. $R(x) = 143x - 0.03x^2$, $R'(x) = 143 - 0.06x$ c. 83, -97. At a production level of 1000 cordless drills, revenue is increasing at a rate of \$83 per drill; at a production level of 4000 cordless drills, revenue is decreasing at a rate of \$97 per drill. d. $P(x) = -0.03x^2 + 78x - 75000$, $P'(x) = -0.06x + 78$ e. 18, -162. At a production level of 1000 cordless drills, profit is increasing at a rate of \$18 per drill; at a production level of 4000 cordless drills, profit is decreasing at a rate of \$162 per drill.

Exercise:

Problem:

A small town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, where t is measured in years.

- Find the rate of change function $P'(t)$ of the population function.
- Find $P'(1)$, $P'(2)$, $P'(3)$, and $P'(4)$. Interpret what the results mean for the town.
- Find $P''(1)$, $P''(2)$, $P''(3)$, and $P''(4)$. Interpret what the results mean for the town's population.

Exercise:**Problem:**

[T] A culture of bacteria grows in number according to the function $N(t) = 3000 \left(1 + \frac{4t}{t^2+100}\right)$, where t is measured in hours.

- Find the rate of change of the number of bacteria.
- Find $N'(0)$, $N'(10)$, $N'(20)$, and $N'(30)$.
- Interpret the results in (b).
- Find $N''(0)$, $N''(10)$, $N''(20)$, and $N''(30)$. Interpret what the answers imply about the bacteria population growth.

Solution:

- a. $N'(t) = 3000 \left(\frac{-4t^2+400}{(t^2+100)^2} \right)$ b. 120, 0, -14.4, -9.6 c. The bacteria population increases from time 0 to 10 hours; afterwards, the bacteria population decreases. d. 0, -6, 0.384, 0.432. The rate at which the bacteria is increasing is decreasing during the first 10 hours. Afterwards, the bacteria population is decreasing at a decreasing rate.

Exercise:**Problem:**

The centripetal force of an object of mass m is given by $F(r) = \frac{mv^2}{r}$, where v is the speed of rotation and r is the distance from the center of rotation.

- a. Find the rate of change of centripetal force with respect to the distance from the center of rotation.
- b. Find the rate of change of centripetal force of an object with mass 1000 kilograms, velocity of 13.89 m/s, and a distance from the center of rotation of 200 meters.

The following questions concern the population (in millions) of London by decade in the 19th century, which is listed in the following table.

Years since 1800	Population (millions)
1	0.8795
11	1.040
21	1.264
31	1.516
41	1.661
51	2.000

Years since 1800	Population (millions)
61	2.634
71	3.272
81	3.911
91	4.422

Population of LondonSource:

http://en.wikipedia.org/wiki/Demographics_of_London.

Exercise:

Problem: [T]

- Using a calculator or a computer program, find the best-fit linear function to measure the population.
- Find the derivative of the equation in a. and explain its physical meaning.
- Find the second derivative of the equation and explain its physical meaning.

Solution:

a. $P(t) = 0.03983 + 0.4280t$ b. $P'(t) = 0.03983$. The population is increasing. c. $P''(t) = 0$. The rate at which the population is increasing is constant.

Exercise:

Problem: [T]

- Using a calculator or a computer program, find the best-fit quadratic curve through the data.

- b. Find the derivative of the equation and explain its physical meaning.
- c. Find the second derivative of the equation and explain its physical meaning.

For the following exercises, consider an astronaut on a large planet in another galaxy. To learn more about the composition of this planet, the astronaut drops an electronic sensor into a deep trench. The sensor transmits its vertical position every second in relation to the astronaut's position. The summary of the falling sensor data is displayed in the following table.

Time after dropping (s)	Position (m)
0	0
1	-1
2	-2
3	-5
4	-7
5	-14

Exercise:

Problem: [T]

- a. Using a calculator or computer program, find the best-fit quadratic curve to the data.
 - b. Find the derivative of the position function and explain its physical meaning.
 - c. Find the second derivative of the position function and explain its physical meaning.
-

Solution:

a. $p(t) = -0.6071x^2 + 0.4357x - 0.3571$ b.

$p'(t) = -1.214x + 0.4357$. This is the velocity of the sensor. c.

$p''(t) = -1.214$. This is the acceleration of the sensor; it is a constant acceleration downward.

Exercise:

Problem: [T]

- a. Using a calculator or computer program, find the best-fit cubic curve to the data.
- b. Find the derivative of the position function and explain its physical meaning.
- c. Find the second derivative of the position function and explain its physical meaning.
- d. Using the result from c. explain why a cubic function is not a good choice for this problem.

The following problems deal with the Holling type I, II, and III equations. These equations describe the ecological event of growth of a predator population given the amount of prey available for consumption.

Exercise:

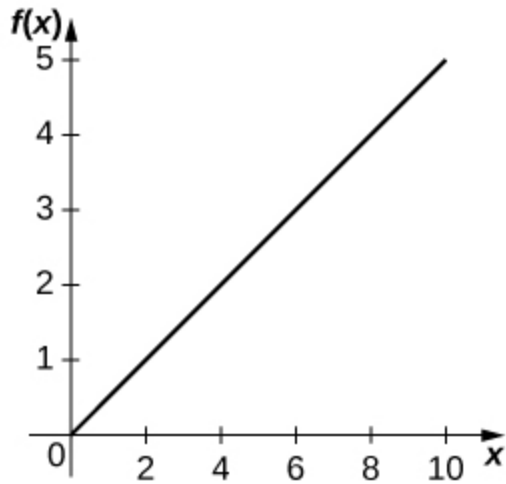
Problem:

[T] The Holling type I equation is described by $f(x) = ax$, where x is the amount of prey available and $a > 0$ is the rate at which the predator meets the prey for consumption.

- Graph the Holling type I equation, given $a = 0.5$.
- Determine the first derivative of the Holling type I equation and explain physically what the derivative implies.
- Determine the second derivative of the Holling type I equation and explain physically what the derivative implies.
- Using the interpretations from b. and c. explain why the Holling type I equation may not be realistic.

Solution:

a.



b. $f'(x) = a$. The more increase in prey, the more growth for predators. c. $f''(x) = 0$. As the amount of prey increases, the rate at which the predator population growth increases is constant. d. This equation assumes that if there is more prey, the predator is able to increase consumption linearly. This assumption is unphysical because

we would expect there to be some saturation point at which there is too much prey for the predator to consume adequately.

Exercise:

Problem:

[T] The Holling type II equation is described by $f(x) = \frac{ax}{n+x}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

- Graph the Holling type II equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type I and II equations?
- Take the first derivative of the Holling type II equation and interpret the physical meaning of the derivative.
- Show that $f(n) = \frac{1}{2}a$ and interpret the meaning of the parameter n .
- Find and interpret the meaning of the second derivative. What makes the Holling type II function more realistic than the Holling type I function?

Exercise:

Problem:

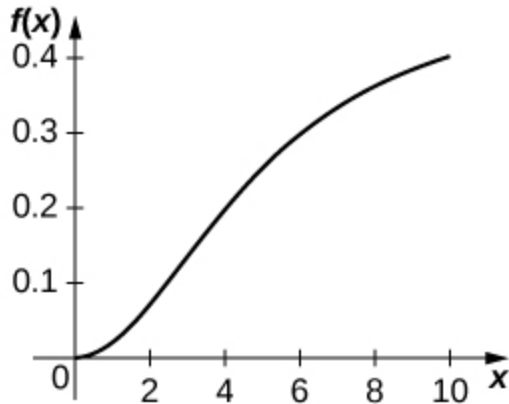
[T] The Holling type III equation is described by $f(x) = \frac{ax^2}{n^2+x^2}$, where x is the amount of prey available and $a > 0$ is the maximum consumption rate of the predator.

- Graph the Holling type III equation given $a = 0.5$ and $n = 5$. What are the differences between the Holling type II and III equations?
- Take the first derivative of the Holling type III equation and interpret the physical meaning of the derivative.
- Find and interpret the meaning of the second derivative (it may help to graph the second derivative).

d. What additional ecological phenomena does the Holling type III function describe compared with the Holling type II function?

Solution:

a.



b. $f'(x) = \frac{2axn^2}{(n^2+x^2)^2}$. When the amount of prey increases, the predator growth increases. c. $f''(x) = \frac{2an^2(n^2-3x^2)}{(n^2+x^2)^3}$. When the amount of prey is extremely small, the rate at which predator growth is increasing is increasing, but when the amount of prey reaches above a certain threshold, the rate at which predator growth is increasing begins to decrease. d. At lower levels of prey, the prey is more easily able to avoid detection by the predator, so fewer prey individuals are consumed, resulting in less predator growth.

Exercise:

Problem:

[T] The populations of the snowshoe hare (in thousands) and the lynx (in hundreds) collected over 7 years from 1937 to 1943 are shown in the following table. The snowshoe hare is the primary prey of the lynx.

Population of snowshoe hare (thousands)	Population of lynx (hundreds)
20	10
55	15
65	55
95	60

Snowshoe Hare and Lynx Populations Source:
<http://www.biotopics.co.uk/newgcse/predatorprey.html>.

- Graph the data points and determine which Holling-type function fits the data best.
- Using the meanings of the parameters a and n , determine values for those parameters by examining a graph of the data. Recall that n measures what prey value results in the half-maximum of the predator value.
- Plot the resulting Holling-type I, II, and III functions on top of the data. Was the result from part a. correct?

Glossary

acceleration

is the rate of change of the velocity, that is, the derivative of velocity

amount of change

the amount of a function $f(x)$ over an interval $[x, x + h]$ is
 $f(x + h) - f(x)$

average rate of change

is a function $f(x)$ over an interval $[x, x + h]$ is $\frac{f(x+h)-f(a)}{b-a}$

marginal cost

is the derivative of the cost function, or the approximate cost of producing one more item

marginal revenue

is the derivative of the revenue function, or the approximate revenue obtained by selling one more item

marginal profit

is the derivative of the profit function, or the approximate profit obtained by producing and selling one more item

population growth rate

is the derivative of the population with respect to time

speed

is the absolute value of velocity, that is, $|v(t)|$ is the speed of an object at time t whose velocity is given by $v(t)$

Derivatives of Trigonometric Functions

- Find the derivatives of the sine and cosine function.
- Find the derivatives of the standard trigonometric functions.
- Calculate the higher-order derivatives of the sine and cosine.

One of the most important types of motion in physics is simple harmonic motion, which is associated with such systems as an object with mass oscillating on a spring. Simple harmonic motion can be described by using either sine or cosine functions. In this section we expand our knowledge of derivative formulas to include derivatives of these and other trigonometric functions. We begin with the derivatives of the sine and cosine functions and then use them to obtain formulas for the derivatives of the remaining four trigonometric functions. Being able to calculate the derivatives of the sine and cosine functions will enable us to find the velocity and acceleration of simple harmonic motion.

Derivatives of the Sine and Cosine Functions

We begin our exploration of the derivative for the sine function by using the formula to make a reasonable guess at its derivative. Recall that for a function $f(x)$,

Equation:

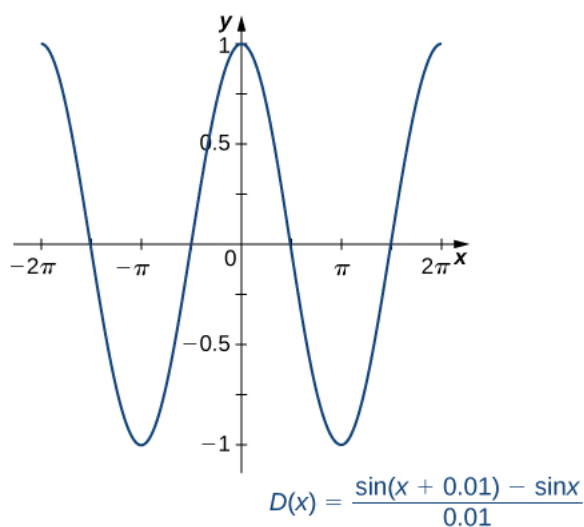
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Consequently, for values of h very close to 0, $f'(x) \approx \frac{f(x+h) - f(x)}{h}$. We see that by using $h = 0.01$,

Equation:

$$\frac{d}{dx}(\sin x) \approx \frac{\sin(x + 0.01) - \sin x}{0.01}$$

By setting $D(x) = \frac{\sin(x+0.01) - \sin x}{0.01}$ and using a graphing utility, we can get a graph of an approximation to the derivative of $\sin x$ ([link](#)).



The graph of the function $D(x)$ looks a lot like a cosine curve.

Upon inspection, the graph of $D(x)$ appears to be very close to the graph of the cosine function. Indeed, we will show that

Equation:

$$\frac{d}{dx}(\sin x) = \cos x.$$

If we were to follow the same steps to approximate the derivative of the cosine function, we would find that

Equation:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Note:

The Derivatives of $\sin x$ and $\cos x$

The derivative of the sine function is the cosine and the derivative of the cosine function is the negative sine.

Equation:

$$\frac{d}{dx}(\sin x) = \cos x$$

Equation:

$$\frac{d}{dx}(\cos x) = -\sin x$$

Proof

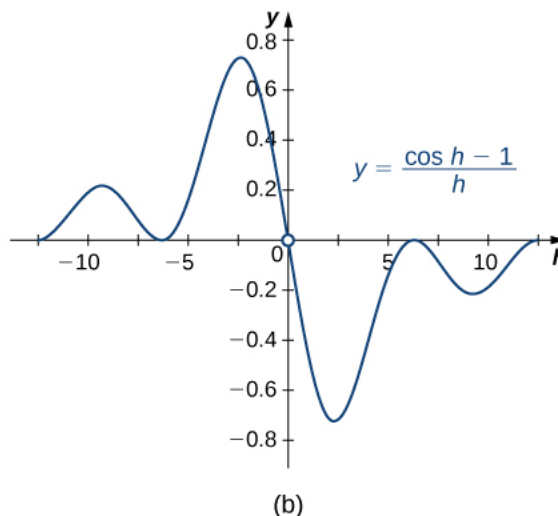
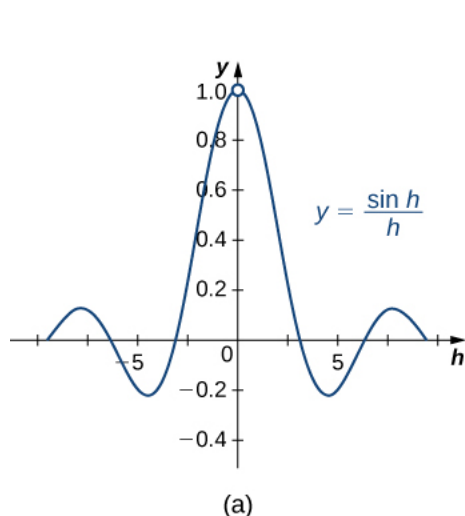
Because the proofs for $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ use similar techniques, we provide only the proof for $\frac{d}{dx}(\sin x) = \cos x$. Before beginning, recall two important trigonometric limits we learned in

[Introduction to Limits](#):

Equation:

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \text{ and } \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0.$$

The graphs of $y = \frac{(\sin h)}{h}$ and $y = \frac{(\cos h - 1)}{h}$ are shown in [\[link\]](#).



These graphs show two important limits needed to establish the derivative formulas for the sine and cosine functions.

We also recall the following trigonometric identity for the sine of the sum of two angles:

Equation:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

Now that we have gathered all the necessary equations and identities, we proceed with the proof.

Equation:

$$\begin{aligned} \frac{d}{dx} \sin x &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right) \\ &= \sin x (0) + \cos x (1) \\ &= \cos x \end{aligned}$$

Apply the definition of the derivative.

Use trig identity for the sine of the sum of two angles

Regroup.

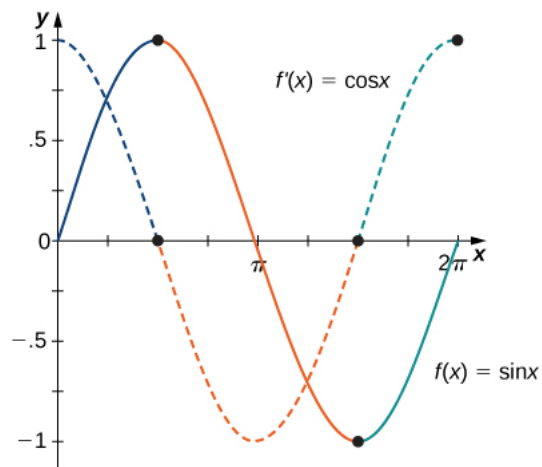
Factor out $\sin x$ and $\cos x$.

Apply trig limit formulas.

Simplify.

□

[\[link\]](#) shows the relationship between the graph of $f(x) = \sin x$ and its derivative $f'(x) = \cos x$. Notice that at the points where $f(x) = \sin x$ has a horizontal tangent, its derivative $f'(x) = \cos x$ takes on the value zero. We also see that where $f(x) = \sin x$ is increasing, $f'(x) = \cos x > 0$ and where $f(x) = \sin x$ is decreasing, $f'(x) = \cos x < 0$.



Where $f(x)$ has a maximum or a minimum, $f'(x) = 0$
that is, $f'(x) = 0$ where $f(x)$ has a horizontal tangent.

These points are noted with dots on the graphs.

Example:

Exercise:

Problem:

Differentiating a Function Containing $\sin x$

Find the derivative of $f(x) = 5x^3 \sin x$.

Solution:

Using the product rule, we have

Equation:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(5x^3) \cdot \sin x + \frac{d}{dx}(\sin x) \cdot 5x^3 \\ &= 15x^2 \cdot \sin x + \cos x \cdot 5x^3. \end{aligned}$$

After simplifying, we obtain

Equation:

$$f'(x) = 15x^2 \sin x + 5x^3 \cos x.$$

Note:

Exercise:

Problem: Find the derivative of $f(x) = \sin x \cos x$.

Solution:

$$f'(x) = \cos^2 x - \sin^2 x$$

Hint

Don't forget to use the product rule.

Example:

Exercise:

Problem:

Finding the Derivative of a Function Containing $\cos x$

Find the derivative of $g(x) = \frac{\cos x}{4x^2}$.

Solution:

By applying the quotient rule, we have

Equation:

$$g'(x) = \frac{(-\sin x)4x^2 - 8x(\cos x)}{(4x^2)^2}.$$

Simplifying, we obtain

Equation:

$$\begin{aligned} g'(x) &= \frac{-4x^2 \sin x - 8x \cos x}{16x^4} \\ &= \frac{-x \sin x - 2 \cos x}{4x^3}. \end{aligned}$$

Note:

Exercise:

Problem: Find the derivative of $f(x) = \frac{x}{\cos x}$.

Solution:

$$\frac{\cos x + x \sin x}{\cos^2 x}$$

Hint

Use the quotient rule.

Example:

Exercise:

Problem:

An Application to Velocity

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 \sin t - t$ for $0 \leq t \leq 2\pi$. At what times is the particle at rest?

Solution:

To determine when the particle is at rest, set $s'(t) = v(t) = 0$. Begin by finding $s'(t)$. We obtain

Equation:

$$s'(t) = 2\cos t - 1,$$

so we must solve

Equation:

$$2\cos t - 1 = 0 \text{ for } 0 \leq t \leq 2\pi.$$

The solutions to this equation are $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$. Thus the particle is at rest at times $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

Note:**Exercise:****Problem:**

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \leq t \leq 2\pi$. At what times is the particle at rest?

Solution:

$$t = \frac{\pi}{3}, t = \frac{2\pi}{3}$$

Hint

Use the previous example as a guide.

Derivatives of Other Trigonometric Functions

Since the remaining four trigonometric functions may be expressed as quotients involving sine, cosine, or both, we can use the quotient rule to find formulas for their derivatives.

Example:**Exercise:****Problem:****The Derivative of the Tangent Function**

Find the derivative of $f(x) = \tan x$.

Solution:

Start by expressing $\tan x$ as the quotient of $\sin x$ and $\cos x$:

Equation:

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

Now apply the quotient rule to obtain

Equation:

$$f'(x) = \frac{\cos x \cos x - (-\sin x)\sin x}{(\cos x)^2}.$$

Simplifying, we obtain

Equation:

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

Recognizing that $\cos^2 x + \sin^2 x = 1$, by the Pythagorean theorem, we now have

Equation:

$$f'(x) = \frac{1}{\cos^2 x}.$$

Finally, use the identity $\sec x = \frac{1}{\cos x}$ to obtain

Equation:

$$f'(x) = \sec^2 x.$$

Note:

Exercise:

Problem: Find the derivative of $f(x) = \cot x$.

Solution:

$$f'(x) = -\csc^2 x$$

Hint

Rewrite $\cot x$ as $\frac{\cos x}{\sin x}$ and use the quotient rule.

The derivatives of the remaining trigonometric functions may be obtained by using similar techniques. We provide these formulas in the following theorem.

Note:

Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The derivatives of the remaining trigonometric functions are as follows:

Equation:

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Equation:

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Equation:

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

Equation:

$$\frac{d}{dx}(\csc x) = -\csc x \cot x.$$

Example:

Exercise:

Problem:

Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Solution:

To find the equation of the tangent line, we need a point and a slope at that point. To find the point, compute

Equation:

$$f\left(\frac{\pi}{4}\right) = \cot \frac{\pi}{4} = 1.$$

Thus the tangent line passes through the point $\left(\frac{\pi}{4}, 1\right)$. Next, find the slope by finding the derivative of $f(x) = \cot x$ and evaluating it at $\frac{\pi}{4}$:

Equation:

$$f'(x) = -\csc^2 x \text{ and } f'\left(\frac{\pi}{4}\right) = -\csc^2\left(\frac{\pi}{4}\right) = -2.$$

Using the point-slope equation of the line, we obtain

Equation:

$$y - 1 = -2\left(x - \frac{\pi}{4}\right)$$

or equivalently,

Equation:

$$y = -2x + 1 + \frac{\pi}{2}.$$

Example:

Exercise:

Problem:

Finding the Derivative of Trigonometric Functions

Find the derivative of $f(x) = \csc x + x \tan x$.

Solution:

To find this derivative, we must use both the sum rule and the product rule. Using the sum rule, we find

Equation:

$$f'(x) = \frac{d}{dx}(\csc x) + \frac{d}{dx}(x \tan x).$$

In the first term, $\frac{d}{dx}(\csc x) = -\csc x \cot x$, and by applying the product rule to the second term we obtain

Equation:

$$\frac{d}{dx}(x \tan x) = (1)(\tan x) + (\sec^2 x)(x).$$

Therefore, we have

Equation:

$$f'(x) = -\csc x \cot x + \tan x + x \sec^2 x.$$

Note:

Exercise:

Problem: Find the derivative of $f(x) = 2 \tan x - 3 \cot x$.

Solution:

$$f'(x) = 2 \sec^2 x + 3 \csc^2 x$$

Hint

Use the rule for differentiating a constant multiple and the rule for differentiating a difference of two functions.

Note:

Exercise:

Problem: Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{6}$.

Solution:

$$\frac{4}{3}$$

Hint

Evaluate the derivative at $x = \frac{\pi}{6}$.

Higher-Order Derivatives

The higher-order derivatives of $\sin x$ and $\cos x$ follow a repeating pattern. By following the pattern, we can find any higher-order derivative of $\sin x$ and $\cos x$.

Example:

Exercise:

Problem:

Finding Higher-Order Derivatives of $y = \sin x$

Find the first four derivatives of $y = \sin x$.

Solution:

Each step in the chain is straightforward:

Equation:

$$\begin{aligned}y &= \sin x \\ \frac{dy}{dx} &= \cos x \\ \frac{d^2y}{dx^2} &= -\sin x \\ \frac{d^3y}{dx^3} &= -\cos x \\ \frac{d^4y}{dx^4} &= \sin x.\end{aligned}$$

Analysis

Once we recognize the pattern of derivatives, we can find any higher-order derivative by determining the step in the pattern to which it corresponds. For example, every fourth derivative of $\sin x$ equals $\sin x$, so

Equation:

$$\begin{aligned}\frac{d^4}{dx^4}(\sin x) &= \frac{d^8}{dx^8}(\sin x) = \frac{d^{12}}{dx^{12}}(\sin x) = \dots = \frac{d^{4n}}{dx^{4n}}(\sin x) = \sin x \\ \frac{d^5}{dx^5}(\sin x) &= \frac{d^9}{dx^9}(\sin x) = \frac{d^{13}}{dx^{13}}(\sin x) = \dots = \frac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x.\end{aligned}$$

Note:

Exercise:

Problem: For $y = \cos x$, find $\frac{d^4y}{dx^4}$.

Solution:

$$\cos x$$

Hint

See the previous example.

Example:**Exercise:****Problem:**

Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Solution:

We can see right away that for the 74th derivative of $\sin x$, $74 = 4(18) + 2$, so

Equation:

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^{72+2}}{dx^{72+2}}(\sin x) = \frac{d^2}{dx^2}(\sin x) = -\sin x.$$

Note:**Exercise:**

Problem: For $y = \sin x$, find $\frac{d^{59}}{dx^{59}}(\sin x)$.

Solution:

$$-\cos x$$

Hint

$$\frac{d^{59}}{dx^{59}}(\sin x) = \frac{d^{4 \cdot 14 + 3}}{dx^{4 \cdot 14 + 3}}(\sin x)$$

Example:**Exercise:****Problem:**

An Application to Acceleration

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 - \sin t$. Find $v(\pi/4)$ and $a(\pi/4)$. Compare these values and decide whether the particle is speeding up or slowing down.

Solution:

First find $v(t) = s'(t)$:

Equation:

$$v(t) = s'(t) = -\cos t.$$

Thus,

Equation:

$$v\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

Next, find $a(t) = v'(t)$. Thus, $a(t) = v'(t) = \sin t$ and we have

Equation:

$$a\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Since $v\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} < 0$ and $a\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} > 0$, we see that velocity and acceleration are acting in opposite directions; that is, the object is being accelerated in the direction opposite to the direction in which it is travelling. Consequently, the particle is slowing down.

Note:

Exercise:

Problem:

A block attached to a spring is moving vertically. Its position at time t is given by $s(t) = 2\sin t$. Find $v\left(\frac{5\pi}{6}\right)$ and $a\left(\frac{5\pi}{6}\right)$. Compare these values and decide whether the block is speeding up or slowing down.

Solution:

$v\left(\frac{5\pi}{6}\right) = -\sqrt{3} < 0$ and $a\left(\frac{5\pi}{6}\right) = -1 < 0$. The block is speeding up.

Hint

Use [\[link\]](#) as a guide.

Key Concepts

- We can find the derivatives of $\sin x$ and $\cos x$ by using the definition of derivative and the limit formulas found earlier. The results are

Equation:

$$\frac{d}{dx} \sin x = \cos x \quad \frac{d}{dx} \cos x = -\sin x.$$

- With these two formulas, we can determine the derivatives of all six basic trigonometric functions.

Key Equations

- Derivative of sine function**

$$\frac{d}{dx} (\sin x) = \cos x$$

- Derivative of cosine function**

$$\frac{d}{dx} (\cos x) = -\sin x$$

- Derivative of tangent function**

$$\frac{d}{dx} (\tan x) = \sec^2 x$$

- Derivative of cotangent function**

$$\frac{d}{dx} (\cot x) = -\csc^2 x$$

- Derivative of secant function**

$$\frac{d}{dx} (\sec x) = \sec x \tan x$$

- **Derivative of cosecant function**

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

For the following exercises, find $\frac{dy}{dx}$ for the given functions.

Exercise:

Problem: $y = x^2 - \sec x + 1$

Solution:

$$\frac{dy}{dx} = 2x - \sec x \tan x$$

Exercise:

Problem: $y = 3 \csc x + \frac{5}{x}$

Exercise:

Problem: $y = x^2 \cot x$

Solution:

$$\frac{dy}{dx} = 2x \cot x - x^2 \csc^2 x$$

Exercise:

Problem: $y = x - x^3 \sin x$

Exercise:

Problem: $y = \frac{\sec x}{x}$

Solution:

$$\frac{dy}{dx} = \frac{x \sec x \tan x - \sec x}{x^2}$$

Exercise:

Problem: $y = \sin x \tan x$

Exercise:

Problem: $y = (x + \cos x)(1 - \sin x)$

Solution:

$$\frac{dy}{dx} = (1 - \sin x)(1 - \sin x) - \cos x(x + \cos x)$$

Exercise:

Problem: $y = \frac{\tan x}{1 - \sec x}$

Exercise:

Problem: $y = \frac{1 - \cot x}{1 + \cot x}$

Solution:

$$\frac{dy}{dx} = \frac{2\csc^2 x}{(1+\cot x)^2}$$

Exercise:

Problem: $y = \cos x (1 + \csc x)$

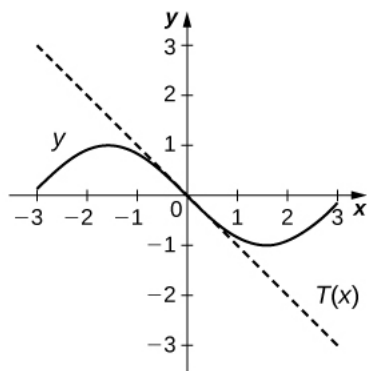
For the following exercises, find the equation of the tangent line to each of the given functions at the indicated values of x . Then use a calculator to graph both the function and the tangent line to ensure the equation for the tangent line is correct.

Exercise:

Problem: [T] $f(x) = -\sin x, x = 0$

Solution:

$$y = -x$$



Exercise:

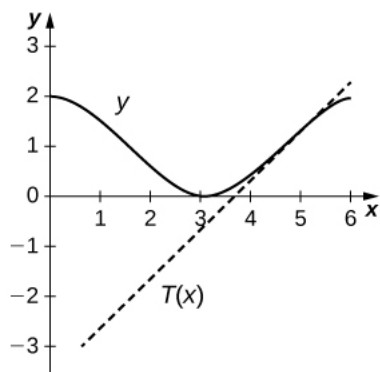
Problem: [T] $f(x) = \csc x, x = \frac{\pi}{2}$

Exercise:

Problem: [T] $f(x) = 1 + \cos x, x = \frac{3\pi}{2}$

Solution:

$$y = x + \frac{2-3\pi}{2}$$



Exercise:

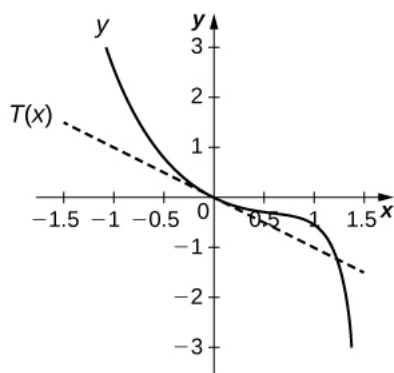
Problem: [T] $f(x) = \sec x, x = \frac{\pi}{4}$

Exercise:

Problem: [T] $f(x) = x^2 - \tan x, x = 0$

Solution:

$$y = -x$$



Exercise:

Problem: [T] $f(x) = 5 \cot x, x = \frac{\pi}{4}$

For the following exercises, find $\frac{d^2y}{dx^2}$ for the given functions.

Exercise:

Problem: $y = x \sin x - \cos x$

Solution:

$$3 \cos x - x \sin x$$

Exercise:

Problem: $y = \sin x \cos x$

Exercise:

Problem: $y = x - \frac{1}{2} \sin x$

Solution:

$$\frac{1}{2} \sin x$$

Exercise:

Problem: $y = \frac{1}{x} + \tan x$

Exercise:

Problem: $y = 2 \csc x$

Solution:

$$\csc(x) (3 \csc^2(x) - 1 + \cot^2(x))$$

Exercise:

Problem: $y = \sec^2 x$

Exercise:

Problem: Find all x values on the graph of $f(x) = -3 \sin x \cos x$ where the tangent line is horizontal.

Solution:

$$\frac{(2n+1)\pi}{4}, \text{ where } n \text{ is an integer}$$

Exercise:

Problem:

Find all x values on the graph of $f(x) = x - 2 \cos x$ for $0 < x < 2\pi$ where the tangent line has slope 2.

Exercise:

Problem:

Let $f(x) = \cot x$. Determine the points on the graph of f for $0 < x < 2\pi$ where the tangent line(s) is (are) parallel to the line $y = -2x$.

Solution:

$$\left(\frac{\pi}{4}, 1\right), \left(\frac{3\pi}{4}, -1\right)$$

Exercise:

Problem:

[T] A mass on a spring bounces up and down in simple harmonic motion, modeled by the function $s(t) = -6 \cos t$ where s is measured in inches and t is measured in seconds. Find the rate at which the spring is oscillating at $t = 5$ s.

Exercise:**Problem:**

Let the position of a swinging pendulum in simple harmonic motion be given by $s(t) = a \cos t + b \sin t$. Find the constants a and b such that when the velocity is 3 cm/s, $s = 0$ and $t = 0$.

Solution:

$$a = 0, b = 3$$

Exercise:**Problem:**

After a diver jumps off a diving board, the edge of the board oscillates with position given by $s(t) = -5 \cos t$ cm at t seconds after the jump.

- Sketch one period of the position function for $t \geq 0$.
- Find the velocity function.
- Sketch one period of the velocity function for $t \geq 0$.
- Determine the times when the velocity is 0 over one period.
- Find the acceleration function.
- Sketch one period of the acceleration function for $t \geq 0$.

Exercise:**Problem:**

The number of hamburgers sold at a fast-food restaurant in Pasadena, California, is given by $y = 10 + 5 \sin x$ where y is the number of hamburgers sold and x represents the number of hours after the restaurant opened at 11 a.m. until 11 p.m., when the store closes. Find y' and determine the intervals where the number of burgers being sold is increasing.

Solution:

$$y' = 5 \cos(x), \text{ increasing on } \left(0, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right), \text{ and } \left(\frac{7\pi}{2}, 12\right)$$

Exercise:**Problem:**

[T] The amount of rainfall per month in Phoenix, Arizona, can be approximated by $y(t) = 0.5 + 0.3 \cos t$, where t is months since January. Find y' and use a calculator to determine the intervals where the amount of rain falling is decreasing.

For the following exercises, use the quotient rule to derive the given equations.

Exercise:

Problem: $\frac{d}{dx}(\cot x) = -\csc^2 x$

Exercise:

Problem: $\frac{d}{dx}(\sec x) = \sec x \tan x$

Exercise:

Problem: $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Exercise:

Problem: Use the definition of derivative and the identity

$$\cos (x+h)=\cos x \cos h-\sin x \sin h \text { to prove that } \frac{d(\cos x)}{d x}=-\sin x .$$

For the following exercises, find the requested higher-order derivative for the given functions.

Exercise:

Problem: $\frac{d^3 y}{d x^3}$ of $y=3 \cos x$

Solution:

$$3 \sin x$$

Exercise:

Problem: $\frac{d^2 y}{d x^2}$ of $y=3 \sin x+x^2 \cos x$

Exercise:

Problem: $\frac{d^4 y}{d x^4}$ of $y=5 \cos x$

Solution:

$$5 \cos x$$

Exercise:

Problem: $\frac{d^2 y}{d x^2}$ of $y=\sec x+\cot x$

Exercise:

Problem: $\frac{d^3 y}{d x^3}$ of $y=x^{10}-\sec x$

Solution:

$$720 x^7-5 \tan (x) \sec ^3(x)-\tan ^3(x) \sec (x)$$

The Chain Rule

- State the chain rule for the composition of two functions.
- Apply the chain rule together with the power rule.
- Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.
- Recognize the chain rule for a composition of three or more functions.
- Describe the proof of the chain rule.

We have seen the techniques for differentiating basic functions (x^n , $\sin x$, $\cos x$, etc.) as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $h(x) = \sin(x^3)$ or $k(x) = \sqrt{3x^2 + 1}$. In this section, we study the rule for finding the derivative of the composition of two or more functions.

Deriving the Chain Rule

When we have a function that is a composition of two or more functions, we could use all of the techniques we have already learned to differentiate it. However, using all of those techniques to break down a function into simpler parts that we are able to differentiate can get cumbersome. Instead, we use the **chain rule**, which states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

To put this rule into context, let's take a look at an example: $h(x) = \sin(x^3)$. We can think of the derivative of this function with respect to x as the rate of change of $\sin(x^3)$ relative to the change in x . Consequently, we want to know how $\sin(x^3)$ changes as x changes. We can think of this event as a chain reaction: As x changes, x^3 changes, which leads to a change in $\sin(x^3)$. This chain reaction gives us hints as to what is involved in computing the derivative of $\sin(x^3)$. First of all, a change in x forcing a change in x^3 suggests that somehow the derivative of x^3 is involved. In addition, the change in x^3 forcing a change in $\sin(x^3)$ suggests that the derivative of $\sin(u)$ with respect to u , where $u = x^3$, is also part of the final derivative.

We can take a more formal look at the derivative of $h(x) = \sin(x^3)$ by setting up the limit that would give us the derivative at a specific value a in the domain of $h(x) = \sin(x^3)$.

Equation:

$$h'(a) = \lim_{x \rightarrow a} \frac{\sin(x^3) - \sin(a^3)}{x - a}.$$

This expression does not seem particularly helpful; however, we can modify it by multiplying and dividing by the expression $x^3 - a^3$ to obtain

Equation:

$$h'(a) = \lim_{x \rightarrow a} \frac{\sin(x^3) - \sin(a^3)}{x^3 - a^3} \cdot \frac{x^3 - a^3}{x - a}.$$

From the definition of the derivative, we can see that the second factor is the derivative of x^3 at $x = a$. That is,

Equation:

$$\lim_{x \rightarrow a} \frac{x^3 - a^3}{x - a} = \frac{d}{dx}(x^3) = 3a^2.$$

However, it might be a little more challenging to recognize that the first term is also a derivative. We can see this by letting $u = x^3$ and observing that as $x \rightarrow a$, $u \rightarrow a^3$:

Equation:

$$\begin{aligned}\lim_{x \rightarrow a} \frac{\sin(x^3) - \sin(a^3)}{x^3 - a^3} &= \lim_{u \rightarrow a^3} \frac{\sin u - \sin(a^3)}{u - a^3} \\ &= \frac{d}{du}(\sin u)_{u=a^3} \\ &= \cos(a^3).\end{aligned}$$

Thus, $h'(a) = \cos(a^3) \cdot 3a^2$.

In other words, if $h(x) = \sin(x^3)$, then $h'(x) = \cos(x^3) \cdot 3x^2$. Thus, if we think of $h(x) = \sin(x^3)$ as the composition $(f \circ g)(x) = f(g(x))$ where $f(x) = \sin x$ and $g(x) = x^3$, then the derivative of $h(x) = \sin(x^3)$ is the product of the derivative of $g(x) = x^3$ and the derivative of the function $f(x) = \sin x$ evaluated at the function $g(x) = x^3$. At this point, we anticipate that for $h(x) = \sin(g(x))$, it is quite likely that $h'(x) = \cos(g(x))g'(x)$. As we determined above, this is the case for $h(x) = \sin(x^3)$.

Now that we have derived a special case of the chain rule, we state the general case and then apply it in a general form to other composite functions. An informal proof is provided at the end of the section.

Note:

Rule: The Chain Rule

Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g(x)$, the derivative of the composite function

Equation:

$$h(x) = (f \circ g)(x) = f(g(x))$$

is given by

Equation:

$$h'(x) = f'(g(x))g'(x).$$

Alternatively, if y is a function of u , and u is a function of x , then

Equation:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Note:

Watch an [animation](#) of the chain rule.

Note:

Problem-Solving Strategy: Applying the Chain Rule

1. To differentiate $h(x) = f(g(x))$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f'(x)$ and evaluate it at $g(x)$ to obtain $f'(g(x))$.
3. Find $g'(x)$.
4. Write $h'(x) = f'(g(x)) \cdot g'(x)$.

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

The Chain and Power Rules Combined

We can now apply the chain rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h(x) = (g(x))^n$, we need to use the chain rule combined with the power rule. To do so, we can think of $h(x) = (g(x))^n$ as $f(g(x))$ where $f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Thus, $f'(g(x)) = n(g(x))^{n-1}$. This leads us to the derivative of a power function using the chain rule,

Equation:

$$h'(x) = n(g(x))^{n-1}g'(x)$$

Note:

Rule: Power Rule for Composition of Functions

For all values of x for which the derivative is defined, if

Equation:

$$h(x) = (g(x))^n.$$

Then

Equation:

$$h'(x) = n(g(x))^{n-1}g'(x).$$

Example:

Exercise:

Problem:

Using the Chain and Power Rules

Find the derivative of $h(x) = \frac{1}{(3x^2+1)^2}$.

Solution:

First, rewrite $h(x) = \frac{1}{(3x^2+1)^2} = (3x^2+1)^{-2}$.

Applying the power rule with $g(x) = 3x^2+1$, we have

Equation:

$$h'(x) = -2(3x^2+1)^{-3}(6x).$$

Rewriting back to the original form gives us

Equation:

$$h'(x) = \frac{-12x}{(3x^2 + 1)^3}.$$

Note:

Exercise:

Problem: Find the derivative of $h(x) = (2x^3 + 2x - 1)^4$.

Solution:

$$h'(x) = 4(2x^3 + 2x - 1)^3(6x + 2) = 8(3x + 1)(2x^3 + 2x - 1)^3$$

Hint

Use [\[link\]](#) with $g(x) = 2x^3 + 2x - 1$

Example:

Exercise:

Problem:

Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3 x$.

Solution:

First recall that $\sin^3 x = (\sin x)^3$, so we can rewrite $h(x) = \sin^3 x$ as $h(x) = (\sin x)^3$.

Applying the power rule with $g(x) = \sin x$, we obtain

Equation:

$$h'(x) = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x.$$

Example:

Exercise:

Problem:

Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $h(x) = \frac{1}{(3x-5)^2}$ at $x = 2$.

Solution:

Because we are finding an equation of a line, we need a point. The x-coordinate of the point is 2. To find the y-coordinate, substitute 2 into $h(x)$. Since $h(2) = \frac{1}{(3(2)-5)^2} = 1$, the point is $(2, 1)$.

For the slope, we need $h'(2)$. To find $h'(x)$, first we rewrite $h(x) = (3x - 5)^{-2}$ and apply the power rule to obtain

Equation:

$$h'(x) = -2(3x - 5)^{-3}(3) = -6(3x - 5)^{-3}.$$

By substituting, we have $h'(2) = -6(3(2) - 5)^{-3} = -6$. Therefore, the line has equation $y - 1 = -6(x - 2)$. Rewriting, the equation of the line is $y = -6x + 13$.

Note:

Exercise:

Problem: Find the equation of the line tangent to the graph of $f(x) = (x^2 - 2)^3$ at $x = -2$.

Solution:

$$y = -48x - 88$$

Hint

Use the preceding example as a guide.

Combining the Chain Rule with Other Rules

Now that we can combine the chain rule and the power rule, we examine how to combine the chain rule with the other rules we have learned. In particular, we can use it with the formulas for the derivatives of trigonometric functions or with the product rule.

Example:

Exercise:

Problem:

Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x) = \cos(g(x))$.

Solution:

Think of $h(x) = \cos(g(x))$ as $f(g(x))$ where $f(x) = \cos x$. Since $f'(x) = -\sin x$, we have $f'(g(x)) = -\sin(g(x))$. Then we do the following calculation.

Equation:

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) && \text{Apply the chain rule.} \\ &= -\sin(g(x))g'(x) && \text{Substitute } f'(g(x)) = -\sin(g(x)). \end{aligned}$$

Thus, the derivative of $h(x) = \cos(g(x))$ is given by $h'(x) = -\sin(g(x))g'(x)$.

In the following example we apply the rule that we have just derived.

Example:

Exercise:

Problem:

Using the Chain Rule on a Cosine Function

Find the derivative of $h(x) = \cos(5x^2)$.

Solution:

Let $g(x) = 5x^2$. Then $g'(x) = 10x$. Using the result from the previous example,

Equation:

$$\begin{aligned}h'(x) &= -\sin(5x^2) \cdot 10x \\ &= -10x \sin(5x^2).\end{aligned}$$

Example:

Exercise:

Problem:

Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h(x) = \sec(4x^5 + 2x)$.

Solution:

Apply the chain rule to $h(x) = \sec(g(x))$ to obtain

Equation:

$$h'(x) = \sec(g(x)) \tan(g(x)) g'(x).$$

In this problem, $g(x) = 4x^5 + 2x$, so we have $g'(x) = 20x^4 + 2$. Therefore, we obtain

Equation:

$$\begin{aligned}h'(x) &= \sec(4x^5 + 2x) \tan(4x^5 + 2x) (20x^4 + 2) \\ &= (20x^4 + 2) \sec(4x^5 + 2x) \tan(4x^5 + 2x).\end{aligned}$$

Note:

Exercise:

Problem: Find the derivative of $h(x) = \sin(7x + 2)$.

Solution:

$$h'(x) = 7 \cos(7x + 2)$$

Hint

Apply the chain rule to $h(x) = \sin g(x)$ first and then use $g(x) = 7x + 2$.

At this point we provide a list of derivative formulas that may be obtained by applying the chain rule in conjunction with the formulas for derivatives of trigonometric functions. Their derivations are similar to those used in [\[link\]](#) and [\[link\]](#). For convenience, formulas are also given in Leibniz's notation, which some students find easier to remember. (We discuss the chain rule using Leibniz's notation at the end of this section.) It is not absolutely necessary to memorize these as separate formulas as they are all applications of the chain rule to previously learned formulas.

Note:

Using the Chain Rule with Trigonometric Functions

For all values of x for which the derivative is defined,

Equation:

$$\frac{d}{dx}(\sin(g(x))) = \cos(g(x))g'(x)$$

$$\frac{d}{dx}(\cos(g(x))) = -\sin(g(x))g'(x)$$

$$\frac{d}{dx}(\tan(g(x))) = \sec^2(g(x))g'(x)$$

$$\frac{d}{dx}(\cot(g(x))) = -\csc^2(g(x))g'(x)$$

$$\frac{d}{dx}(\sec(g(x))) = \sec(g(x))\tan(g(x))g'(x)$$

$$\frac{d}{dx}(\csc(g(x))) = -\csc(g(x))\cot(g(x))g'(x)$$

$$\frac{d}{dx}\sin u = \cos u \frac{du}{dx}$$

$$\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$$

$$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}$$

$$\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}$$

$$\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx}$$

$$\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}$$

Example:

Exercise:

Problem:

Combining the Chain Rule with the Product Rule

Find the derivative of $h(x) = (2x + 1)^5(3x - 2)^7$.

Solution:

First apply the product rule, then apply the chain rule to each term of the product.

Equation:

$$h'(x) = \frac{d}{dx}((2x + 1)^5) \cdot (3x - 2)^7 + \frac{d}{dx}((3x - 2)^7) \cdot (2x + 1)^5$$

$$= 5(2x + 1)^4 \cdot 2 \cdot (3x - 2)^7 + 7(3x - 2)^6 \cdot 3 \cdot (2x + 1)^5$$

$$= 10(2x + 1)^4(3x - 2)^7 + 21(3x - 2)^6(2x + 1)^5$$

$$= (2x + 1)^4(3x - 2)^6(10(3x - 2) + 21(2x + 1))$$

$$= (2x + 1)^4(3x - 2)^6(72x - 49)$$

Apply the product rule.

Apply the chain rule.

Simplify.

Factor out $(2x + 1)^4(3x - 2)^6$.

Simplify.

Note:

Exercise:

Problem: Find the derivative of $h(x) = \frac{x}{(2x+3)^3}$.

Solution:

$$h'(x) = \frac{3-4x}{(2x+3)^4}$$

Hint

Start out by applying the quotient rule. Remember to use the chain rule to differentiate the denominator.

Composites of Three or More Functions

We can now combine the chain rule with other rules for differentiating functions, but when we are differentiating the composition of three or more functions, we need to apply the chain rule more than once. If we look at this situation in general terms, we can generate a formula, but we do not need to remember it, as we can simply apply the chain rule multiple times.

In general terms, first we let

Equation:

$$k(x) = h(f(g(x))).$$

Then, applying the chain rule once we obtain

Equation:

$$k'(x) = \frac{d}{dx}(h(f(g(x)))) = h'(f(g(x))) \cdot \frac{d}{dx}f(g(x)).$$

Applying the chain rule again, we obtain

Equation:

$$k'(x) = h'(f(g(x)))f'(g(x))g'(x).$$

Note:

Rule: Chain Rule for a Composition of Three Functions

For all values of x for which the function is differentiable, if

Equation:

$$k(x) = h(f(g(x))),$$

then

Equation:

$$k'(x) = h'(f(g(x)))f'(g(x))g'(x).$$

In other words, we are applying the chain rule twice.

Notice that the derivative of the composition of three functions has three parts. (Similarly, the derivative of the composition of four functions has four parts, and so on.) Also, *remember, we can always work from the outside in, taking one derivative at a time.*

Example:

Exercise:

Problem:

Differentiating a Composite of Three Functions

Find the derivative of $k(x) = \cos^4(7x^2 + 1)$.

Solution:

First, rewrite $k(x)$ as

Equation:

$$k(x) = (\cos(7x^2 + 1))^4.$$

Then apply the chain rule several times.

Equation:

$$\begin{aligned} k'(x) &= 4(\cos(7x^2 + 1))^3 \left(\frac{d}{dx}(\cos(7x^2 + 1)) \right) && \text{Apply the chain rule.} \\ &= 4(\cos(7x^2 + 1))^3 (-\sin(7x^2 + 1)) \left(\frac{d}{dx}(7x^2 + 1) \right) && \text{Apply the chain rule.} \\ &= 4(\cos(7x^2 + 1))^3 (-\sin(7x^2 + 1)) (14x) && \text{Apply the chain rule.} \\ &= -56x \sin(7x^2 + 1) \cos^3(7x^2 + 1) && \text{Simplify.} \end{aligned}$$

Note:

Exercise:

Problem: Find the derivative of $h(x) = \sin^6(x^3)$.

Solution:

$$h'(x) = 18x^2 \sin^5(x^3) \cos(x^3)$$

Hint

Rewrite $h(x) = \sin^6(x^3) = (\sin(x^3))^6$ and use [\[link\]](#) as a guide.

Example:

Exercise:

Problem:

Using the Chain Rule in a Velocity Problem

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sin(2t) + \cos(3t)$. What is the velocity of the particle at time $t = \frac{\pi}{6}$?

Solution:

To find $v(t)$, the velocity of the particle at time t , we must differentiate $s(t)$. Thus,

Equation:

$$v(t) = s'(t) = 2\cos(2t) - 3\sin(3t).$$

Substituting $t = \frac{\pi}{6}$ into $v(t)$, we obtain $v\left(\frac{\pi}{6}\right) = -2$.

Note:

Exercise:

Problem:

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sin(4t)$. Find its acceleration at time t .

Solution:

$$a(t) = -16\sin(4t)$$

Hint

Acceleration is the second derivative of position.

Proof

At this point, we present a very informal proof of the chain rule. For simplicity's sake we ignore certain issues: For example, we assume that $g(x) \neq g(a)$ for $x \neq a$ in some open interval containing a . We begin by applying the limit definition of the derivative to the function $h(x)$ to obtain $h'(a)$:

Equation:

$$h'(a) = \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{x - a}.$$

Rewriting, we obtain

Equation:

$$h'(a) = \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a}.$$

Although it is clear that

Equation:

$$\lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} = g'(a),$$

it is not obvious that

Equation:

$$\lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = f'(g(a)).$$

To see that this is true, first recall that since g is differentiable at a , g is also continuous at a . Thus,

Equation:

$$\lim_{x \rightarrow a} g(x) = g(a).$$

Next, make the substitution $y = g(x)$ and $b = g(a)$ and use change of variables in the limit to obtain

Equation:

$$\lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = \lim_{y \rightarrow b} \frac{f(y) - f(b)}{y - b} = f'(b) = f'(g(a)).$$

Finally,

Equation:

$$h'(a) = \lim_{x \rightarrow a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} = f'(g(a))g'(a).$$

□

Example:

Exercise:

Problem:

Using the Chain Rule with Functional Values

Let $h(x) = f(g(x))$. If $g(1) = 4$, $g'(1) = 3$, and $f'(4) = 7$, find $h'(1)$.

Solution:

Use the chain rule, then substitute.

Equation:

$h'(1)$	$= f'(g(1))g'(1)$	Apply the chain rule.
	$= f'(4) \cdot 3$	Substitute $g(1) = 4$ and $g'(1) = 3$.
	$= 7 \cdot 3$	Substitute $f'(4) = 7$.
	$= 21$	Simplify.

Note:

Exercise:

Problem: Given $h(x) = f(g(x))$. If $g(2) = -3$, $g'(2) = 4$, and $f'(-3) = 7$, find $h'(2)$.

Solution:

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Hint

Follow [\[link\]](#).

The Chain Rule Using Leibniz's Notation

As with other derivatives that we have seen, we can express the chain rule using Leibniz's notation. This notation for the chain rule is used heavily in physics applications.

For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = f(u)$. Thus,

Equation:

$$h'(x) = \frac{dy}{dx}, f'(g(x)) = f'(u) = \frac{dy}{du} \text{ and } g'(x) = \frac{du}{dx}.$$

Consequently,

Equation:

$$\frac{dy}{dx} = h'(x) = f'(g(x))g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Note:

Rule: Chain Rule Using Leibniz's Notation

If y is a function of u , and u is a function of x , then

Equation:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example:

Exercise:

Problem:

Taking a Derivative Using Leibniz's Notation, Example 1

Find the derivative of $y = \left(\frac{x}{3x+2}\right)^5$.

Solution:

First, let $u = \frac{x}{3x+2}$. Thus, $y = u^5$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$. Using the quotient rule,

Equation:

$$\frac{du}{dx} = \frac{2}{(3x+2)^2}$$

and

Equation:

$$\frac{dy}{du} = 5u^4.$$

Finally, we put it all together.

Equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 5u^4 \cdot \frac{2}{(3x+2)^2} \\ &= 5\left(\frac{x}{3x+2}\right)^4 \cdot \frac{2}{(3x+2)^2} \\ &= \frac{10x^4}{(3x+2)^6} \end{aligned}$$

Apply the chain rule.

Substitute $\frac{dy}{du} = 5u^4$ and $\frac{du}{dx} = \frac{2}{(3x+2)^2}$.

Substitute $u = \frac{x}{3x+2}$.

Simplify.

It is important to remember that, when using the Leibniz form of the chain rule, the final answer must be expressed entirely in terms of the original variable given in the problem.

Example:

Exercise:

Problem:

Taking a Derivative Using Leibniz's Notation, Example 2

Find the derivative of $y = \tan(4x^2 - 3x + 1)$.

Solution:

First, let $u = 4x^2 - 3x + 1$. Then $y = \tan u$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$:

Equation:

$$\frac{du}{dx} = 8x - 3 \text{ and } \frac{dy}{du} = \sec^2 u.$$

Finally, we put it all together.

Equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \sec^2 u \cdot (8x - 3) \\ &= \sec^2(4x^2 - 3x + 1) \cdot (8x - 3) \end{aligned}$$

Apply the chain rule.

Use $\frac{du}{dx} = 8x - 3$ and $\frac{dy}{du} = \sec^2 u$.

Substitute $u = 4x^2 - 3x + 1$.

Note:**Exercise:****Problem:**

Use Leibniz's notation to find the derivative of $y = \cos(x^3)$. Make sure that the final answer is expressed entirely in terms of the variable x .

Solution:

$$\frac{dy}{dx} = -3x^2 \sin(x^3)$$

Hint

Let $u = x^3$.

Key Concepts

- The chain rule allows us to differentiate compositions of two or more functions. It states that for $h(x) = f(g(x))$,

Equation:

$$h'(x) = f'(g(x))g'(x).$$

In Leibniz's notation this rule takes the form

Equation:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- We can use the chain rule with other rules that we have learned, and we can derive formulas for some of them.
- The chain rule combines with the power rule to form a new rule:

Equation:

$$\text{If } h(x) = (g(x))^n, \text{ then } h'(x) = n(g(x))^{n-1}g'(x).$$

- When applied to the composition of three functions, the chain rule can be expressed as follows: If $h(x) = f(g(k(x)))$, then $h'(x) = f'(g(k(x)))g'(k(x))k'(x)$.

Key Equations

- The chain rule**

$$h'(x) = f'(g(x))g'(x)$$

- The power rule for functions**

$$h'(x) = n(g(x))^{n-1}g'(x)$$

For the following exercises, given $y = f(u)$ and $u = g(x)$, find $\frac{dy}{dx}$ by using Leibniz's notation for the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$.

Exercise:

Problem: $y = 3u - 6, u = 2x^2$

Exercise:

Problem: $y = 6u^3, u = 7x - 4$

Solution:

$$18u^2 \cdot 7 = 18(7x - 4)^2 \cdot 7$$

Exercise:

Problem: $y = \sin u, u = 5x - 1$

Exercise:

Problem: $y = \cos u, u = \frac{-x}{8}$

Solution:

$$-\sin u \cdot \frac{-1}{8} = -\sin\left(\frac{-x}{8}\right) \cdot \frac{-1}{8}$$

Exercise:

Problem: $y = \tan u, u = 9x + 2$

Exercise:

Problem: $y = \sqrt{4u + 3}, u = x^2 - 6x$

Solution:

$$\frac{8x-24}{2\sqrt{4u+3}} = \frac{4x-12}{\sqrt{4x^2-24x+3}}$$

For each of the following exercises,

- decompose each function in the form $y = f(u)$ and $u = g(x)$, and
- find $\frac{dy}{dx}$ as a function of x .

Exercise:

Problem: $y = (3x - 2)^6$

Exercise:

Problem: $y = (3x^2 + 1)^3$

Solution:

a. $u = 3x^2 + 1$; b. $18x(3x^2 + 1)^2$

Exercise:

Problem: $y = \sin^5(x)$

Exercise:

Problem: $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$

Solution:

a. $f(u) = u^7, u = \frac{x}{7} + \frac{7}{x}$; b. $7\left(\frac{x}{7} + \frac{7}{x}\right)^6 \cdot \left(\frac{1}{7} - \frac{7}{x^2}\right)$

Exercise:

Problem: $y = \tan(\sec x)$

Exercise:

Problem: $y = \csc(\pi x + 1)$

Solution:

a. $f(u) = \csc u, u = \pi x + 1$; b. $-\pi \csc(\pi x + 1) \cdot \cot(\pi x + 1)$

Exercise:

Problem: $y = \cot^2 x$

Exercise:

Problem: $y = -6\sin^{-3}x$

Solution:

a. $f(u) = -6u^{-3}, u = \sin x$, b. $18\sin^{-4}x \cdot \cos x$

For the following exercises, find $\frac{dy}{dx}$ for each function.

Exercise:

Problem: $y = (3x^2 + 3x - 1)^4$

Exercise:

Problem: $y = (5 - 2x)^{-2}$

Solution:

$$\frac{4}{(5-2x)^3}$$

Exercise:

Problem: $y = \cos^3(\pi x)$

Exercise:

Problem: $y = (2x^3 - x^2 + 6x + 1)^3$

Solution:

$$6(2x^3 - x^2 + 6x + 1)^2(3x^2 - x + 3)$$

Exercise:

Problem: $y = \frac{1}{\sin^2(x)}$

Exercise:

Problem: $y = (\tan x + \sin x)^{-3}$

Solution:

$$-3(\tan x + \sin x)^{-4} \cdot (\sec^2 x + \cos x)$$

Exercise:

Problem: $y = x^2 \cos^4 x$

Exercise:

Problem: $y = \sin(\cos 7x)$

Solution:

$$-7 \cos(\cos 7x) \cdot \sin 7x$$

Exercise:

Problem: $y = \sqrt{6 + \sec \pi x^2}$

Exercise:

Problem: $y = \cot^3(4x + 1)$

Solution:

$$-12 \cot^2(4x + 1) \cdot \csc^2(4x + 1)$$

Exercise:

Problem: Let $y = [f(x)]^3$ and suppose that $f'(1) = 4$ and $\frac{dy}{dx} = 10$ for $x = 1$. Find $f(1)$.

Exercise:

Problem: Let $y = (f(x) + 5x^2)^4$ and suppose that $f(-1) = -4$ and $\frac{dy}{dx} = 3$ when $x = -1$. Find $f'(-1)$

Solution:

$$10 \frac{3}{4}$$

Exercise:

Problem: Let $y = (f(u) + 3x)^2$ and $u = x^3 - 2x$. If $f(4) = 6$ and $\frac{dy}{dx} = 18$ when $x = 2$, find $f'(4)$.

Exercise:

Problem:

[T] Find the equation of the tangent line to $y = -\sin\left(\frac{x}{2}\right)$ at the origin. Use a calculator to graph the function and the tangent line together.

Solution:

$$y = \frac{-1}{2}x$$

Exercise:**Problem:**

[T] Find the equation of the tangent line to $y = \left(3x + \frac{1}{x}\right)^2$ at the point $(1, 16)$. Use a calculator to graph the function and the tangent line together.

Exercise:

Problem: Find the x -coordinates at which the tangent line to $y = \left(x - \frac{6}{x}\right)^8$ is horizontal.

Solution:

$$x = \pm\sqrt{6}$$

Exercise:**Problem:**

[T] Find an equation of the line that is normal to $g(\theta) = \sin^2(\pi\theta)$ at the point $\left(\frac{1}{4}, \frac{1}{2}\right)$. Use a calculator to graph the function and the normal line together.

For the following exercises, use the information in the following table to find $h'(a)$ at the given value for a .

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	5	0	2
1	1	-2	3	0
2	4	4	1	-1
3	3	-3	2	3

Exercise:

Problem: $h(x) = f(g(x))$; $a = 0$

Solution:

Exercise:

Problem: $h(x) = g(f(x)); a = 0$

Exercise:

Problem: $h(x) = (x^4 + g(x))^{-2}; a = 1$

Solution:

$$-\frac{1}{8}$$

Exercise:

Problem: $h(x) = \left(\frac{f(x)}{g(x)}\right)^2; a = 3$

Exercise:

Problem: $h(x) = f(x + f(x)); a = 1$

Solution:

$$-4$$

Exercise:

Problem: $h(x) = (1 + g(x))^3; a = 2$

Exercise:

Problem: $h(x) = g(2 + f(x^2)); a = 1$

Solution:

$$-12$$

Exercise:

Problem: $h(x) = f(g(\sin x)); a = 0$

Exercise:

Problem:

[T] The position function of a freight train is given by $s(t) = 100(t + 1)^{-2}$, with s in meters and t in seconds. At time $t = 6$ s, find the train's

- a. velocity and
 - b. acceleration.
 - c. Using a. and b. is the train speeding up or slowing down?
-

Solution:

a. $-\frac{200}{343}$ m/s, b. $\frac{600}{2401}$ m/s², c. The train is slowing down since velocity and acceleration have opposite signs.

Exercise:

Problem:

[T] A mass hanging from a vertical spring is in simple harmonic motion as given by the following position function, where t is measured in seconds and s is in inches:

$$s(t) = -3 \cos\left(\pi t + \frac{\pi}{4}\right).$$

- Determine the position of the spring at $t = 1.5$ s.
- Find the velocity of the spring at $t = 1.5$ s.

Exercise:**Problem:**

[T] The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C = 0.0001x^3 - 0.02x^2 + 3x + 300$. In t weeks production is estimated to be $x = 1600 + 100t$ boxes.

- Find the marginal cost $C'(x)$.
- Use Leibniz's notation for the chain rule, $\frac{dC}{dt} = \frac{dC}{dx} \cdot \frac{dx}{dt}$, to find the rate with respect to time t that the cost is changing.
- Use b. to determine how fast costs are increasing when $t = 2$ weeks. Include units with the answer.

Solution:

- $C'(x) = 0.0003x^2 - 0.04x + 3$
- $\frac{dC}{dt} = 100 \cdot (0.0003x^2 - 0.04x + 3)$
- Approximately \$90,300 per week

Exercise:**Problem:**

[T] The formula for the area of a circle is $A = \pi r^2$, where r is the radius of the circle. Suppose a circle is expanding, meaning that both the area A and the radius r (in inches) are expanding.

- Suppose $r = 2 - \frac{100}{(t+7)^2}$ where t is time in seconds. Use the chain rule $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the area is expanding.
- Use a. to find the rate at which the area is expanding at $t = 4$ s.

Exercise:**Problem:**

[T] The formula for the volume of a sphere is $S = \frac{4}{3}\pi r^3$, where r (in feet) is the radius of the sphere. Suppose a spherical snowball is melting in the sun.

- Suppose $r = \frac{1}{(t+1)^2} - \frac{1}{12}$ where t is time in minutes. Use the chain rule $\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$ to find the rate at which the snowball is melting.
- Use a. to find the rate at which the volume is changing at $t = 1$ min.

Solution:

- $\frac{dS}{dt} = -\frac{8\pi r^2}{(t+1)^3}$
- The volume is decreasing at a rate of $-\frac{\pi}{36}$ ft³/min.

Exercise:

Problem:

[T] The daily temperature in degrees Fahrenheit of Phoenix in the summer can be modeled by the function $T(x) = 94 - 10 \cos \left[\frac{\pi}{12}(x - 2) \right]$, where x is hours after midnight. Find the rate at which the temperature is changing at 4 p.m.

Exercise:**Problem:**

[T] The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function $D(t) = 5 \sin \left(\frac{\pi}{6}t - \frac{7\pi}{6} \right) + 8$, where t is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m.

Solution:

~2.3 ft/hr

Glossary

chain rule

the chain rule defines the derivative of a composite function as the derivative of the outer function evaluated at the inner function times the derivative of the inner function

Introduction

class="introduction"

As a rocket
is being
launched, at
what rate
should the
angle of a
video
camera
change to
continue
viewing the
rocket?

(credit:
modification
n of work
by Steve
Jurvetson,
Wikimedia
Commons)



A rocket is being launched from the ground and cameras are recording the event. A video camera is located on the ground a certain distance from the launch pad. At what rate should the angle of inclination (the angle the camera makes with the ground) change to allow the camera to record the flight of the rocket as it heads upward? (See [\[link\]](#).)

A rocket launch involves two related quantities that change over time. Being able to solve this type of problem is just one application of derivatives introduced in this chapter. We also look at derivatives involving exponential functions, logarithms, and inverse functions.

Derivatives of Inverse Functions

- Calculate the derivative of an inverse function.
- Recognize the derivatives of the standard inverse trigonometric functions.

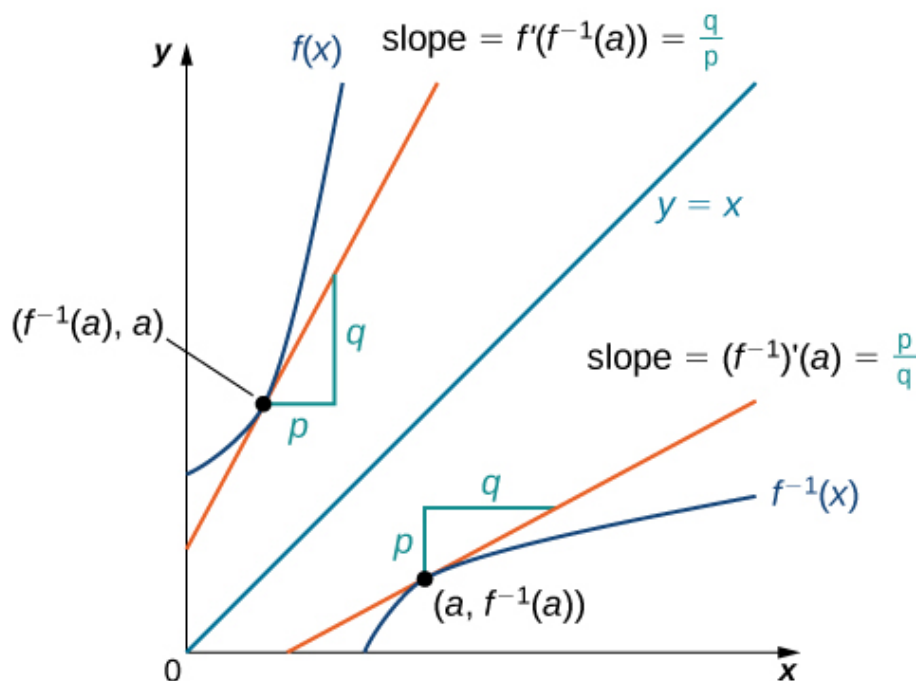
In this section we explore the relationship between the derivative of a function and the derivative of its inverse. For functions whose derivatives we already know, we can use this relationship to find derivatives of inverses without having to use the limit definition of the derivative. In particular, we will apply the formula for derivatives of inverse functions to trigonometric functions. This formula may also be used to extend the power rule to rational exponents.

The Derivative of an Inverse Function

We begin by considering a function and its inverse. If $f(x)$ is both invertible and differentiable, it seems reasonable that the inverse of $f(x)$ is also differentiable. [\[link\]](#) shows the relationship between a function $f(x)$ and its inverse $f^{-1}(x)$. Look at the point $(a, f^{-1}(a))$ on the graph of $f^{-1}(x)$ having a tangent line with a slope of $(f^{-1})'(a) = \frac{p}{q}$. This point corresponds to a point $(f^{-1}(a), a)$ on the graph of $f(x)$ having a tangent line with a slope of $f'(f^{-1}(a)) = \frac{q}{p}$. Thus, if $f^{-1}(x)$ is differentiable at a , then it must be the case that

Equation:

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$



The tangent lines of a function and its inverse are related; so, too, are the derivatives of these functions.

We may also derive the formula for the derivative of the inverse by first recalling that $x = f(f^{-1}(x))$. Then by differentiating both sides of this equation (using the chain rule on the right), we obtain

Equation:

$$1 = f'(f^{-1}(x)) (f^{-1})'(x).$$

Solving for $(f^{-1})'(x)$, we obtain

Equation:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

We summarize this result in the following theorem.

Note:**Inverse Function Theorem**

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'(f^{-1}(x)) \neq 0$,

Equation:

$$\frac{dy}{dx} = \frac{d}{dx}(f^{-1}(x)) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

Equation:

$$g(x) = \frac{1}{f'(g(x))}.$$

Example:**Exercise:****Problem:****Applying the Inverse Function Theorem**

Use the inverse function theorem to find the derivative of $g(x) = \frac{x+2}{x}$. Compare the resulting derivative to that obtained by differentiating the function directly.

Solution:

The inverse of $g(x) = \frac{x+2}{x}$ is $f(x) = \frac{2}{x-1}$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

Equation:

$$f'(x) = \frac{-2}{(x-1)^2} \text{ and } f'(g(x)) = \frac{-2}{(g(x)-1)^2} = \frac{-2}{\left(\frac{x+2}{x}-1\right)^2} = -\frac{x^2}{2}.$$

Finally,

Equation:

$$g'(x) = \frac{1}{f'(g(x))} = -\frac{2}{x^2}.$$

We can verify that this is the correct derivative by applying the quotient rule to $g(x)$ to obtain

Equation:

$$g'(x) = -\frac{2}{x^2}.$$

Note:

Exercise:

Problem:

Use the inverse function theorem to find the derivative of $g(x) = \frac{1}{x+2}$.
Compare the result obtained by differentiating $g(x)$ directly.

Solution:

$$g'(x) = -\frac{1}{(x+2)^2}$$

Hint

Use the preceding example as a guide.

Example:

Exercise:

Problem:

Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \sqrt[3]{x}$.

Solution:

The function $g(x) = \sqrt[3]{x}$ is the inverse of the function $f(x) = x^3$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

Equation:

$$f'(x) = 3x^2 \text{ and } f'(g(x)) = 3(\sqrt[3]{x})^2 = 3x^{2/3}.$$

Finally,

Equation:

$$g'(x) = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}.$$

Note:**Exercise:****Problem:**

Find the derivative of $g(x) = \sqrt[5]{x}$ by applying the inverse function theorem.

Solution:

$$g(x) = \frac{1}{5}x^{-4/5}$$

Hint

$g(x)$ is the inverse of $f(x) = x^5$.

From the previous example, we see that we can use the inverse function theorem to extend the power rule to exponents of the form $\frac{1}{n}$, where n is a

positive integer. This extension will ultimately allow us to differentiate x^q , where q is any rational number.

Note:

Extending the Power Rule to Rational Exponents

The power rule may be extended to rational exponents. That is, if n is a positive integer, then

Equation:

$$\frac{d}{dx} \left(x^{1/n} \right) = \frac{1}{n} x^{(1/n)-1}.$$

Also, if n is a positive integer and m is an arbitrary integer, then

Equation:

$$\frac{d}{dx} \left(x^{m/n} \right) = \frac{m}{n} x^{(m/n)-1}.$$

Proof

The function $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

Equation:

$$f'(x) = nx^{n-1} \text{ and } f'(g(x)) = n(x^{1/n})^{n-1} = nx^{(n-1)/n}.$$

Finally,

Equation:

$$g'(x) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n} x^{(1-n)/n} = \frac{1}{n} x^{(1/n)-1}.$$

To differentiate $x^{m/n}$ we must rewrite it as $(x^{1/n})^m$ and apply the chain rule.

Thus,

Equation:

$$\frac{d}{dx} \left(x^{m/n} \right) = \frac{d}{dx} \left(\left(x^{1/n} \right)^m \right) = m \left(x^{1/n} \right)^{m-1} \cdot \frac{1}{n} x^{(1/n)-1} = \frac{m}{n} x^{(m/n)-1}.$$

□

Example:

Exercise:

Problem:

Applying the Power Rule to a Rational Power

Find the equation of the line tangent to the graph of $y = x^{2/3}$ at $x = 8$.

Solution:

First find $\frac{dy}{dx}$ and evaluate it at $x = 8$. Since

Equation:

$$\frac{dy}{dx} = \frac{2}{3} x^{-1/3} \text{ and } \left. \frac{dy}{dx} \right|_{x=8} = \frac{1}{3}$$

the slope of the tangent line to the graph at $x = 8$ is $\frac{1}{3}$.

Substituting $x = 8$ into the original function, we obtain $y = 4$. Thus, the tangent line passes through the point $(8, 4)$. Substituting into the point-slope formula for a line, we obtain the tangent line

Equation:

$$y = \frac{1}{3}x + \frac{4}{3}.$$

Note:

Exercise:

Problem: Find the derivative of $s(t) = \sqrt{2t + 1}$.

Solution:

$$s'(t) = (2t + 1)^{-1/2}$$

Hint

Use the chain rule.

Derivatives of Inverse Trigonometric Functions

We now turn our attention to finding derivatives of inverse trigonometric functions. These derivatives will prove invaluable in the study of integration later in this text. The derivatives of inverse trigonometric functions are quite surprising in that their derivatives are actually algebraic functions. Previously, derivatives of algebraic functions have proven to be algebraic functions and derivatives of trigonometric functions have been shown to be trigonometric functions. Here, for the first time, we see that the derivative of a function need not be of the same type as the original function.

Example:

Exercise:

Problem:

Derivative of the Inverse Sine Function

Use the inverse function theorem to find the derivative of $g(x) = \sin^{-1}x$.

Solution:

Since for x in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $f(x) = \sin x$ is the inverse of $g(x) = \sin^{-1}x$, begin by finding $f'(x)$. Since

Equation:

$$f'(x) = \cos x \text{ and } f'(g(x)) = \cos(\sin^{-1}x) = \sqrt{1-x^2},$$

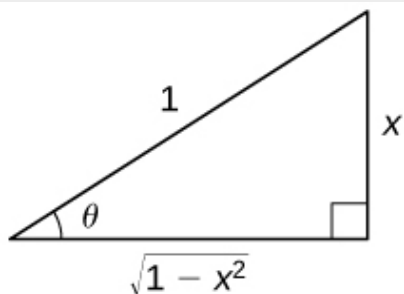
we see that

Equation:

$$g'(x) = \frac{d}{dx}(\sin^{-1}x) = \frac{1}{f'(g(x))} = \frac{1}{\sqrt{1-x^2}}.$$

Analysis

To see that $\cos(\sin^{-1}x) = \sqrt{1-x^2}$, consider the following argument. Set $\sin^{-1}x = \theta$. In this case, $\sin \theta = x$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. We begin by considering the case where $0 < \theta < \frac{\pi}{2}$. Since θ is an acute angle, we may construct a right triangle having acute angle θ , a hypotenuse of length 1 and the side opposite angle θ having length x . From the Pythagorean theorem, the side adjacent to angle θ has length $\sqrt{1-x^2}$. This triangle is shown in [\[link\]](#). Using the triangle, we see that $\cos(\sin^{-1}x) = \cos \theta = \sqrt{1-x^2}$.



Using a right triangle having acute angle θ , a hypotenuse of length 1, and the side opposite angle θ having length x , we can see that $\cos(\sin^{-1}x) = \cos \theta = \sqrt{1-x^2}$.

In the case where $-\frac{\pi}{2} < \theta < 0$, we make the observation that $0 < -\theta < \frac{\pi}{2}$ and hence

Equation:

$$\cos (\sin ^{-1} x)=\cos \theta=\cos (-\theta)=\sqrt{1-x^2}.$$

Now if $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$, $x = 1$ or $x = -1$, and since in either case $\cos \theta = 0$ and $\sqrt{1-x^2} = 0$, we have

Equation:

$$\cos (\sin ^{-1} x)=\cos \theta=\sqrt{1-x^2}.$$

Consequently, in all cases, $\cos (\sin ^{-1} x)=\sqrt{1-x^2}$.

Example:

Exercise:

Problem:

Applying the Chain Rule to the Inverse Sine Function

Apply the chain rule to the formula derived in [\[link\]](#) to find the derivative of $h(x) = \sin^{-1}(g(x))$ and use this result to find the derivative of $h(x) = \sin^{-1}(2x^3)$.

Solution:

Applying the chain rule to $h(x) = \sin^{-1}(g(x))$, we have

Equation:

$$h'(x) = \frac{1}{\sqrt{1-(g(x))^2}} g'(x).$$

Now let $g(x) = 2x^3$, so $g'(x) = 6x^2$. Substituting into the previous result, we obtain

Equation:

$$\begin{aligned} h'(x) &= \frac{1}{\sqrt{1-4x^6}} \cdot 6x^2 \\ &= \frac{6x^2}{\sqrt{1-4x^6}}. \end{aligned}$$

Note:

Exercise:

Problem:

Use the inverse function theorem to find the derivative of $g(x) = \tan^{-1}x$.

Solution:

$$g'(x) = \frac{1}{1+x^2}$$

Hint

The inverse of $g(x)$ is $f(x) = \tan x$. Use [\[link\]](#) as a guide.

The derivatives of the remaining inverse trigonometric functions may also be found by using the inverse function theorem. These formulas are provided in the following theorem.

Note:

Derivatives of Inverse Trigonometric Functions

Equation:

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - (x)^2}}$$

Equation:

$$\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1 - (x)^2}}$$

Equation:

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + (x)^2}$$

Equation:

$$\frac{d}{dx} \cot^{-1} x = \frac{-1}{1 + (x)^2}$$

Equation:

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x| \sqrt{(x)^2 - 1}}$$

Equation:

$$\frac{d}{dx} \csc^{-1} x = \frac{-1}{|x| \sqrt{(x)^2 - 1}}$$

Example:

Exercise:

Problem:

Applying Differentiation Formulas to an Inverse Tangent Function

Find the derivative of $f(x) = \tan^{-1}(x^2)$.

Solution:

Let $g(x) = x^2$, so $g'(x) = 2x$. Substituting into [\[link\]](#), we obtain

Equation:

$$f'(x) = \frac{1}{1 + (x^2)^2} \cdot (2x).$$

Simplifying, we have

Equation:

$$f'(x) = \frac{2x}{1 + x^4}.$$

Example:

Exercise:

Problem:

Applying Differentiation Formulas to an Inverse Sine Function

Find the derivative of $h(x) = x^2 \sin^{-1}x$.

Solution:

By applying the product rule, we have

Equation:

$$h'(x) = 2x \sin^{-1}x + \frac{1}{\sqrt{1 - x^2}} \cdot x^2.$$

Note:

Exercise:

Problem: Find the derivative of $h(x) = \cos^{-1}(3x - 1)$.

Solution:

$$h'(x) = \frac{-3}{\sqrt{6x-9x^2}}$$

Hint

Use [\[link\]](#), with $g(x) = 3x - 1$

Example:

Exercise:

Problem:

Applying the Inverse Tangent Function

The position of a particle at time t is given by $s(t) = \tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity of the particle at time $t = 1$.

Solution:

Begin by differentiating $s(t)$ in order to find $v(t)$. Thus,

Equation:

$$v(t) = s'(t) = \frac{1}{1 + \left(\frac{1}{t}\right)^2} \cdot \frac{-1}{t^2}.$$

Simplifying, we have

Equation:

$$v(t) = -\frac{1}{t^2 + 1}.$$

Thus, $v(1) = -\frac{1}{2}$.

Note:

Exercise:

Problem:

Find the equation of the line tangent to the graph of $f(x) = \sin^{-1}x$ at $x = 0$.

Solution:

$$y = x$$

Hint

$f'(0)$ is the slope of the tangent line.

Key Concepts

- The inverse function theorem allows us to compute derivatives of inverse functions without using the limit definition of the derivative.
- We can use the inverse function theorem to develop differentiation formulas for the inverse trigonometric functions.

Key Equations

- **Inverse function theorem**

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$ whenever $f'(f^{-1}(x)) \neq 0$ and $f(x)$ is differentiable.

- **Power rule with rational exponents**

$$\frac{d}{dx}(x^{m/n}) = \frac{m}{n}x^{(m/n)-1}.$$

- **Derivative of inverse sine function**

$$\frac{d}{dx} \sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

- **Derivative of inverse cosine function**

$$\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-(x)^2}}$$

- **Derivative of inverse tangent function**

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+(x)^2}$$

- **Derivative of inverse cotangent function**

$$\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+(x)^2}$$

- **Derivative of inverse secant function**

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{(x)^2-1}}$$

- **Derivative of inverse cosecant function**

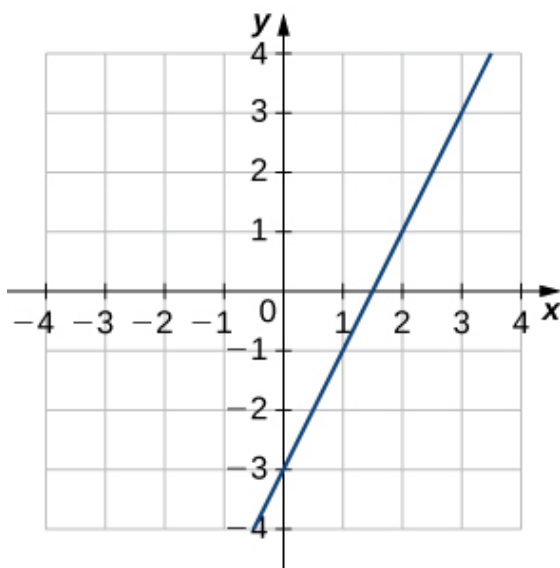
$$\frac{d}{dx} \csc^{-1} x = \frac{-1}{|x|\sqrt{(x)^2-1}}$$

For the following exercises, use the graph of $y = f(x)$ to

- sketch the graph of $y = f^{-1}(x)$, and
- use part a. to estimate $(f^{-1})'(1)$.

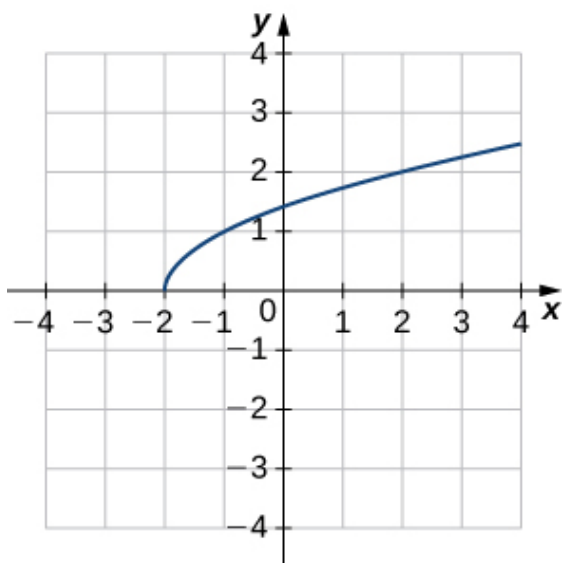
Exercise:

Problem:



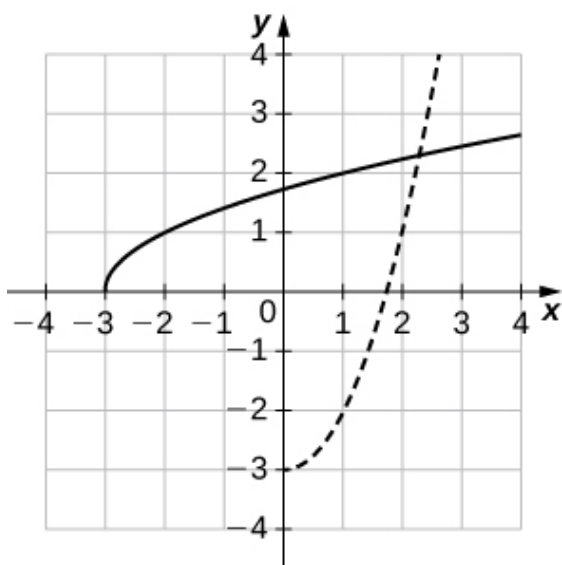
Exercise:

Problem:



Solution:

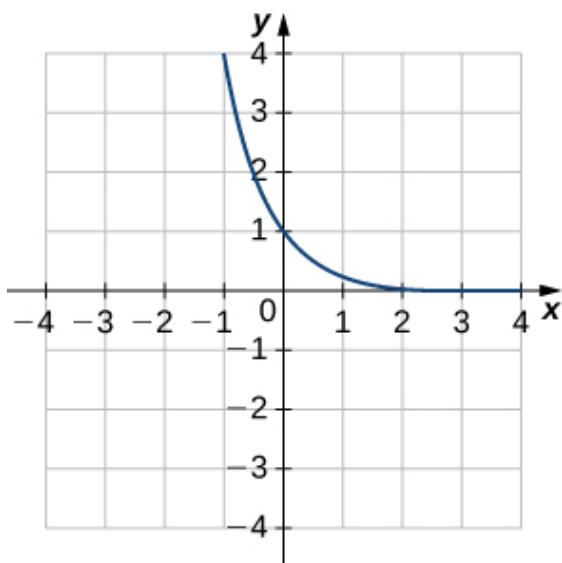
a.



b. $(f^{-1})'(1) \sim 2$

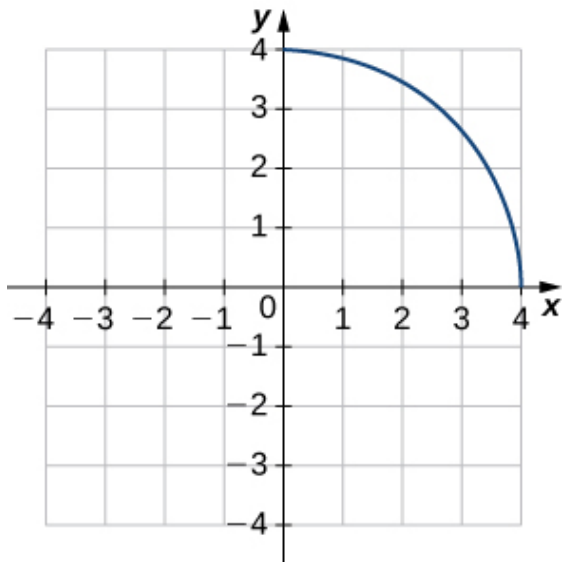
Exercise:

Problem:



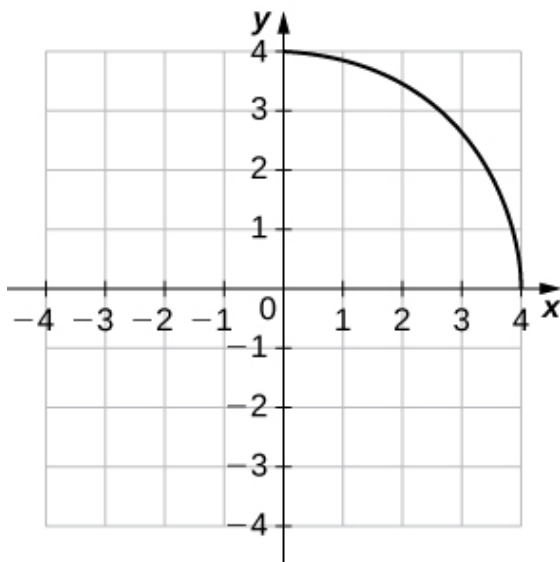
Exercise:

Problem:



Solution:

a.



b. $(f^{-1})'(1) \sim -1/\sqrt{3}$

For the following exercises, use the functions $y = f(x)$ to find

a. $\frac{df}{dx}$ at $x = a$ and

b. $x = f^{-1}(y)$.

c. Then use part b. to find $\frac{df^{-1}}{dy}$ at $y = f(a)$.

Exercise:

Problem: $f(x) = 6x - 1, x = -2$

Exercise:

Problem: $f(x) = 2x^3 - 3, x = 1$

Solution:

a. 6, b. $x = f^{-1}(y) = \left(\frac{y+3}{2}\right)^{1/3}$, c. $\frac{1}{6}$

Exercise:

Problem: $f(x) = 9 - x^2, 0 \leq x \leq 3, x = 2$

Exercise:

Problem: $f(x) = \sin x, x = 0$

Solution:

a. 1, b. $x = f^{-1}(y) = \sin^{-1}y$, c. 1

For each of the following functions, find $(f^{-1})'(a)$.

Exercise:

Problem: $f(x) = x^2 + 3x + 2, x \geq -1, a = 2$

Exercise:

Problem: $f(x) = x^3 + 2x + 3, a = 0$

Solution:

$$\frac{1}{5}$$

Exercise:

Problem: $f(x) = x + \sqrt{x}, a = 2$

Exercise:

Problem: $f(x) = x - \frac{2}{x}, x < 0, a = 1$

Solution:

$$\frac{1}{3}$$

Exercise:

Problem: $f(x) = x + \sin x, a = 0$

Exercise:

Problem: $f(x) = \tan x + 3x^2, a = 0$

Solution:

1

For each of the given functions $y = f(x)$,

- find the slope of the tangent line to its inverse function f^{-1} at the indicated point P , and
- find the equation of the tangent line to the graph of f^{-1} at the indicated point.

Exercise:

Problem: $f(x) = \frac{4}{1+x^2}, P(2, 1)$

Exercise:

Problem: $f(x) = \sqrt{x-4}, P(2, 8)$

Solution:

a. 4, b. $y = 4x$

Exercise:

Problem: $f(x) = (x^3 + 1)^4, P(16, 1)$

Exercise:

Problem: $f(x) = -x^3 - x + 2, P(-8, 2)$

Solution:

a. $-\frac{1}{96}$, b. $y = -\frac{1}{13}x + \frac{18}{13}$

Exercise:

Problem: $f(x) = x^5 + 3x^3 - 4x - 8, P(-8, 1)$

For the following exercises, find $\frac{dy}{dx}$ for the given function.

Exercise:

Problem: $y = \sin^{-1}(x^2)$

Solution:

$$\frac{2x}{\sqrt{1-x^4}}$$

Exercise:

Problem: $y = \cos^{-1}(\sqrt{x})$

Exercise:

Problem: $y = \sec^{-1}\left(\frac{1}{x}\right)$

Solution:

$$\frac{-1}{\sqrt{1-x^2}}$$

Exercise:

Problem: $y = \sqrt{\csc^{-1}x}$

Exercise:

Problem: $y = (1 + \tan^{-1}x)^3$

Solution:

$$\frac{3(1+\tan^{-1}x)^2}{1+x^2}$$

Exercise:

Problem: $y = \cos^{-1}(2x) \cdot \sin^{-1}(2x)$

Exercise:

Problem: $y = \frac{1}{\tan^{-1}(x)}$

Solution:

$$\frac{-1}{(1+x^2)(\tan^{-1}x)^2}$$

Exercise:

Problem: $y = \sec^{-1}(-x)$

Exercise:

Problem: $y = \cot^{-1}\sqrt{4-x^2}$

Solution:

$$\frac{x}{(5-x^2)\sqrt{4-x^2}}$$

Exercise:

Problem: $y = x \cdot \csc^{-1}x$

For the following exercises, use the given values to find $(f^{-1})'(a)$.

Exercise:

Problem: $f(\pi) = 0, f'(\pi) = -1, a = 0$

Solution:

$$-1$$

Exercise:

Problem: $f(6) = 2, f'(6) = \frac{1}{3}, a = 2$

Exercise:

Problem: $f\left(\frac{1}{3}\right) = -8, f'\left(\frac{1}{3}\right) = 2, a = -8$

Solution:

$$\frac{1}{2}$$

Exercise:

Problem: $f\left(\sqrt{3}\right) = \frac{1}{2}, f'\left(\sqrt{3}\right) = \frac{2}{3}, a = \frac{1}{2}$

Exercise:

Problem: $f(1) = -3, f'(1) = 10, a = -3$

Solution:

$$\frac{1}{10}$$

Exercise:

Problem: $f(1) = 0, f'(1) = -2, a = 0$

Exercise:

Problem:

[T] The position of a moving hockey puck after t seconds is $s(t) = \tan^{-1}t$ where s is in meters.

- Find the velocity of the hockey puck at any time t .
- Find the acceleration of the puck at any time t .
- Evaluate a. and b. for $t = 2, 4$, and 6 seconds.

d. What conclusion can be drawn from the results in c.?

Solution:

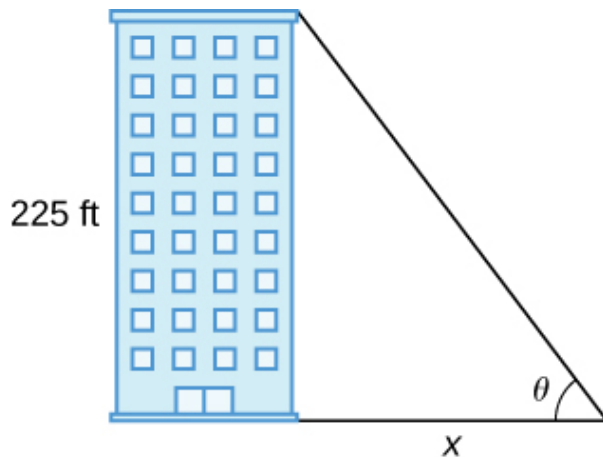
a. $v(t) = \frac{1}{1+t^2}$ b. $a(t) = \frac{-2t}{(1+t^2)^2}$ c.

(a) 0.2, 0.06, 0.03; (b) -0.16, -0.028, -0.0088 d. The hockey puck is decelerating/slowing down at 2, 4, and 6 seconds.

Exercise:

Problem:

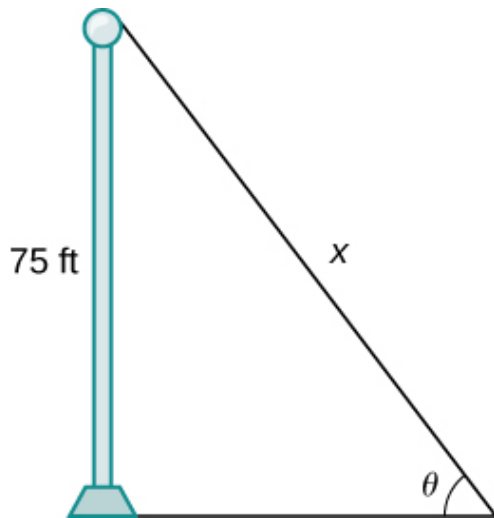
[T] A building that is 225 feet tall casts a shadow of various lengths x as the day goes by. An angle of elevation θ is formed by lines from the top and bottom of the building to the tip of the shadow, as seen in the following figure. Find the rate of change of the angle of elevation $\frac{d\theta}{dx}$ when $x = 272$ feet.



Exercise:

Problem:

[T] A pole stands 75 feet tall. An angle θ is formed when wires of various lengths of x feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 90 feet is attached.



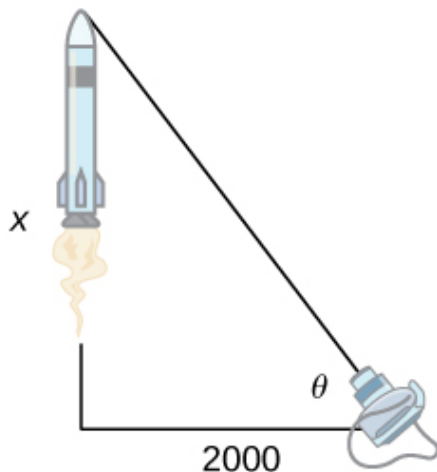
Solution:

-0.0168 radians per foot

Exercise:

Problem:

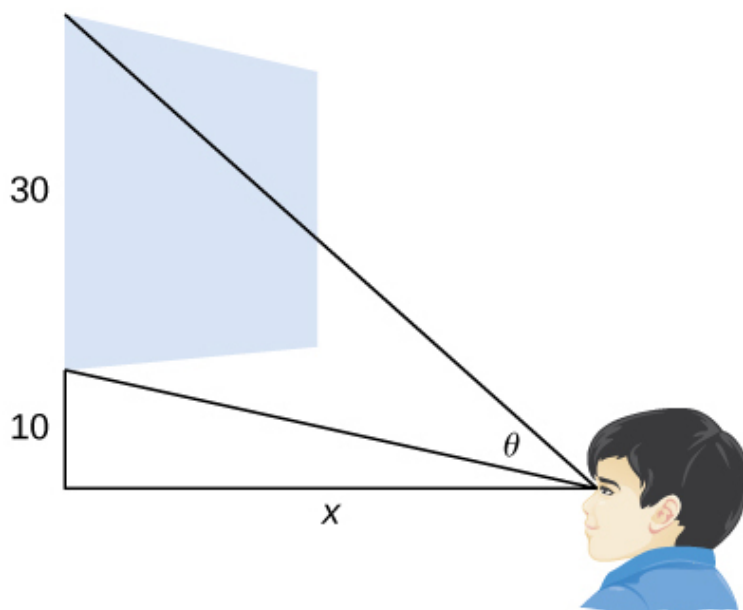
[T] A television camera at ground level is 2000 feet away from the launching pad of a space rocket that is set to take off vertically, as seen in the following figure. The angle of elevation of the camera can be found by $\theta = \tan^{-1}\left(\frac{x}{2000}\right)$, where x is the height of the rocket. Find the rate of change of the angle of elevation after launch when the camera and the rocket are 5000 feet apart.



Exercise:**Problem:**

[T] A local movie theater with a 30-foot-high screen that is 10 feet above a person's eye level when seated has a viewing angle θ (in radians) given by $\theta = \cot^{-1} \frac{x}{40} - \cot^{-1} \frac{x}{10}$,

where x is the distance in feet away from the movie screen that the person is sitting, as shown in the following figure.



- Find $\frac{d\theta}{dx}$.
- Evaluate $\frac{d\theta}{dx}$ for $x = 5, 10, 15$, and 20 .
- Interpret the results in b..
- Evaluate $\frac{d\theta}{dx}$ for $x = 25, 30, 35$, and 40 .
- Interpret the results in d. At what distance x should the person stand to maximize his or her viewing angle?

Solution:

- a. $\frac{d\theta}{dx} = \frac{10}{100+x^2} - \frac{40}{1600+x^2}$ b. $\frac{18}{325}, \frac{9}{340}, \frac{42}{4745}, 0$ c. As a person moves farther away from the screen, the viewing angle is increasing, which implies that as he or she moves farther away, his or her screen vision is

widening. d. $-\frac{54}{12905}$, $-\frac{3}{500}$, $-\frac{198}{29945}$, $-\frac{9}{1360}$ e. As the person moves beyond 20 feet from the screen, the viewing angle is decreasing. The optimal distance the person should stand for maximizing the viewing angle is 20 feet.

Implicit Differentiation

- Find the derivative of a complicated function by using implicit differentiation.
- Use implicit differentiation to determine the equation of a tangent line.

We have already studied how to find equations of tangent lines to functions and the rate of change of a function at a specific point. In all these cases we had the explicit equation for the function and differentiated these functions explicitly. Suppose instead that we want to determine the equation of a tangent line to an arbitrary curve or the rate of change of an arbitrary curve at a point. In this section, we solve these problems by finding the derivatives of functions that define y implicitly in terms of x .

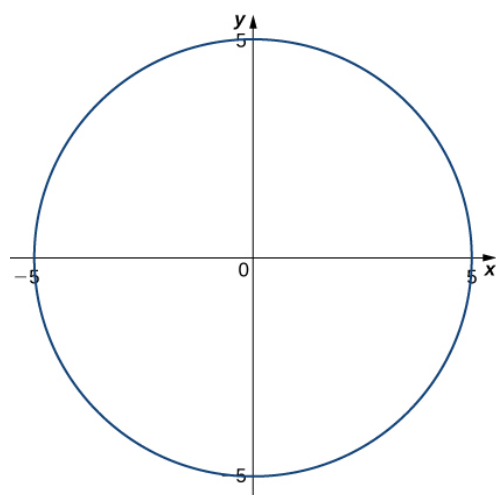
Implicit Differentiation

In most discussions of math, if the dependent variable y is a function of the independent variable x , we express y in terms of x . If this is the case, we say that y is an explicit function of x . For example, when we write the equation $y = x^2 + 1$, we are defining y explicitly in terms of x . On the other hand, if the relationship between the function y and the variable x is expressed by an equation where y is not expressed entirely in terms of x , we say that the equation defines y implicitly in terms of x . For example, the equation $y - x^2 = 1$ defines the function $y = x^2 + 1$ implicitly.

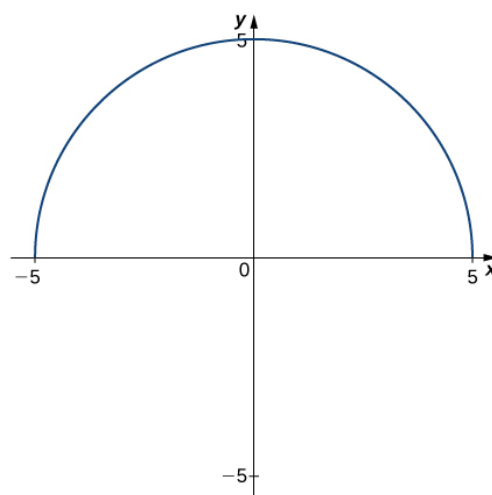
Implicit differentiation allows us to find slopes of tangents to curves that are clearly not functions (they fail the vertical line test). We are using the idea that portions of y are functions that satisfy the given equation, but that y is not actually a function of x .

In general, an equation defines a function implicitly if the function satisfies that equation. An equation may define many different functions implicitly. For example, the functions

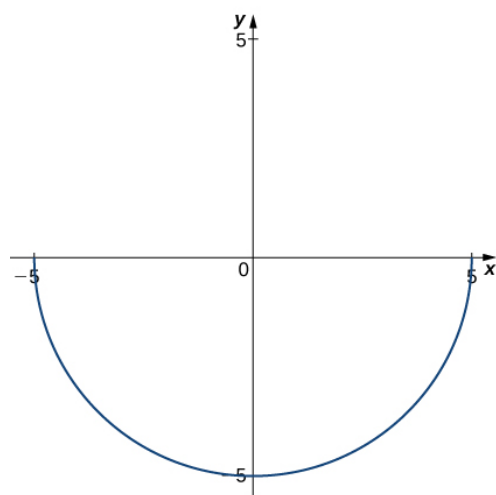
$y = \sqrt{25 - x^2}$ and $y = \begin{cases} \sqrt{25 - x^2} & \text{if } -25 \leq x < 0 \\ -\sqrt{25 - x^2} & \text{if } 0 \leq x \leq 25 \end{cases}$, which are illustrated in [\[link\]](#), are just three of the many functions defined implicitly by the equation $x^2 + y^2 = 25$.



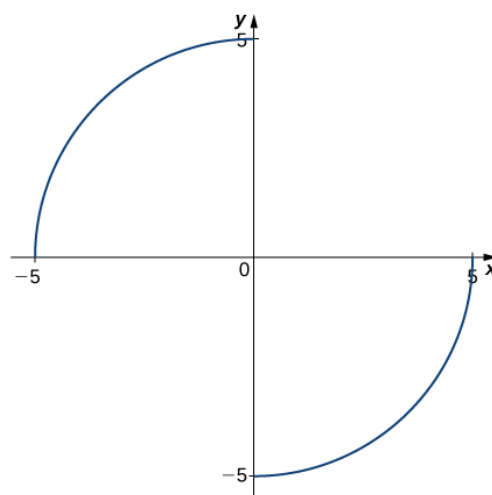
$$x^2 + y^2 = 25$$



$$y = \sqrt{25 - x^2}$$



$$y = -\sqrt{25 - x^2}$$



$$y = \begin{cases} \sqrt{25 - x^2} & \text{if } -5 < x < 0 \\ -\sqrt{25 - x^2} & \text{if } 0 < x < 5 \end{cases}$$

The equation $x^2 + y^2 = 25$ defines many functions implicitly.

If we want to find the slope of the line tangent to the graph of $x^2 + y^2 = 25$ at the point $(3, 4)$, we could evaluate the derivative of the function $y = \sqrt{25 - x^2}$ at $x = 3$. On the other hand, if we want the slope of the tangent line at the point $(3, -4)$, we could use the derivative of $y = -\sqrt{25 - x^2}$. However, it is not always easy to solve for a function defined implicitly by an equation. Fortunately, the technique of **implicit differentiation** allows us to find the derivative of an implicitly defined function without ever solving for the function explicitly. The process of finding $\frac{dy}{dx}$ using implicit differentiation is described in the following problem-solving strategy.

Note:

Problem-Solving Strategy: Implicit Differentiation

To perform implicit differentiation on an equation that defines a function y implicitly in terms of a variable x , use the following steps:

1. Take the derivative of both sides of the equation. Keep in mind that y is a function of x . Consequently, whereas $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$ because we must use the chain rule to differentiate $\sin y$ with respect to x .
2. Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and all terms that do not contain $\frac{dy}{dx}$ are on the right.
3. Factor out $\frac{dy}{dx}$ on the left.
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

Example:

Exercise:

Problem:

Using Implicit Differentiation

Assuming that y is defined implicitly by the equation $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Solution:

Follow the steps in the problem-solving strategy.

Equation:

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$	Step 1. Differentiate both sides of the equation.
$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$	Step 1.1. Use the sum rule on the left.
$2x + 2y \frac{dy}{dx} = 0$	On the right $\frac{d}{dx}(25) = 0$.
$2y \frac{dy}{dx} = -2x$	Step 1.2. Take the derivatives, so $\frac{d}{dx}(x^2) = 2x$ and $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$.
$\frac{dy}{dx} = -\frac{x}{y}$	Step 2. Keep the terms with $\frac{dy}{dx}$ on the left. Move the remaining terms to the right.
	Step 4. Divide both sides of the equation by 2y. (Step 3 does not apply in this case.)

Analysis

Note that the resulting expression for $\frac{dy}{dx}$ is in terms of both the independent variable x and the dependent variable y . Although in some cases it may be possible to express $\frac{dy}{dx}$ in terms of x only, it is generally not possible to do so.

Example:

Exercise:

Problem:

Using Implicit Differentiation and the Product Rule

Assuming that y is defined implicitly by the equation $x^3 \sin y + y = 4x + 3$, find $\frac{dy}{dx}$.

Solution:

Equation:

$$\frac{d}{dx}(x^3 \sin y + y) = \frac{d}{dx}(4x + 3)$$

$$\frac{d}{dx}(x^3 \sin y) + \frac{d}{dx}(y) = 4$$

$$\left(\frac{d}{dx}(x^3) \cdot \sin y + \frac{d}{dx}(\sin y) \cdot x^3\right) + \frac{dy}{dx} = 4$$

$$3x^2 \sin y + \left(\cos y \frac{dy}{dx}\right) \cdot x^3 + \frac{dy}{dx} = 4$$

$$x^3 \cos y \frac{dy}{dx} + \frac{dy}{dx} = 4 - 3x^2 \sin y$$

$$\frac{dy}{dx}(x^3 \cos y + 1) = 4 - 3x^2 \sin y$$

$$\frac{dy}{dx} = \frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}$$

Step 1: Differentiate both sides of the equation.

Step 1.1: Apply the sum rule on the left.

On the right, $\frac{d}{dx}(4x + 3) = 4$.

Step 1.2: Use the product rule to find

$\frac{d}{dx}(x^3 \sin y)$. Observe that $\frac{d}{dx}(y) = \frac{dy}{dx}$.

Step 1.3: We know $\frac{d}{dx}(x^3) = 3x^2$. Use the chain rule to obtain $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$.

Step 2: Keep all terms containing $\frac{dy}{dx}$ on the left. Move all other terms to the right.

Step 3: Factor out $\frac{dy}{dx}$ on the left.

Step 4: Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by $x^3 \cos y + 1$.

Example:

Exercise:

Problem:

Using Implicit Differentiation to Find a Second Derivative

Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 25$.

Solution:

In [\[link\]](#), we showed that $\frac{dy}{dx} = -\frac{x}{y}$. We can take the derivative of both sides of this equation to find $\frac{d^2y}{dx^2}$.

Equation:

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dy} \left(-\frac{x}{y} \right) \\ &= -\frac{\left(1 \cdot y - x \frac{dy}{dx}\right)}{y^2} \\ &= \frac{-y + x \frac{dy}{dx}}{y^2} \\ &= \frac{-y + x \left(-\frac{x}{y}\right)}{y^2} \\ &= \frac{-y^2 - x^2}{y^3} \end{aligned}$$

Differentiate both sides of $\frac{dy}{dx} = -\frac{x}{y}$.

Use the quotient rule to find $\frac{d}{dy} \left(-\frac{x}{y} \right)$.

Simplify.

Substitute $\frac{dy}{dx} = -\frac{x}{y}$.

Simplify.

At this point we have found an expression for $\frac{d^2y}{dx^2}$. If we choose, we can simplify the expression further by recalling that $x^2 + y^2 = 25$ and making this substitution in the numerator to obtain $\frac{d^2y}{dx^2} = -\frac{25}{y^3}$.

Note:

Exercise:

Problem: Find $\frac{dy}{dx}$ for y defined implicitly by the equation $4x^5 + \tan y = y^2 + 5x$.

Solution:

$$\frac{dy}{dx} = \frac{5-20x^4}{\sec^2 y - 2y}$$

Hint

Follow the problem solving strategy, remembering to apply the chain rule to differentiate \tan and y^2 .

Finding Tangent Lines Implicitly

Now that we have seen the technique of implicit differentiation, we can apply it to the problem of finding equations of tangent lines to curves described by equations.

Example:

Exercise:

Problem:

Finding a Tangent Line to a Circle

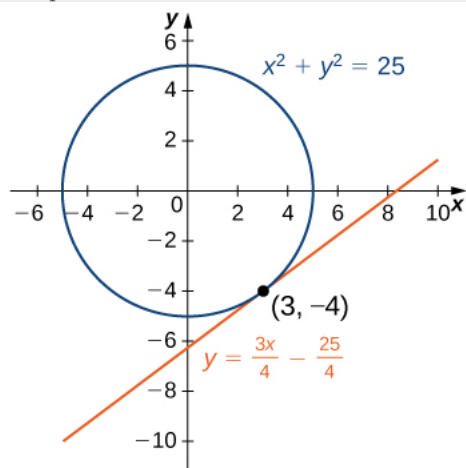
Find the equation of the line tangent to the curve $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution:

Although we could find this equation without using implicit differentiation, using that method makes it much easier. In [\[link\]](#), we found $\frac{dy}{dx} = -\frac{x}{y}$.

The slope of the tangent line is found by substituting $(3, -4)$ into this expression. Consequently, the slope of the tangent line is $\left. \frac{dy}{dx} \right|_{(3, -4)} = -\frac{3}{-4} = \frac{3}{4}$.

Using the point $(3, -4)$ and the slope $\frac{3}{4}$ in the point-slope equation of the line, we obtain the equation $y = \frac{3}{4}x - \frac{25}{4}$ ([\[link\]](#)).



The line $y = \frac{3}{4}x - \frac{25}{4}$ is tangent to $x^2 + y^2 = 25$ at

The line $y - 4 = -\frac{4}{3}(x - 3)$ is tangent to $x^3 + y^3 = 20$ at the point $(3, -4)$.

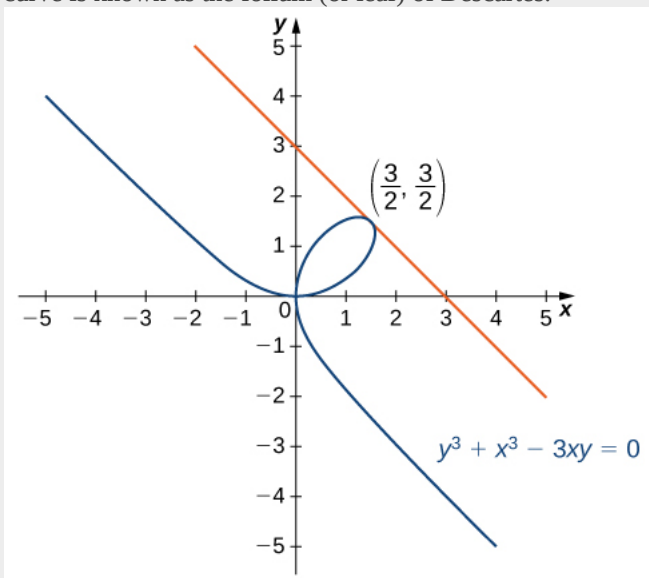
Example:

Exercise:

Problem:

Finding the Equation of the Tangent Line to a Curve

Find the equation of the line tangent to the graph of $y^3 + x^3 - 3xy = 0$ at the point $(\frac{3}{2}, \frac{3}{2})$ ([link](#)). This curve is known as the folium (or leaf) of Descartes.



Finding the tangent line to the folium of Descartes at $(\frac{3}{2}, \frac{3}{2})$.

Solution:

Begin by finding $\frac{dy}{dx}$.

Equation:

$$\begin{aligned}\frac{d}{dx}(y^3 + x^3 - 3xy) &= \frac{d}{dx}(0) \\ 3y^2 \frac{dy}{dx} + 3x^2 - \left(3y + \frac{dy}{dx} 3x\right) &= 0 \\ \frac{dy}{dx} &= \frac{3y - 3x^2}{3y^2 - 3x}.\end{aligned}$$

Next, substitute $(\frac{3}{2}, \frac{3}{2})$ into $\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$ to find the slope of the tangent line:

Equation:

$$\left. \frac{dy}{dx} \right|_{\left(\frac{3}{2}, \frac{3}{2}\right)} = -1.$$

Finally, substitute into the point-slope equation of the line to obtain

Equation:

$$y = -x + 3.$$

Example:

Exercise:

Problem:

Applying Implicit Differentiation

In a simple video game, a rocket travels in an elliptical orbit whose path is described by the equation $4x^2 + 25y^2 = 100$. The rocket can fire missiles along lines tangent to its path. The object of the game is to destroy an incoming asteroid traveling along the positive x -axis toward $(0, 0)$. If the rocket fires a missile when it is located at $\left(3, \frac{8}{3}\right)$, where will it intersect the x -axis?

Solution:

To solve this problem, we must determine where the line tangent to the graph of

$4x^2 + 25y^2 = 100$ at $\left(3, \frac{8}{3}\right)$ intersects the x -axis. Begin by finding $\frac{dy}{dx}$ implicitly.

Differentiating, we have

Equation:

$$8x + 50y \frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we have

Equation:

$$\frac{dy}{dx} = -\frac{4x}{25y}.$$

The slope of the tangent line is $\left. \frac{dy}{dx} \right|_{\left(3, \frac{8}{3}\right)} = -\frac{9}{50}$. The equation of the tangent line is $y = -\frac{9}{50}x + \frac{183}{200}$. To determine where the line intersects the x -axis, solve $0 = -\frac{9}{50}x + \frac{183}{200}$. The solution is $x = \frac{61}{3}$. The missile intersects the x -axis at the point $\left(\frac{61}{3}, 0\right)$.

Note:

Exercise:

Problem: Find the equation of the line tangent to the hyperbola $x^2 - y^2 = 16$ at the point $(5, 3)$.

Solution:

$$y = \frac{5}{3}x - \frac{16}{3}$$

Hint

$$\frac{dy}{dx} = \frac{x}{y}$$

Key Concepts

- We use implicit differentiation to find derivatives of implicitly defined functions (functions defined by equations).
- By using implicit differentiation, we can find the equation of a tangent line to the graph of a curve.

For the following exercises, use implicit differentiation to find $\frac{dy}{dx}$.

Exercise:

Problem: $x^2 - y^2 = 4$

Exercise:

Problem: $6x^2 + 3y^2 = 12$

Solution:

$$\frac{dy}{dx} = \frac{-2x}{y}$$

Exercise:

Problem: $x^2y = y - 7$

Exercise:

Problem: $3x^3 + 9xy^2 = 5x^3$

Solution:

$$\frac{dy}{dx} = \frac{x}{3y} - \frac{y}{2x}$$

Exercise:

Problem: $xy - \cos(xy) = 1$

Exercise:

Problem: $y\sqrt{x+4} = xy + 8$

Solution:

$$\frac{dy}{dx} = \frac{y - \frac{y}{2\sqrt{x+4}}}{\sqrt{x+4} - x}$$

Exercise:

Problem: $-xy - 2 = \frac{x}{7}$

Exercise:

Problem: $y \sin(xy) = y^2 + 2$

Solution:

$$\frac{dy}{dx} = \frac{y^2 \cos(xy)}{2y - \sin(xy) - xy \cos(xy)}$$

Exercise:

Problem: $(xy)^2 + 3x = y^2$

Exercise:

Problem: $x^3y + xy^3 = -8$

Solution:

$$\frac{dy}{dx} = \frac{-3x^2y - y^3}{x^3 + 3xy^2}$$

For the following exercises, find the equation of the tangent line to the graph of the given equation at the indicated point. Use a calculator or computer software to graph the function and the tangent line.

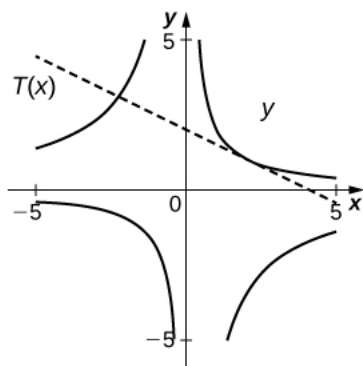
Exercise:

Problem: [T] $x^4y - xy^3 = -2, (-1, -1)$

Exercise:

Problem: [T] $x^2y^2 + 5xy = 14, (2, 1)$

Solution:



$$y = -\frac{1}{2}x + 2$$

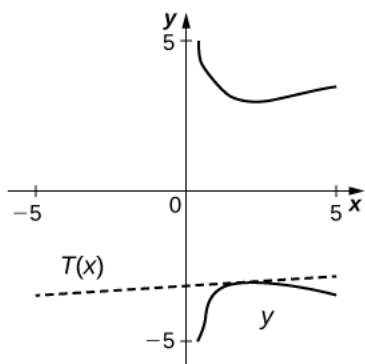
Exercise:

Problem: [T] $\tan(xy) = y, \left(\frac{\pi}{4}, 1\right)$

Exercise:

Problem: [T] $xy^2 + \sin(\pi y) - 2x^2 = 10, (2, -3)$

Solution:



$$y = \frac{1}{\pi+12}x - \frac{3\pi+38}{\pi+12}$$

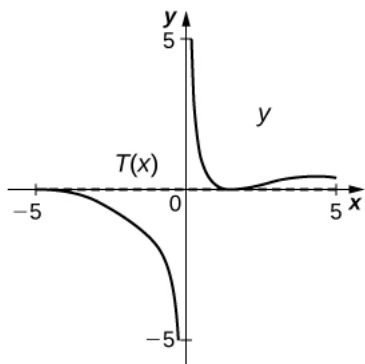
Exercise:

Problem: [T] $\frac{x}{y} + 5x - 7 = -\frac{3}{4}y, (1, 2)$

Exercise:

Problem: [T] $xy + \sin(x) = 1, (\frac{\pi}{2}, 0)$

Solution:

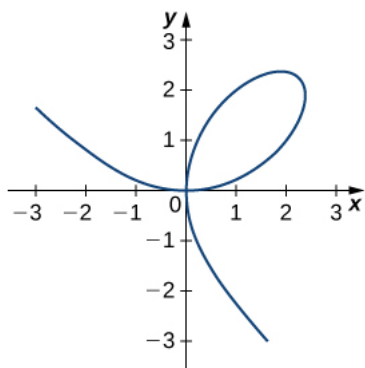


$$y = 0$$

Exercise:

Problem:

[T] The graph of a folium of Descartes with equation $2x^3 + 2y^3 - 9xy = 0$ is given in the following graph.



- Find the equation of the tangent line at the point $(2, 1)$. Graph the tangent line along with the folium.
- Find the equation of the normal line to the tangent line in a. at the point $(2, 1)$.

Exercise:

Problem: For the equation $x^2 + 2xy - 3y^2 = 0$,

- Find the equation of the normal to the tangent line at the point $(1, 1)$.
- At what other point does the normal line in a. intersect the graph of the equation?

Solution:

- $y = -x + 2$
- $(3, -1)$

Exercise:

Problem: Find all points on the graph of $y^3 - 27y = x^2 - 90$ at which the tangent line is vertical.

Exercise:

Problem: For the equation $x^2 + xy + y^2 = 7$,

- Find the x -intercept(s).
- Find the slope of the tangent line(s) at the x -intercept(s).
- What does the value(s) in b. indicate about the tangent line(s)?

Solution:

- $(\pm\sqrt{7}, 0)$
- -2
- They are parallel since the slope is the same at both intercepts.

Exercise:

Problem:

Find the equation of the tangent line to the graph of the equation $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{6}$ at the point $(0, \frac{1}{2})$.

Exercise:

Problem:

Find the equation of the tangent line to the graph of the equation $\tan^{-1}(x + y) = x^2 + \frac{\pi}{4}$ at the point $(0, 1)$.

Solution:

$$y = -x + 1$$

Exercise:

Problem: Find y' and y'' for $x^2 + 6xy - 2y^2 = 3$.

Exercise:

Problem:

[T] The number of cell phones produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $60x^{3/4}y^{1/4} = 3240$.

- Find $\frac{dy}{dx}$ and evaluate at the point $(81, 16)$.
- Interpret the result of a.

Solution:

- -0.5926
- When \$81 is spent on labor and \$16 is spent on capital, the amount spent on capital is decreasing by \$0.5926 per \$1 spent on labor.

Exercise:

Problem:

[T] The number of cars produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $30x^{1/3}y^{2/3} = 360$.

(Both x and y are measured in thousands of dollars.)

- Find $\frac{dy}{dx}$ and evaluate at the point $(27, 8)$.
- Interpret the result of a.

Exercise:

Problem:

The volume of a right circular cone of radius x and height y is given by $V = \frac{1}{3}\pi x^2 y$. Suppose that the volume of the cone is $85\pi \text{ cm}^3$. Find $\frac{dy}{dx}$ when $x = 4$ and $y = 16$.

Solution:

$$-8$$

For the following exercises, consider a closed rectangular box with a square base with side x and height y .

Exercise:

Problem: Find an equation for the surface area of the rectangular box, $S(x, y)$.

Exercise:

Problem: If the surface area of the rectangular box is 78 square feet, find $\frac{dy}{dx}$ when $x = 3$ feet and $y = 5$ feet.

Solution:

$$-2.67$$

For the following exercises, use implicit differentiation to determine y' . Does the answer agree with the formulas we have previously determined?

Exercise:

Problem: $x = \sin y$

Exercise:

Problem: $x = \cos y$

Solution:

$$y' = -\frac{1}{\sqrt{1-x^2}}$$

Exercise:

Problem: $x = \tan y$

Glossary

implicit differentiation

is a technique for computing $\frac{dy}{dx}$ for a function defined by an equation, accomplished by differentiating both sides of the equation (remembering to treat the variable y as a function) and solving for $\frac{dy}{dx}$

Related Rates

- Express changing quantities in terms of derivatives.
- Find relationships among the derivatives in a given problem.
- Use the chain rule to find the rate of change of one quantity that depends on the rate of change of other quantities.

We have seen that for quantities that are changing over time, the rates at which these quantities change are given by derivatives. If two related quantities are changing over time, the rates at which the quantities change are related. For example, if a balloon is being filled with air, both the radius of the balloon and the volume of the balloon are increasing. In this section, we consider several problems in which two or more related quantities are changing and we study how to determine the relationship between the rates of change of these quantities.

Setting up Related-Rates Problems

In many real-world applications, related quantities are changing with respect to time. For example, if we consider the balloon example again, we can say that the rate of change in the volume, V , is related to the rate of change in the radius, r . In this case, we say that $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are **related rates** because V is related to r . Here we study several examples of related quantities that are changing with respect to time and we look at how to calculate one rate of change given another rate of change.

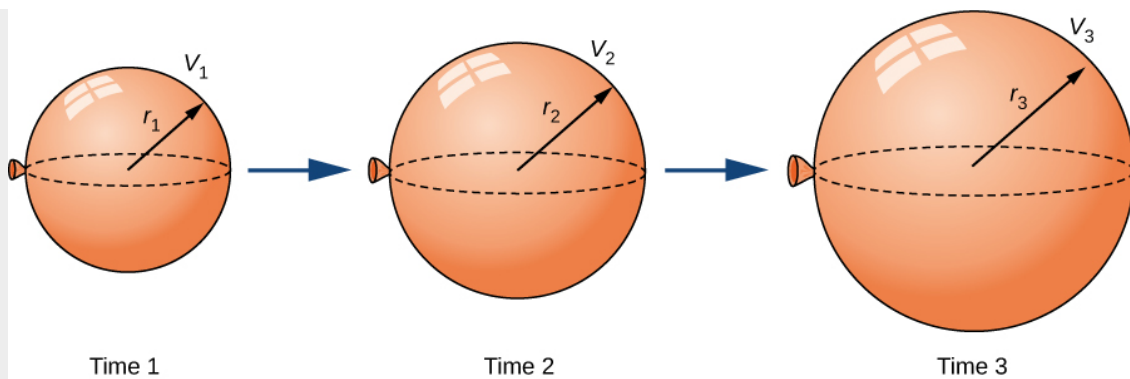
Example:

Exercise:

Problem:

Inflating a Balloon

A spherical balloon is being filled with air at the constant rate of $2 \text{ cm}^3/\text{sec}$ ([\[link\]](#)). How fast is the radius increasing when the radius is 3 cm?



As the balloon is being filled with air, both the radius and the volume are increasing with respect to time.

Solution:

The volume of a sphere of radius r centimeters is

Equation:

$$V = \frac{4}{3}\pi r^3 \text{ cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is

Equation:

$$V(t) = \frac{4}{3}\pi [r(t)]^3 \text{ cm}^3.$$

Differentiating both sides of this equation with respect to time and applying the chain rule, we see that the rate of change in the volume is related to the rate of change in the radius by the equation

Equation:

$$V'(t) = 4\pi [r(t)]^2 r'(t).$$

The balloon is being filled with air at the constant rate of $2 \text{ cm}^3/\text{sec}$, so $V'(t) = 2 \text{ cm}^3/\text{sec}$. Therefore,

Equation:

$$2 \text{ cm}^3/\text{sec} = \left(4\pi[r(t)]^2 \text{ cm}^2\right) \cdot (r'(t) \text{ cm/s}),$$

which implies

Equation:

$$r'(t) = \frac{1}{2\pi[r(t)]^2} \text{ cm/sec}.$$

When the radius $r = 3 \text{ cm}$,

Equation:

$$r'(t) = \frac{1}{18\pi} \text{ cm/sec}.$$

Note:

Exercise:

Problem:

What is the instantaneous rate of change of the radius when $r = 6 \text{ cm}$?

Solution:

$$\frac{1}{72\pi} \text{ cm/sec, or approximately } 0.0044 \text{ cm/sec}$$

Hint

$$\frac{dr}{dt} = \frac{1}{2\pi r^2}$$

Before looking at other examples, let's outline the problem-solving strategy we will be using to solve related-rates problems.

Note:**Problem-Solving Strategy: Solving a Related-Rates Problem**

1. Assign symbols to all variables involved in the problem. Draw a figure if applicable.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an equation relating the variables introduced in step 1.
4. Using the chain rule, differentiate both sides of the equation found in step 3 with respect to the independent variable. This new equation will relate the derivatives.
5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change.

Note that when solving a related-rates problem, it is crucial not to substitute known values too soon. For example, if the value for a changing quantity is substituted into an equation before both sides of the equation are differentiated, then that quantity will behave as a constant and its derivative will not appear in the new equation found in step 4. We examine this potential error in the following example.

Examples of the Process

Let's now implement the strategy just described to solve several related-rates problems. The first example involves a plane flying overhead. The relationship we are studying is between the speed of the plane and the rate

at which the distance between the plane and a person on the ground is changing.

Example:

Exercise:

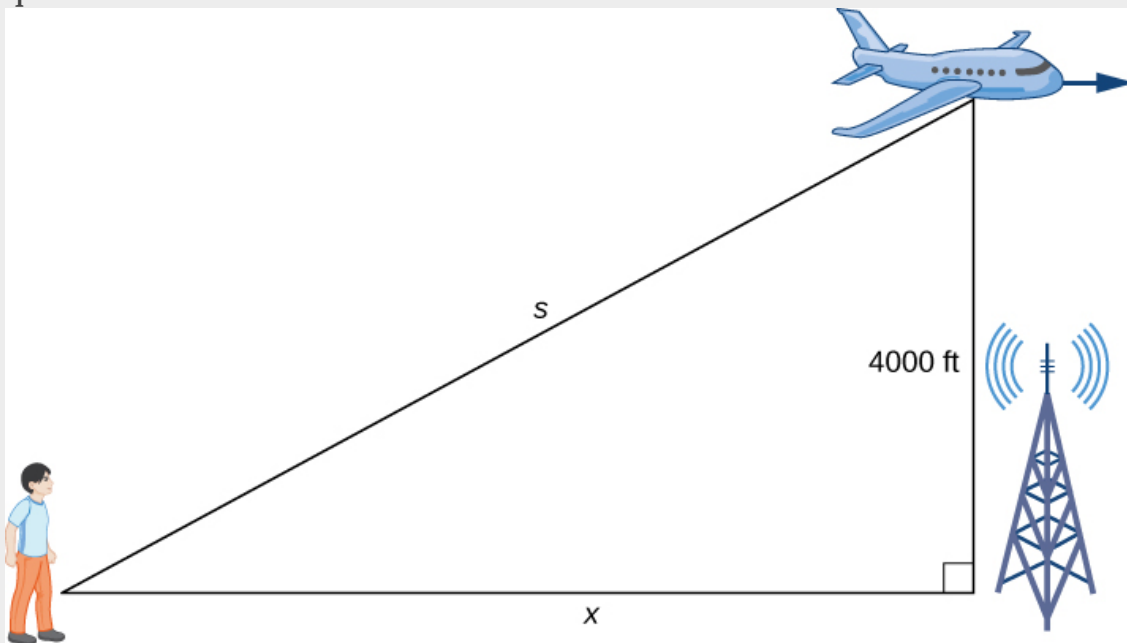
Problem:

An Airplane Flying at a Constant Elevation

An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 ft/sec, at what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

Solution:

Step 1. Draw a picture, introducing variables to represent the different quantities involved.



An airplane is flying at a constant height of 4000 ft. The distance

between the person and the airplane and the person and the place on the ground directly below the airplane are changing. We denote those quantities with the variables s and x , respectively.

As shown, x denotes the distance between the man and the position on the ground directly below the airplane. The variable s denotes the distance between the man and the plane. Note that both x and s are functions of time. We do not introduce a variable for the height of the plane because it remains at a constant elevation of 4000 ft. Since an object's height above the ground is measured as the shortest distance between the object and the ground, the line segment of length 4000 ft is perpendicular to the line segment of length x feet, creating a right triangle.

Step 2. Since x denotes the horizontal distance between the man and the point on the ground below the plane, dx/dt represents the speed of the plane. We are told the speed of the plane is 600 ft/sec. Therefore, $\frac{dx}{dt} = 600$ ft/sec. Since we are asked to find the rate of change in the distance between the man and the plane when the plane is directly above the radio tower, we need to find ds/dt when $x = 3000$ ft.

Step 3. From the figure, we can use the Pythagorean theorem to write an equation relating x and s :

Equation:

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Step 4. Differentiating this equation with respect to time and using the fact that the derivative of a constant is zero, we arrive at the equation

Equation:

$$x \frac{dx}{dt} = s \frac{ds}{dt}.$$

Step 5. Find the rate at which the distance between the man and the plane is increasing when the plane is directly over the radio tower. That is, find $\frac{ds}{dt}$ when $x = 3000$ ft. Since the speed of the plane is 600 ft/sec, we know that $\frac{dx}{dt} = 600$ ft/sec. We are not given an explicit value for s ; however, since we are trying to find $\frac{ds}{dt}$ when $x = 3000$ ft, we can use the Pythagorean theorem to determine the distance s when $x = 3000$ and the height is 4000 ft. Solving the equation

Equation:

$$3000^2 + 4000^2 = s^2$$

for s , we have $s = 5000$ ft at the time of interest. Using these values, we conclude that ds/dt is a solution of the equation

Equation:

$$(3000)(600) = (5000) \cdot \frac{ds}{dt}.$$

Therefore,

Equation:

$$\frac{ds}{dt} = \frac{3000 \cdot 600}{5000} = 360 \text{ ft/sec.}$$

Note: When solving related-rates problems, it is important not to substitute values for the variables too soon. For example, in step 3, we related the variable quantities $x(t)$ and $s(t)$ by the equation

Equation:

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Since the plane remains at a constant height, it is not necessary to introduce a variable for the height, and we are allowed to use the

constant 4000 to denote that quantity. However, the other two quantities are changing. If we mistakenly substituted $x(t) = 3000$ into the equation before differentiating, our equation would have been

Equation:

$$3000^2 + 4000^2 = [s(t)]^2.$$

After differentiating, our equation would become

Equation:

$$0 = s(t) \frac{ds}{dt}.$$

As a result, we would incorrectly conclude that $\frac{ds}{dt} = 0$.

Note:

Exercise:

Problem:

What is the speed of the plane if the distance between the person and the plane is increasing at the rate of 300 ft/sec?

Solution:

500 ft/sec

Hint

$$\frac{ds}{dt} = 300 \text{ ft/sec}$$

We now return to the problem involving the rocket launch from the beginning of the chapter.

Example:

Exercise:

Problem:

Chapter Opener: A Rocket Launch

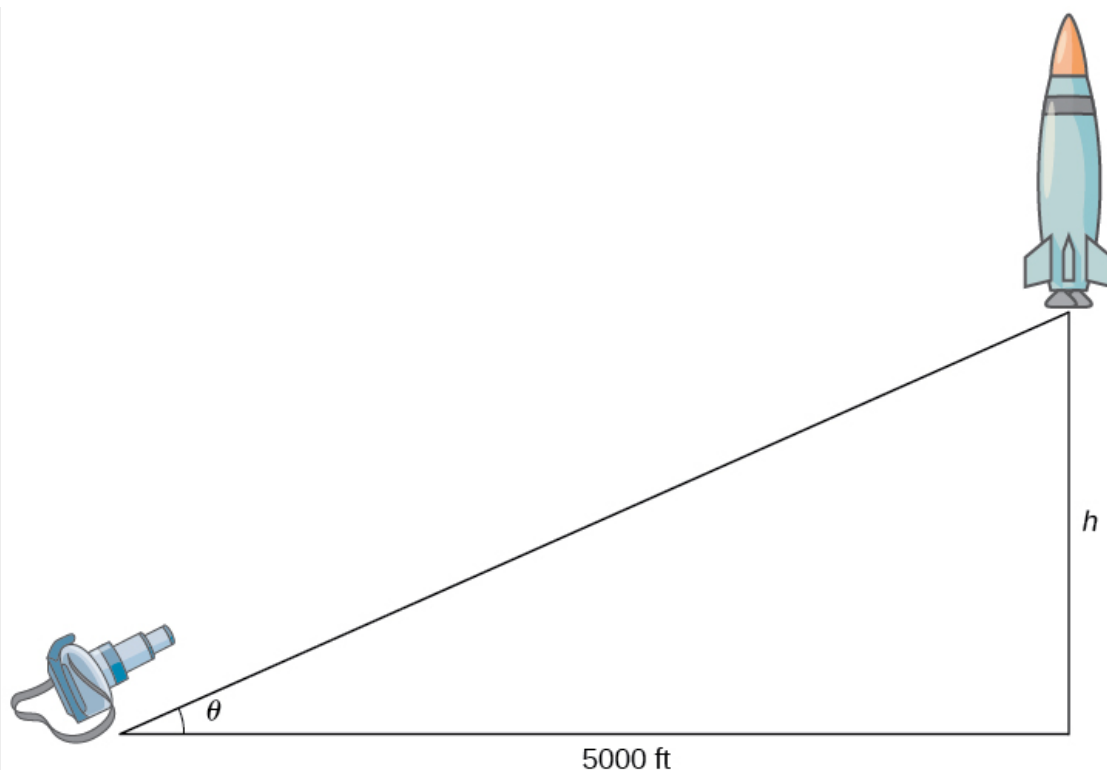


(credit: modification of work by
Steve Jurvetson, Wikimedia
Commons)

A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec. Find the necessary rate of change of the camera's angle as a function of time so that it stays focused on the rocket.

Solution:

Step 1. Draw a picture introducing the variables.



A camera is positioned 5000 ft from the launch pad of the rocket. The height of the rocket and the angle of the camera are changing with respect to time. We denote those quantities with the variables h and θ , respectively.

Let h denote the height of the rocket above the launch pad and θ be the angle between the camera lens and the ground.

Step 2. We are trying to find the rate of change in the angle of the camera with respect to time when the rocket is 1000 ft off the ground. That is, we need to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At that time, we know the velocity of the rocket is $\frac{dh}{dt} = 600$ ft/sec.

Step 3. Now we need to find an equation relating the two quantities that are changing with respect to time: h and θ . How can we create such an equation? Using the fact that we have drawn a right triangle, it is natural to think about trigonometric functions. Recall that $\tan \theta$ is

the ratio of the length of the opposite side of the triangle to the length of the adjacent side. Thus, we have

Equation:

$$\tan \theta = \frac{h}{5000}.$$

This gives us the equation

Equation:

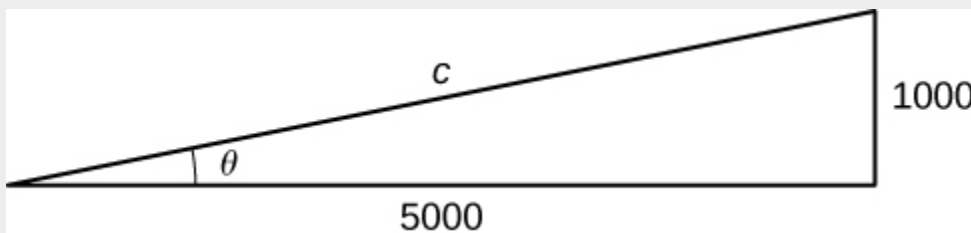
$$h = 5000 \tan \theta.$$

Step 4. Differentiating this equation with respect to time t , we obtain

Equation:

$$\frac{dh}{dt} = 5000 \sec^2 \theta \frac{d\theta}{dt}.$$

Step 5. We want to find $\frac{d\theta}{dt}$ when $h = 1000$ ft. At this time, we know that $\frac{dh}{dt} = 600$ ft/sec. We need to determine $\sec^2 \theta$. Recall that $\sec \theta$ is the ratio of the length of the hypotenuse to the length of the adjacent side. We know the length of the adjacent side is 5000 ft. To determine the length of the hypotenuse, we use the Pythagorean theorem, where the length of one leg is 5000 ft, the length of the other leg is $h = 1000$ ft, and the length of the hypotenuse is c feet as shown in the following figure.



We see that

Equation:

$$1000^2 + 5000^2 = c^2$$

and we conclude that the hypotenuse is

Equation:

$$c = 1000\sqrt{26} \text{ ft.}$$

Therefore, when $h = 1000$, we have

Equation:

$$\sec^2\theta = \left(\frac{1000\sqrt{26}}{5000} \right)^2 = \frac{26}{25}.$$

Recall from step 4 that the equation relating $\frac{d\theta}{dt}$ to our known values is

Equation:

$$\frac{dh}{dt} = 5000 \sec^2\theta \frac{d\theta}{dt}.$$

When $h = 1000$ ft, we know that $\frac{dh}{dt} = 600$ ft/sec and $\sec^2\theta = \frac{26}{25}$.

Substituting these values into the previous equation, we arrive at the equation

Equation:

$$600 = 5000 \left(\frac{26}{25} \right) \frac{d\theta}{dt}.$$

Therefore, $\frac{d\theta}{dt} = \frac{3}{26}$ rad/sec.

Note:**Exercise:****Problem:**

What rate of change is necessary for the elevation angle of the camera if the camera is placed on the ground at a distance of 4000 ft from the launch pad and the velocity of the rocket is 500 ft/sec when the rocket is 2000 ft off the ground?

Solution:

$$\frac{1}{10} \text{ rad/sec}$$

Hint

Find $\frac{d\theta}{dt}$ when $h = 2000$ ft. At that time, $\frac{dh}{dt} = 500$ ft/sec.

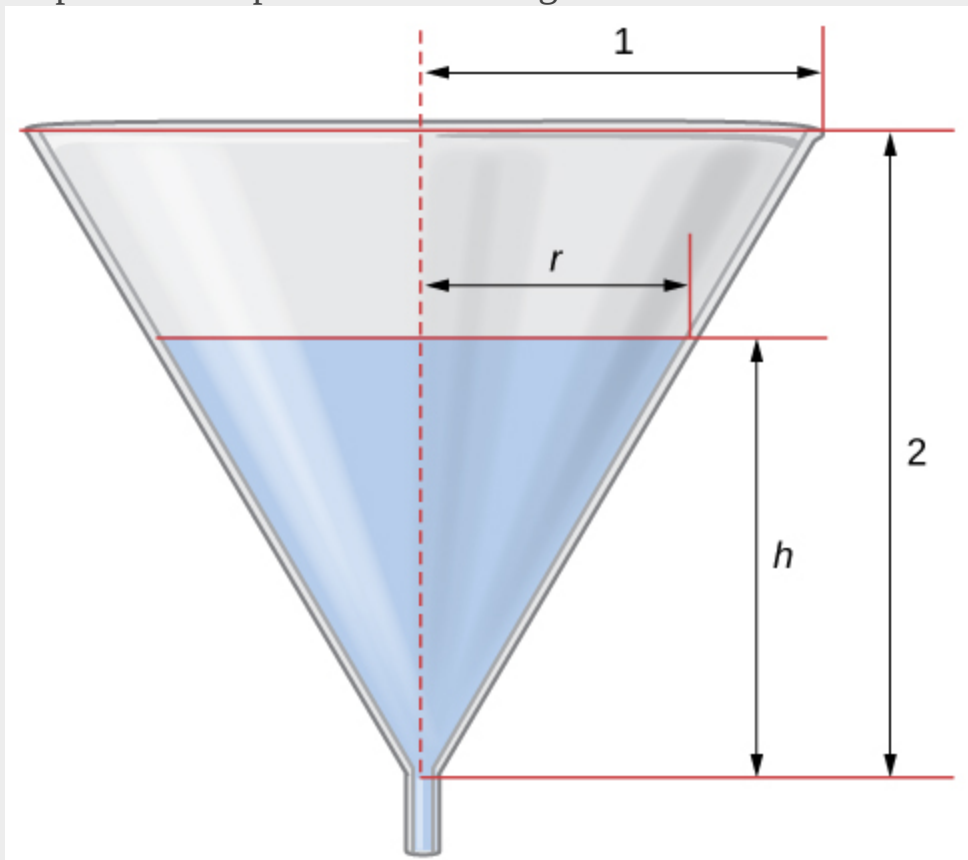
In the next example, we consider water draining from a cone-shaped funnel. We compare the rate at which the level of water in the cone is decreasing with the rate at which the volume of water is decreasing.

Example:**Exercise:****Problem:****Water Draining from a Funnel**

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03 \text{ ft}^3/\text{sec}$. The height of the funnel is 2 ft and the radius at the top of the funnel is 1 ft. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}$ ft?

Solution:

Step 1: Draw a picture introducing the variables.



Water is draining from a funnel of height 2 ft and radius 1 ft. The height of the water and the radius of water are changing over time. We denote these quantities with the variables h and r , respectively.

Let h denote the height of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

Step 2: We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. We know that $\frac{dV}{dt} = -0.03$ ft/sec.

Step 3: The volume of water in the cone is

Equation:

$$V = \frac{1}{3}\pi r^2 h.$$

From the figure, we see that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same. Therefore, $\frac{r}{h} = \frac{1}{2}$ or $r = \frac{h}{2}$. Using this fact, the equation for volume can be simplified to

Equation:

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3.$$

Step 4: Applying the chain rule while differentiating both sides of this equation with respect to time t , we obtain

Equation:

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

Step 5: We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. Since water is leaving at the rate of $0.03 \text{ ft}^3/\text{sec}$, we know that $\frac{dV}{dt} = -0.03 \text{ ft}^3/\text{sec}$.

Therefore,

Equation:

$$-0.03 = \frac{\pi}{4} \left(\frac{1}{2}\right)^2 \frac{dh}{dt},$$

which implies

Equation:

$$-0.03 = \frac{\pi}{16} \frac{dh}{dt}.$$

It follows that

Equation:

$$\frac{dh}{dt} = -\frac{0.48}{\pi} = -0.153 \text{ ft/sec.}$$

Note:

Exercise:

Problem:

At what rate is the height of the water changing when the height of the water is $\frac{1}{4}$ ft?

Solution:

-0.61 ft/sec

Hint

We need to find $\frac{dh}{dt}$ when $h = \frac{1}{4}$.

Key Concepts

- To solve a related rates problem, first draw a picture that illustrates the relationship between the two or more related quantities that are changing with respect to time.
- In terms of the quantities, state the information given and the rate to be found.
- Find an equation relating the quantities.
- Use differentiation, applying the chain rule as necessary, to find an equation that relates the rates.

- Be sure not to substitute a variable quantity for one of the variables until after finding an equation relating the rates.

For the following exercises, find the quantities for the given equation.

Exercise:

Problem: Find $\frac{dy}{dt}$ at $x = 1$ and $y = x^2 + 3$ if $\frac{dx}{dt} = 4$.

Solution:

8

Exercise:

Problem: Find $\frac{dx}{dt}$ at $x = -2$ and $y = 2x^2 + 1$ if $\frac{dy}{dt} = -1$.

Exercise:

Problem:

Find $\frac{dz}{dt}$ at $(x, y) = (1, 3)$ and $z^2 = x^2 + y^2$ if $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = 3$.

Solution:

$\frac{13}{\sqrt{10}}$

For the following exercises, sketch the situation if necessary and used related rates to solve for the quantities.

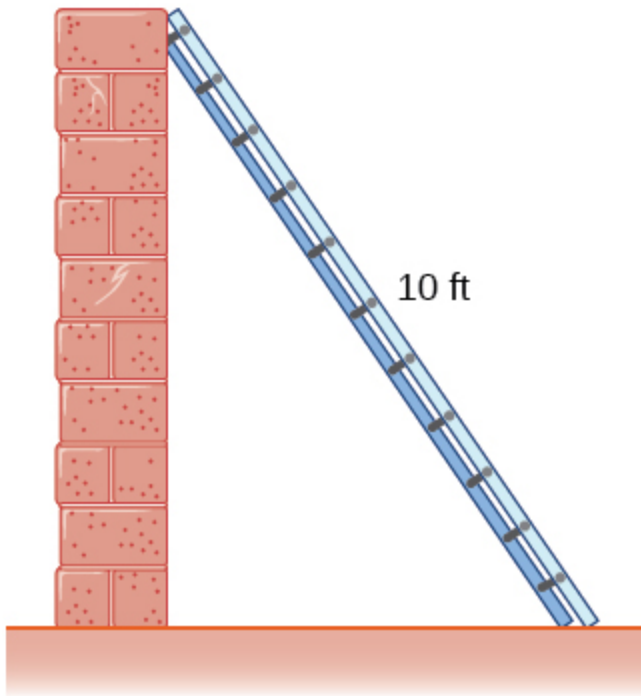
Exercise:

Problem:

[T] If two electrical resistors are connected in parallel, the total resistance (measured in ohms, denoted by the Greek capital letter omega, Ω) is given by the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 is increasing at a rate of $0.5 \Omega/\text{min}$ and R_2 decreases at a rate of $1.1 \Omega/\text{min}$, at what rate does the total resistance change when $R_1 = 20 \Omega$ and $R_2 = 50 \Omega$?

Exercise:**Problem:**

A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 ft/sec , how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?



Solution:

$$2\sqrt{3} \text{ ft/sec}$$

Exercise:

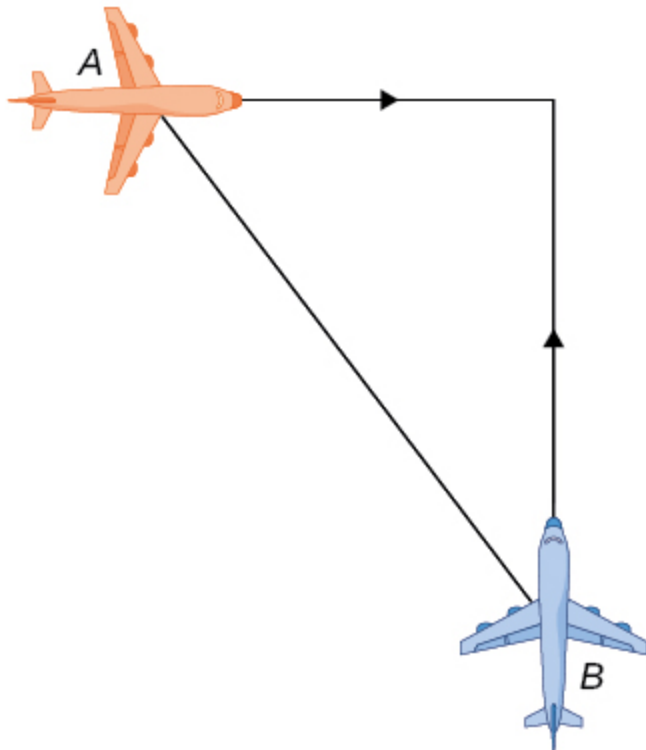
Problem:

A 25-ft ladder is leaning against a wall. If we push the ladder toward the wall at a rate of 1 ft/sec, and the bottom of the ladder is initially 20 ft away from the wall, how fast does the ladder move up the wall 5 sec after we start pushing?

Exercise:

Problem:

Two airplanes are flying in the air at the same height: airplane *A* is flying east at 250 mi/h and airplane *B* is flying north at 300 mi/h. If they are both heading to the same airport, located 30 miles east of airplane *A* and 40 miles north of airplane *B*, at what rate is the distance between the airplanes changing?



Solution:

The distance is decreasing at 390 mi/h.

Exercise:

Problem:

You and a friend are riding your bikes to a restaurant that you think is east; your friend thinks the restaurant is north. You both leave from the same point, with you riding at 16 mph east and your friend riding 12 mph north. After you traveled 4 mi, at what rate is the distance between you changing?

Exercise:

Problem:

Two buses are driving along parallel freeways that are 5 mi apart, one heading east and the other heading west. Assuming that each bus drives a constant 55 mph, find the rate at which the distance between the buses is changing when they are 13 mi apart, heading toward each other.

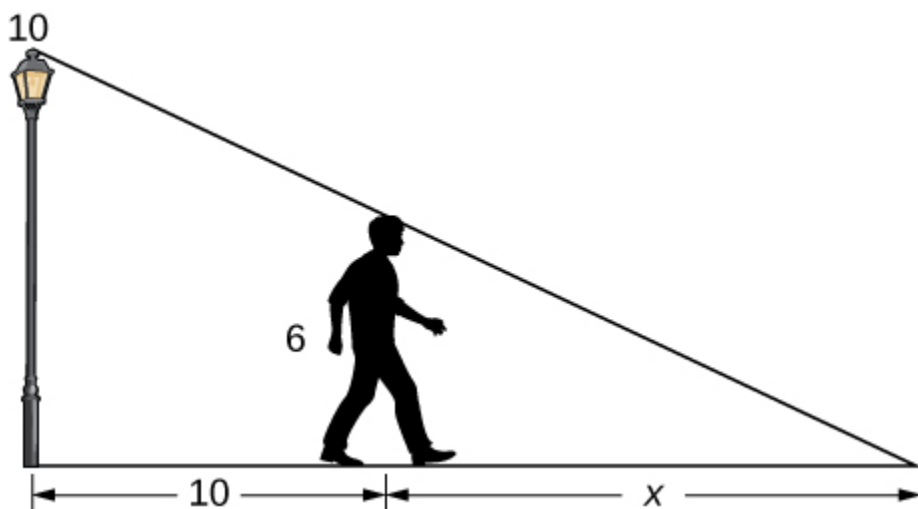
Solution:

The distance between them shrinks at a rate of $\frac{1320}{13} \approx 101.5$ mph.

Exercise:

Problem:

A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of 3 ft/sec. What is the rate that the tip of the shadow moves away from the pole when the person is 10 ft away from the pole?



Exercise:

Problem:

Using the previous problem, what is the rate at which the tip of the shadow moves away from the person when the person is 10 ft from the pole?

Solution:

$$\frac{9}{2} \text{ ft/sec}$$

Exercise:

Problem:

A 5-ft-tall person walks toward a wall at a rate of 2 ft/sec. A spotlight is located on the ground 40 ft from the wall. How fast does the height of the person's shadow on the wall change when the person is 10 ft from the wall?

Exercise:

Problem:

Using the previous problem, what is the rate at which the shadow changes when the person is 10 ft from the wall, if the person is walking away from the wall at a rate of 2 ft/sec?

Solution:

It grows at a rate $\frac{4}{9}$ ft/sec

Exercise:**Problem:**

A helicopter starting on the ground is rising directly into the air at a rate of 25 ft/sec. You are running on the ground starting directly under the helicopter at a rate of 10 ft/sec. Find the rate of change of the distance between the helicopter and yourself after 5 sec.

Exercise:**Problem:**

Using the previous problem, what is the rate at which the distance between you and the helicopter is changing when the helicopter has risen to a height of 60 ft in the air, assuming that, initially, it was 30 ft above you?

Solution:

The distance is increasing at $\frac{(135\sqrt{26})}{26}$ ft/sec

For the following exercises, draw and label diagrams to help solve the related-rates problems.

Exercise:**Problem:**

The side of a cube increases at a rate of $\frac{1}{2}$ m/sec. Find the rate at which the volume of the cube increases when the side of the cube is 4 m.

Exercise:

Problem:

The volume of a cube decreases at a rate of 10 m/sec. Find the rate at which the side of the cube changes when the side of the cube is 2 m.

Solution:

$$-\frac{5}{6} \text{ m/sec}$$

Exercise:**Problem:**

The radius of a circle increases at a rate of 2 m/sec. Find the rate at which the area of the circle increases when the radius is 5 m.

Exercise:**Problem:**

The radius of a sphere decreases at a rate of 3 m/sec. Find the rate at which the surface area decreases when the radius is 10 m.

Solution:

$$240\pi \text{ m}^2/\text{sec}$$

Exercise:**Problem:**

The radius of a sphere increases at a rate of 1 m/sec. Find the rate at which the volume increases when the radius is 20 m.

Exercise:**Problem:**

The radius of a sphere is increasing at a rate of 9 cm/sec. Find the radius of the sphere when the volume and the radius of the sphere are increasing at the same numerical rate.

Solution:

$$\frac{1}{2\sqrt{\pi}} \text{ cm}$$

Exercise:**Problem:**

The base of a triangle is shrinking at a rate of 1 cm/min and the height of the triangle is increasing at a rate of 5 cm/min. Find the rate at which the area of the triangle changes when the height is 22 cm and the base is 10 cm.

Exercise:**Problem:**

A triangle has two constant sides of length 3 ft and 5 ft. The angle between these two sides is increasing at a rate of 0.1 rad/sec. Find the rate at which the area of the triangle is changing when the angle between the two sides is $\pi/6$.

Solution:

The area is increasing at a rate $\frac{(3\sqrt{3})}{8} \text{ ft}^2/\text{sec}$.

Exercise:**Problem:**

A triangle has a height that is increasing at a rate of 2 cm/sec and its area is increasing at a rate of 4 cm²/sec. Find the rate at which the base of the triangle is changing when the height of the triangle is 4 cm and the area is 20 cm².

For the following exercises, consider a right cone that is leaking water. The dimensions of the conical tank are a height of 16 ft and a radius of 5 ft.

Exercise:

Problem:

How fast does the depth of the water change when the water is 10 ft high if the cone leaks water at a rate of $10 \text{ ft}^3/\text{min}$?

Solution:

The depth of the water decreases at $\frac{128}{125\pi} \text{ ft}/\text{min}$.

Exercise:**Problem:**

Find the rate at which the surface area of the water changes when the water is 10 ft high if the cone leaks water at a rate of $10 \text{ ft}^3/\text{min}$.

Exercise:**Problem:**

If the water level is decreasing at a rate of $3 \text{ in.}/\text{min}$ when the depth of the water is 8 ft, determine the rate at which water is leaking out of the cone.

Solution:

The volume is decreasing at a rate of $\frac{(25\pi)}{16} \text{ ft}^3/\text{min}$.

Exercise:**Problem:**

A vertical cylinder is leaking water at a rate of $1 \text{ ft}^3/\text{sec}$. If the cylinder has a height of 10 ft and a radius of 1 ft, at what rate is the height of the water changing when the height is 6 ft?

Exercise:

Problem:

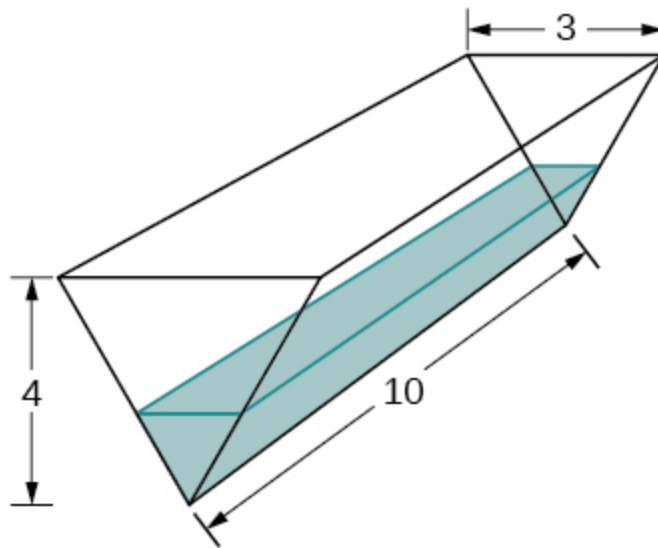
A cylinder is leaking water but you are unable to determine at what rate. The cylinder has a height of 2 m and a radius of 2 m. Find the rate at which the water is leaking out of the cylinder if the rate at which the height is decreasing is 10 cm/min when the height is 1 m.

Solution:

The water flows out at rate $\frac{(2\pi)}{5} \text{ m}^3/\text{min}$.

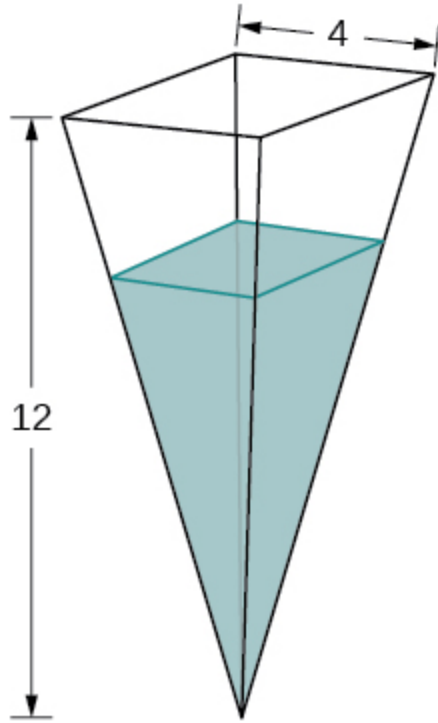
Exercise:**Problem:**

A trough has ends shaped like isosceles triangles, with width 3 m and height 4 m, and the trough is 10 m long. Water is being pumped into the trough at a rate of $5 \text{ m}^3/\text{min}$. At what rate does the height of the water change when the water is 1 m deep?

**Exercise:**

Problem:

A tank is shaped like an upside-down square pyramid, with base of 4 m by 4 m and a height of 12 m (see the following figure). How fast does the height increase when the water is 2 m deep if water is being pumped in at a rate of $\frac{2}{3}$ m/sec?



Solution:

$$\frac{3}{2} \text{ m/sec}$$

For the following problems, consider a pool shaped like the bottom half of a sphere, that is being filled at a rate of $25 \text{ ft}^3/\text{min}$. The radius of the pool is 10 ft.

Exercise:

Problem:

Find the rate at which the depth of the water is changing when the water has a depth of 5 ft.

Exercise:**Problem:**

Find the rate at which the depth of the water is changing when the water has a depth of 1 ft.

Solution:

$$\frac{25}{19\pi} \text{ ft/min}$$

Exercise:**Problem:**

If the height is increasing at a rate of 1 in./sec when the depth of the water is 2 ft, find the rate at which water is being pumped in.

Exercise:**Problem:**

Gravel is being unloaded from a truck and falls into a pile shaped like a cone at a rate of $10 \text{ ft}^3/\text{min}$. The radius of the cone base is three times the height of the cone. Find the rate at which the height of the gravel changes when the pile has a height of 5 ft.

Solution:

$$\frac{2}{45\pi} \text{ ft/min}$$

Exercise:

Problem:

Using a similar setup from the preceding problem, find the rate at which the gravel is being unloaded if the pile is 5 ft high and the height is increasing at a rate of 4 in./min.

For the following exercises, draw the situations and solve the related-rate problems.

Exercise:**Problem:**

You are stationary on the ground and are watching a bird fly horizontally at a rate of 10 m/sec. The bird is located 40 m above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is 9 m?

Solution:

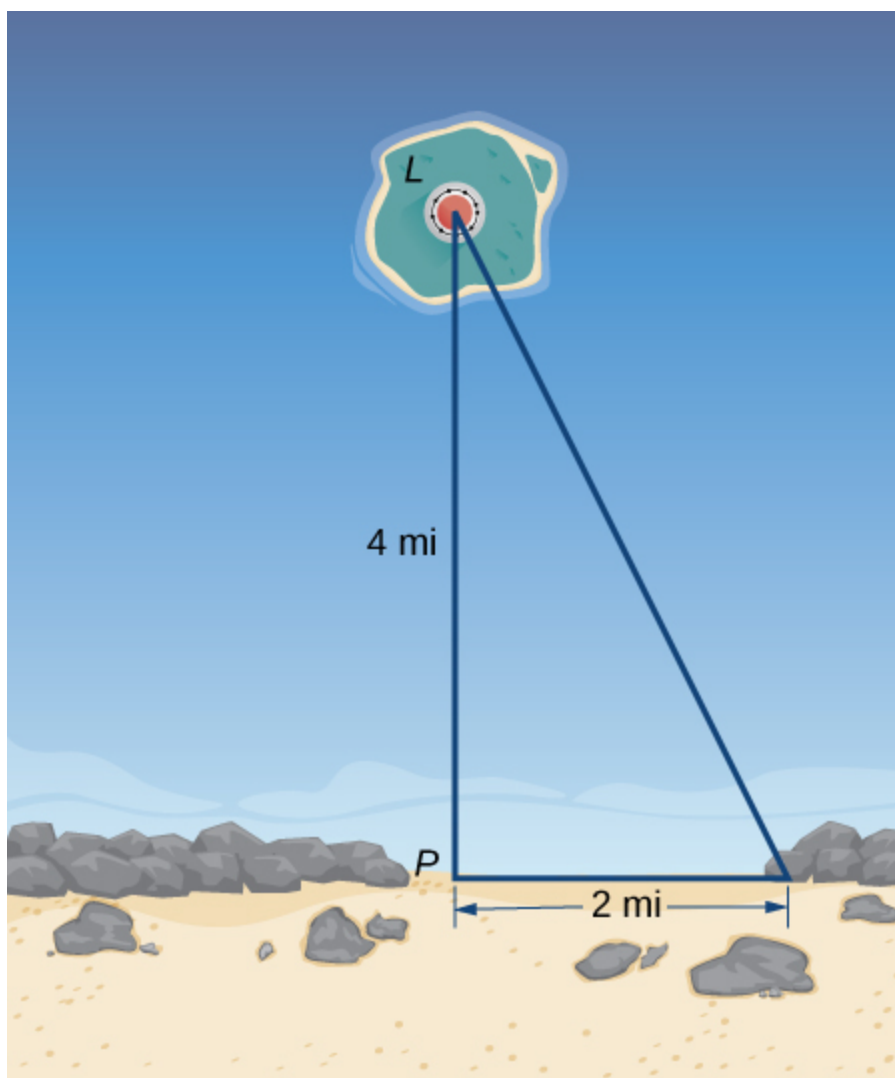
The angle decreases at $\frac{400}{1681}$ rad/sec.

Exercise:**Problem:**

You stand 40 ft from a bottle rocket on the ground and watch as it takes off vertically into the air at a rate of 20 ft/sec. Find the rate at which the angle of elevation changes when the rocket is 30 ft in the air.

Exercise:**Problem:**

A lighthouse, L , is on an island 4 mi away from the closest point, P , on the beach (see the following image). If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach 2 mi away from the closest point on the beach?



Solution:

$$100\pi/\text{min}$$

Exercise:

Problem:

Using the same setup as the previous problem, determine at what rate the beam of light moves across the beach 1 mi away from the closest point on the beach.

Exercise:

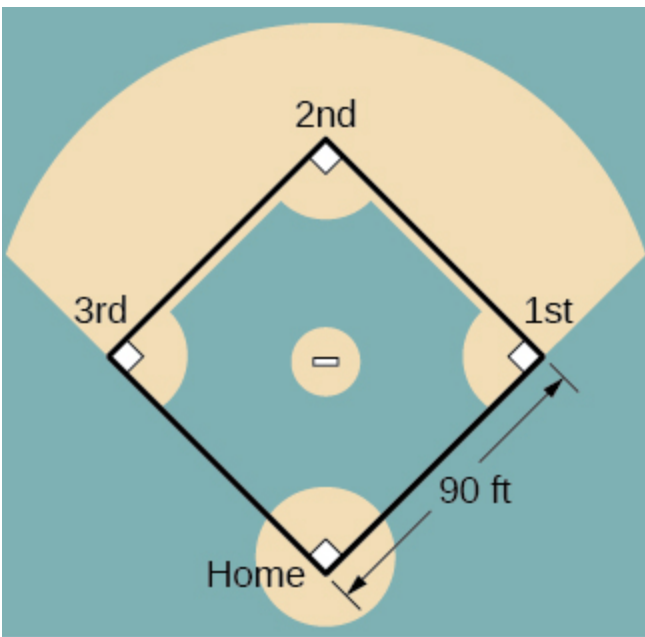
Problem:

You are walking to a bus stop at a right-angle corner. You move north at a rate of 2 m/sec and are 20 m south of the intersection. The bus travels west at a rate of 10 m/sec away from the intersection – you have missed the bus! What is the rate at which the angle between you and the bus is changing when you are 20 m south of the intersection and the bus is 10 m west of the intersection?

Solution:

The angle is changing at a rate of $\frac{21}{25}$ rad/sec.

For the following exercises, refer to the figure of baseball diamond, which has sides of 90 ft.

**Exercise:****Problem:**

[T] A batter hits a ball toward third base at 75 ft/sec and runs toward first base at a rate of 24 ft/sec. At what rate does the distance between the ball and the batter change when 2 sec have passed?

Exercise:**Problem:**

[T] A batter hits a ball toward second base at 80 ft/sec and runs toward first base at a rate of 30 ft/sec. At what rate does the distance between the ball and the batter change when the runner has covered one-third of the distance to first base? (*Hint:* Recall the law of cosines.)

Solution:

The distance is increasing at a rate of 62.50 ft/sec.

Exercise:**Problem:**

[T] A batter hits the ball and runs toward first base at a speed of 22 ft/sec. At what rate does the distance between the runner and second base change when the runner has run 30 ft?

Exercise:**Problem:**

[T] Runners start at first and second base. When the baseball is hit, the runner at first base runs at a speed of 18 ft/sec toward second base and the runner at second base runs at a speed of 20 ft/sec toward third base. How fast is the distance between runners changing 1 sec after the ball is hit?

Solution:

The distance is decreasing at a rate of 11.99 ft/sec.

Glossary

related rates

are rates of change associated with two or more related quantities that are changing over time

Derivatives of Exponential and Logarithmic Functions

- Find the derivative of exponential functions.
- Find the derivative of logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in [Introduction to Functions and Graphs](#), exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B(x) = b^x$, $b > 0$, is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of b^n , where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^0 = 1$, $b^{-n} = \frac{1}{b^n}$, for a positive integer n , and $b^{s/t} = (\sqrt[t]{b})^s$ for positive integers s and t . These definitions leave open the question of the value of b^r where r is an arbitrary real number. By assuming the continuity of $B(x) = b^x$, $b > 0$, we may interpret b^r as $\lim_{x \rightarrow r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

Equation:

$$4^3 < 4^\pi < 4^4, 4^{3.1} < 4^\pi < 4^{3.2}, 4^{3.14} < 4^\pi < 4^{3.15}, \\ 4^{3.141} < 4^\pi < 4^{3.142}, 4^{3.1415} < 4^\pi < 4^{3.1416}, \dots$$

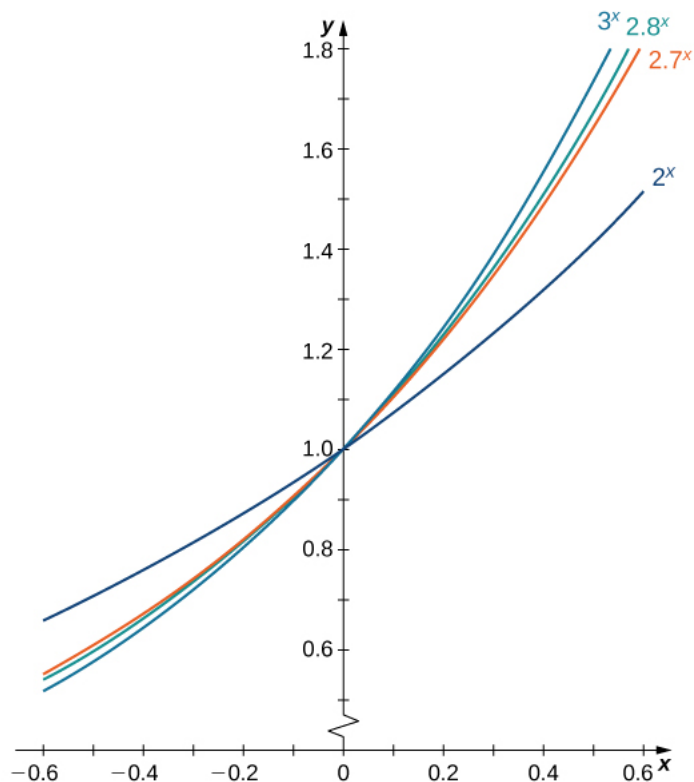
As we see in the following table, $4^\pi \approx 77.88$.

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^4	256

Approximating a Value of 4^π

We also assume that for $B(x) = b^x$, $b > 0$, the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1$. We define e to be this unique value, as we did in [Introduction to Functions and Graphs](#). [\[link\]](#) provides graphs of the functions $y = 2^x$, $y = 3^x$, $y = 2.7^x$, and $y = 2.8^x$. A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function**.



The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of e , we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

Equation:

$$B'(0) = \lim_{x \rightarrow 0} \frac{b^x - b^0}{x - 0} = \lim_{x \rightarrow 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$ to obtain the estimate

Equation:

$$\frac{b^{-0.00001} - 1}{-0.00001} < B'(0) < \frac{b^{0.00001} - 1}{0.00001}.$$

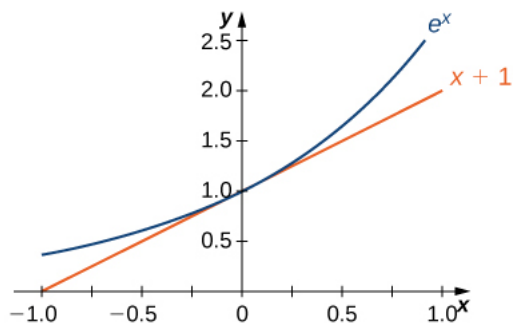
See the following table.

b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$	b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$
2	$0.693145 < B'(0) < 0.69315$	2.7183	$1.000002 < B'(0) < 1.000012$
2.7	$0.993247 < B'(0) < 0.993257$	2.719	$1.000259 < B'(0) < 1.000269$
2.71	$0.996944 < B'(0) < 0.996954$	2.72	$1.000627 < B'(0) < 1.000637$
2.718	$0.999891 < B'(0) < 0.999901$	2.8	$1.029614 < B'(0) < 1.029625$
2.7182	$0.999965 < B'(0) < 0.999975$	3	$1.098606 < B'(0) < 1.098618$

Estimating a Value of e

The evidence from the table suggests that $2.7182 < e < 2.7183$.

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in [\[link\]](#). This line is tangent to the graph of $E(x) = e^x$ at $x = 0$.



The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x$, $b > 0$. Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

Equation:

$$B'(0) = \lim_{h \rightarrow 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \rightarrow 0} \frac{b^h - 1}{h}.$$

Turning to $B'(x)$, we obtain the following.

Equation:

$$\begin{aligned}
 B'(x) &= \lim_{h \rightarrow 0} \frac{b^{x+h} - b^x}{h} && \text{Apply the limit definition of the derivative.} \\
 &= \lim_{h \rightarrow 0} \frac{b^x b^h - b^x}{h} && \text{Note that } b^{x+h} = b^x b^h. \\
 &= \lim_{h \rightarrow 0} \frac{b^x (b^h - 1)}{h} && \text{Factor out } b^x. \\
 &= b^x \lim_{h \rightarrow 0} \frac{b^h - 1}{h} && \text{Apply a property of limits.} \\
 &= b^x B'(0) && \text{Use } B'(0) = \lim_{h \rightarrow 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \rightarrow 0} \frac{b^h - 1}{h}.
 \end{aligned}$$

We see that on the basis of the assumption that $B(x) = b^x$ is differentiable at 0, $B(x)$ is not only differentiable everywhere, but its derivative is

Equation:

$$B'(x) = b^x B'(0).$$

For $E(x) = e^x$, $E'(0) = 1$. Thus, we have $E'(x) = e^x$. (The value of $B'(0)$ for an arbitrary function of the form $B(x) = b^x$, $b > 0$, will be derived later.)

Note:

Derivative of the Natural Exponential Function

Let $E(x) = e^x$ be the natural exponential function. Then

Equation:

$$E'(x) = e^x.$$

In general,

Equation:

$$\frac{d}{dx} (e^{g(x)}) = e^{g(x)} g'(x).$$

Example:

Exercise:

Problem:

Derivative of an Exponential Function

Find the derivative of $f(x) = e^{\tan(2x)}$.

Solution:

Using the derivative formula and the chain rule,

Equation:

$$\begin{aligned}
 f'(x) &= e^{\tan(2x)} \frac{d}{dx} (\tan(2x)) \\
 &= e^{\tan(2x)} \sec^2(2x) \cdot 2.
 \end{aligned}$$

Example:

Exercise:

Problem:

Combining Differentiation Rules

Find the derivative of $y = \frac{e^{x^2}}{x}$.

Solution:

Use the derivative of the natural exponential function, the quotient rule, and the chain rule.

Equation:

$$\begin{aligned} y' &= \frac{(e^{x^2} \cdot 2)x \cdot x - 1 \cdot e^{x^2}}{x^2} && \text{Apply the quotient rule.} \\ &= \frac{e^{x^2}(2x^2 - 1)}{x^2} && \text{Simplify.} \end{aligned}$$

Note:

Exercise:

Problem: Find the derivative of $h(x) = xe^{2x}$.

Solution:

$$h'(x) = e^{2x} + 2xe^{2x}$$

Hint

Don't forget to use the product rule.

Example:

Exercise:

Problem:

Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ is constant.

Solution:

First find $A'(t)$. By using the chain rule, we have $A'(t) = 300e^{0.3t}$. Thus, the ratio of the rate of change of the population to the population is given by

Equation:

$$A'(t) = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

The ratio of the rate of change of the population to the population is the constant 0.3.

Note:

Exercise:

Problem:

If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A(t)$ after 4 days?

Solution:

996

Hint

Find $A'(4)$.

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Note:

The Derivative of the Natural Logarithmic Function

If $x > 0$ and $y = \ln x$, then

Equation:

$$\frac{dy}{dx} = \frac{1}{x}.$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g'(x) > 0$, the derivative of $h(x) = \ln(g(x))$ is given by

Equation:

$$h'(x) = \frac{1}{g(x)} g'(x).$$

Proof

If $x > 0$ and $y = \ln x$, then $e^y = x$. Differentiating both sides of this equation results in the equation

Equation:

$$e^y \frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ yields

Equation:

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally, we substitute $x = e^y$ to obtain

Equation:

$$\frac{dy}{dx} = \frac{1}{x}.$$

We may also derive this result by applying the inverse function theorem, as follows. Since $y = g(x) = \ln x$ is the inverse of $f(x) = e^x$, by applying the inverse function theorem we have

Equation:

$$\frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

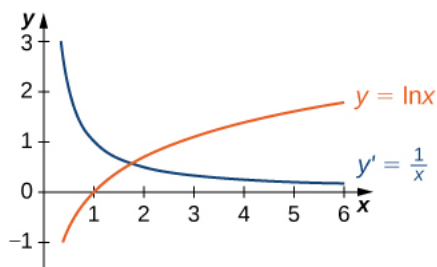
Using this result and applying the chain rule to $h(x) = \ln(g(x))$ yields

Equation:

$$h'(x) = \frac{1}{g(x)} g'(x).$$

□

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in [\[link\]](#).



The function $y = \ln x$ is increasing on $(0, +\infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, +\infty)$.

Example:

Exercise:

Problem:

Taking a Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln(x^3 + 3x - 4)$.

Solution:

Use [link](#) directly.

Equation:

$$\begin{aligned} f'(x) &= \frac{1}{x^3+3x-4} \cdot (3x^2 + 3) && \text{Use } g(x) = x^3 + 3x - 4 \text{ in } h'(x) = \frac{1}{g(x)} g'(x). \\ &= \frac{3x^2+3}{x^3+3x-4} && \text{Rewrite.} \end{aligned}$$

Example:

Exercise:

Problem:

Using Properties of Logarithms in a Derivative

Find the derivative of $f(x) = \ln\left(\frac{x^2 \sin x}{2x+1}\right)$.

Solution:

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms prior to finding the derivative, we can make the problem much simpler.

Equation:

$$\begin{aligned} f(x) &= \ln\left(\frac{x^2 \sin x}{2x+1}\right) = 2 \ln x + \ln(\sin x) - \ln(2x+1) && \text{Apply properties of logarithms.} \\ f'(x) &= \frac{2}{x} + \cot x - \frac{2}{2x+1} && \text{Apply sum rule and } h'(x) = \frac{1}{g(x)} g'(x). \end{aligned}$$

Note:

Exercise:

Problem: Differentiate: $f(x) = \ln(3x + 2)^5$.

Solution:

$$f'(x) = \frac{15}{3x+2}$$

Hint

Use a property of logarithms to simplify before taking the derivative.

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = \log_b x$ and $y = b^x$ for $b > 0, b \neq 1$.

Note:**Derivatives of General Exponential and Logarithmic Functions**

Let $b > 0, b \neq 1$, and let $g(x)$ be a differentiable function.

i. If $y = \log_b x$, then

Equation:

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

More generally, if $h(x) = \log_b(g(x))$, then for all values of x for which $g(x) > 0$,

Equation:

$$h'(x) = \frac{g'(x)}{g(x) \ln b}.$$

ii. If $y = b^x$, then

Equation:

$$\frac{dy}{dx} = b^x \ln b.$$

More generally, if $h(x) = b^{g(x)}$, then

Equation:

$$h'(x) = b^{g(x)} g'(x) \ln b.$$

Proof

If $y = \log_b x$, then $b^y = x$. It follows that $\ln(b^y) = \ln x$. Thus $y \ln b = \ln x$. Solving for y , we have $y = \frac{\ln x}{\ln b}$.

Differentiating and keeping in mind that $\ln b$ is a constant, we see that

Equation:

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

The derivative in [\[link\]](#) now follows from the chain rule.

If $y = b^x$, then $\ln y = x \ln b$. Using implicit differentiation, again keeping in mind that $\ln b$ is constant, it follows that $\frac{1}{y} \frac{dy}{dx} = \ln b$. Solving for $\frac{dy}{dx}$ and substituting $y = b^x$, we see that

Equation:

$$\frac{dy}{dx} = y \ln b = b^x \ln b.$$

The more general derivative ([\[link\]](#)) follows from the chain rule.

□

Example:

Exercise:

Problem:

Applying Derivative Formulas

Find the derivative of $h(x) = \frac{3^x}{3^x + 2}$.

Solution:

Use the quotient rule and [\[link\]](#).

Equation:

$$\begin{aligned} h'(x) &= \frac{3^x \ln 3 (3^x + 2) - 3^x \ln 3 (3^x)}{(3^x + 2)^2} && \text{Apply the quotient rule.} \\ &= \frac{2 \cdot 3^x \ln 3}{(3^x + 2)^2} && \text{Simplify.} \end{aligned}$$

Example:

Exercise:

Problem:

Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Solution:

To find the slope, we must evaluate $\frac{dy}{dx}$ at $x = 1$. Using [\[link\]](#), we see that

Equation:

$$\frac{dy}{dx} = \frac{3}{\ln 2 (3x + 1)}.$$

By evaluating the derivative at $x = 1$, we see that the tangent line has slope

Equation:

$$\left. \frac{dy}{dx} \right|_{x=1} = \frac{3}{4 \ln 2} = \frac{3}{\ln 16}.$$

Note:

Exercise:

Problem: Find the slope for the line tangent to $y = 3^x$ at $x = 2$.

Solution:

$$9 \ln(3)$$

Hint

Evaluate the derivative at $x = 2$.

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g(x))^n$ for certain values of n , as well as functions of the form $y = b^{g(x)}$, where $b > 0$ and $b \neq 1$. Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^\pi$. These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$. We outline this technique in the following problem-solving strategy.

Note:

Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y} \frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

Example:

Exercise:

Problem:

Using Logarithmic Differentiation

Find the derivative of $y = (2x^4 + 1)^{\tan x}$.

Solution:

Use logarithmic differentiation to find this derivative.

Equation:

$$\ln y = \ln(2x^4 + 1)^{\tan x}$$

$$\ln y = \tan x \ln(2x^4 + 1)$$

$$\frac{1}{y} \frac{dy}{dx} = \sec^2 x \ln(2x^4 + 1) + \frac{8x^3}{2x^4 + 1} \cdot \tan x$$

$$\frac{dy}{dx} = y \cdot \left(\sec^2 x \ln(2x^4 + 1) + \frac{8x^3}{2x^4 + 1} \cdot \tan x \right)$$

$$\frac{dy}{dx} = (2x^4 + 1)^{\tan x} \left(\sec^2 x \ln(2x^4 + 1) + \frac{8x^3}{2x^4 + 1} \cdot \tan x \right)$$

Step 1. Take the natural logarithm of both sides.

Step 2. Expand using properties of logarithms.

Step 3. Differentiate both sides. Use the product rule on the right.

Step 4. Multiply by y on both sides.

Step 5. Substitute $y = (2x^4 + 1)^{\tan x}$.

Example:

Exercise:**Problem:**
Using Logarithmic Differentiation

Find the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Solution:

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.

Equation:

$\ln y = \ln \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$	Step 1. Take the natural logarithm of both sides.
$\ln y = \ln x + \frac{1}{2} \ln(2x+1) - x \ln e - 3 \ln \sin x$	Step 2. Expand using properties of logarithms.
$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \frac{1}{2x+1} - 1 - 3 \frac{\cos x}{\sin x}$	Step 3. Differentiate both sides.
$\frac{dy}{dx} = y \left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3 \cot x \right)$	Step 4. Multiply by y on both sides.
$\frac{dy}{dx} = \frac{x\sqrt{2x+1}}{e^x \sin^3 x} \left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3 \cot x \right)$	Step 5. Substitute $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Example:**Exercise:****Problem:**
Extending the Power Rule

Find the derivative of $y = x^r$ where r is an arbitrary real number.

Solution:

The process is the same as in [\[link\]](#), though with fewer complications.

Equation:

$\ln y = \ln x^r$	Step 1. Take the natural logarithm of both sides.
$\ln y = r \ln x$	Step 2. Expand using properties of logarithms.
$\frac{1}{y} \frac{dy}{dx} = r \frac{1}{x}$	Step 3. Differentiate both sides.
$\frac{dy}{dx} = y \frac{r}{x}$	Step 4. Multiply by y on both sides.
$\frac{dy}{dx} = x^r \frac{r}{x}$	Step 5. Substitute $y = x^r$.
$\frac{dy}{dx} = r x^{r-1}$	Simplify.

Note:**Exercise:**

Problem: Use logarithmic differentiation to find the derivative of $y = x^x$.

Solution:

$$\frac{dy}{dx} = x^x (1 + \ln x)$$

Hint

Follow the problem solving strategy.

Note:

Exercise:

Problem: Find the derivative of $y = (\tan x)^\pi$.

Solution:

$$y' = \pi(\tan x)^{\pi-1} \sec^2 x$$

Hint

Use the result from [\[link\]](#).

Key Concepts

- On the basis of the assumption that the exponential function $y = b^x, b > 0$ is continuous everywhere and differentiable at 0, this function is differentiable everywhere and there is a formula for its derivative.
- We can use a formula to find the derivative of $y = \ln x$, and the relationship $\log_b x = \frac{\ln x}{\ln b}$ allows us to extend our differentiation formulas to include logarithms with arbitrary bases.
- Logarithmic differentiation allows us to differentiate functions of the form $y = g(x)^{f(x)}$ or very complex functions by taking the natural logarithm of both sides and exploiting the properties of logarithms before differentiating.

Key Equations

- Derivative of the natural exponential function**

$$\frac{d}{dx} (e^{g(x)}) = e^{g(x)} g'(x)$$
- Derivative of the natural logarithmic function**

$$\frac{d}{dx} (\ln g(x)) = \frac{1}{g(x)} g'(x)$$
- Derivative of the general exponential function**

$$\frac{d}{dx} (b^{g(x)}) = b^{g(x)} g'(x) \ln b$$
- Derivative of the general logarithmic function**

$$\frac{d}{dx} (\log_b g(x)) = \frac{g'(x)}{g(x) \ln b}$$

For the following exercises, find $f'(x)$ for each function.

Exercise:

Problem: $f(x) = x^2 e^x$

Solution:

$$2xe^x + x^2 e^x$$

Exercise:

Problem: $f(x) = \frac{e^{-x}}{x}$

Exercise:

Problem: $f(x) = e^{x^3 \ln x}$

Solution:

$$e^{x^3 \ln x} (3x^2 \ln x + x^2)$$

Exercise:

Problem: $f(x) = \sqrt{e^{2x} + 2x}$

Exercise:

Problem: $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Solution:

$$\frac{4}{(e^x + e^{-x})^2}$$

Exercise:

Problem: $f(x) = \frac{10^x}{\ln 10}$

Exercise:

Problem: $f(x) = 2^{4x} + 4x^2$

Solution:

$$2^{4x+2} \cdot \ln 2 + 8x$$

Exercise:

Problem: $f(x) = 3^{\sin 3x}$

Exercise:

Problem: $f(x) = x^\pi \cdot \pi^x$

Solution:

$$\pi x^{\pi-1} \cdot \pi^x + x^\pi \cdot \pi^x \ln \pi$$

Exercise:

Problem: $f(x) = \ln(4x^3 + x)$

Exercise:

Problem: $f(x) = \ln \sqrt{5x - 7}$

Solution:

$$\frac{5}{2(5x-7)}$$

Exercise:

Problem: $f(x) = x^2 \ln 9x$

Exercise:

Problem: $f(x) = \log(\sec x)$

Solution:

$$\frac{\tan x}{\ln 10}$$

Exercise:

Problem: $f(x) = \log_7(6x^4 + 3)^5$

Exercise:

Problem: $f(x) = 2^x \cdot \log_3 7^{x^2-4}$

Solution:

$$2^x \cdot \ln 2 \cdot \log_3 7^{x^2-4} + 2^x \cdot \frac{2x \ln 7}{\ln 3}$$

For the following exercises, use logarithmic differentiation to find $\frac{dy}{dx}$.

Exercise:

Problem: $y = x^{\sqrt{x}}$

Exercise:

Problem: $y = (\sin 2x)^{4x}$

Solution:

$$(\sin 2x)^{4x} [4 \cdot \ln(\sin 2x) + 8x \cdot \cot 2x]$$

Exercise:

Problem: $y = (\ln x)^{\ln x}$

Exercise:

Problem: $y = x^{\log_2 x}$

Solution:

$$x^{\log_2 x} \cdot \frac{2 \ln x}{x \ln 2}$$

Exercise:

Problem: $y = (x^2 - 1)^{\ln x}$

Exercise:

Problem: $y = x^{\cot x}$

Solution:

$$x^{\cot x} \cdot \left[-\csc^2 x \cdot \ln x + \frac{\cot x}{x} \right]$$

Exercise:

Problem: $y = \frac{x+11}{\sqrt[3]{x^2-4}}$

Exercise:

Problem: $y = x^{-1/2}(x^2 + 3)^{2/3}(3x - 4)^4$

Solution:

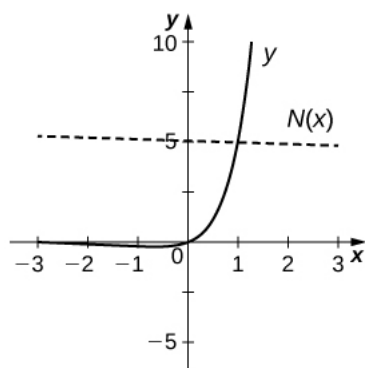
$$x^{-1/2}(x^2 + 3)^{2/3}(3x - 4)^4 \cdot \left[\frac{-1}{2x} + \frac{4x}{3(x^2+3)} + \frac{12}{3x-4} \right]$$

Exercise:

Problem: [T] Find an equation of the tangent line to the graph of $f(x) = 4xe^{(x^2-1)}$ at the point where $x = -1$. Graph both the function and the tangent line.

Exercise:**Problem:**

[T] Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

Solution:

$$y = \frac{-1}{5+5\ln 5}x + \left(5 + \frac{1}{5+5\ln 5}\right)$$

Exercise:

Problem:

[T] Find the equation of the tangent line to the graph of $x^3 - x \ln y + y^3 = 2x + 5$ at the point where $x = 2$.
 (Hint: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

Exercise:

Problem: Consider the function $y = x^{1/x}$ for $x > 0$.

- Determine the points on the graph where the tangent line is horizontal.
- Determine the points on the graph where $y' > 0$ and those where $y' < 0$.

Solution:

- $x = e^{-2.718}$
- $(e, \infty), (0, e)$

Exercise:

Problem: The formula $I(t) = \frac{\sin t}{e^t}$ is the formula for a decaying alternating current.

- Complete the following table with the appropriate values.

t	$\frac{\sin t}{e^t}$
0	(i)
$\frac{\pi}{2}$	(ii)
π	(iii)
$\frac{3\pi}{2}$	(iv)
2π	(v)
2π	(vi)
3π	(vii)
$\frac{7\pi}{2}$	(viii)
4π	(ix)

- Using only the values in the table, determine where the tangent line to the graph of $I(t)$ is horizontal.

Exercise:

Problem:

[T] The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

- Write the exponential function that relates the total population as a function of t .
- Use a. to determine the rate at which the population is increasing in t years.
- Use b. to determine the rate at which the population is increasing in 10 years.

Solution:

a. $P = 500,000(1.05)^t$ individuals b. $P'(t) = 24395 \cdot (1.05)^t$ individuals per year c. 39,737 individuals per year

Exercise:**Problem:**

[T] An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

- Write the exponential function that relates the amount of substance remaining as a function of t , measured in hours.
- Use a. to determine the rate at which the substance is decaying in t hours.
- Use b. to determine the rate of decay at $t = 4$ hours.

Exercise:**Problem:**

[T] The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1961 is modeled by the function

$$N(t) = 5.3e^{0.093t^2 - 0.87t}, (0 \leq t \leq 4),$$

where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.

- Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.
- Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in the United States.

Solution:

a. At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1963 there were approximately 723 cases of the disease in the United States. b. At the beginning of 1960 the number of cases of the disease was decreasing at rate of -4.611 thousand per year; at the beginning of 1963, the number of cases of the disease was decreasing at a rate of -0.2808 thousand per year.

Exercise:**Problem:**

[T] The *relative rate of change* of a differentiable function $y = f(x)$ is given by $\frac{100 \cdot f'(x)}{f(x)}\%$. One model for population growth is a Gompertz growth function, given by $P(x) = ae^{-b \cdot e^{-cx}}$ where a , b , and c are constants.

- Find the relative rate of change formula for the generic Gompertz function.
- Use a. to find the relative rate of change of a population in $x = 20$ months when $a = 204$, $b = 0.0198$, and $c = 0.15$.
- Briefly interpret what the result of b. means.

For the following exercises, use the population of New York City from 1790 to 1860, given in the following table.

Years since 1790	Population
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

New York City Population Over TimeSource: http://en.wikipedia.org/wiki/Largest_cities_in_the_United_States_by_population_by_decade.

Exercise:

Problem: [T] Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

Solution:

$$p = 35741(1.045)^t$$

Exercise:

Problem:

[T] Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

Exercise:

Problem:

[T] Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

Solution:

Years since 1790	P''
0	69.25
10	107.5
20	167.0
30	259.4
40	402.8
50	625.5
60	971.4
70	1508.5

Exercise:

Problem: [T] Using the tables of first and second derivatives and the best fit, answer the following questions:

- Will the model be accurate in predicting the future population of New York City? Why or why not?
- Estimate the population in 2010. Was the prediction correct from a.?

Chapter Review Exercises

True or False? Justify the answer with a proof or a counterexample.

Exercise:

Problem: Every function has a derivative.

Solution:

False.

Exercise:

Problem: A continuous function has a continuous derivative.

Exercise:

Problem: A continuous function has a derivative.

Solution:

False

Exercise:

Problem: If a function is differentiable, it is continuous.

Use the limit definition of the derivative to exactly evaluate the derivative.

Exercise:

Problem: $f(x) = \sqrt{x+4}$

Solution:

$$\frac{1}{2\sqrt{x+4}}$$

Exercise:

Problem: $f(x) = \frac{3}{x}$

Find the derivatives of the following functions.

Exercise:

Problem: $f(x) = 3x^3 - \frac{4}{x^2}$

Solution:

$$9x^2 + \frac{8}{x^3}$$

Exercise:

Problem: $f(x) = (4 - x^2)^3$

Exercise:

Problem: $f(x) = e^{\sin x}$

Solution:

$$e^{\sin x} \cos x$$

Exercise:

Problem: $f(x) = \ln(x+2)$

Exercise:

Problem: $f(x) = x^2 \cos x + x \tan(x)$

Solution:

$$x \sec^2(x) + 2x \cos(x) + \tan(x) - x^2 \sin(x)$$

Exercise:

Problem: $f(x) = \sqrt{3x^2 + 2}$

Exercise:

Problem: $f(x) = \frac{x}{4} \sin^{-1}(x)$

Solution:

$$\frac{1}{4} \left(\frac{x}{\sqrt{1-x^2}} + \sin^{-1}(x) \right)$$

Exercise:

Problem: $x^2y = (y + 2) + xy \sin(x)$

Find the following derivatives of various orders.

Exercise:

Problem: First derivative of $y = x \ln(x) \cos x$

Solution:

$$\cos x \cdot (\ln x + 1) - x \ln(x) \sin x$$

Exercise:

Problem: Third derivative of $y = (3x + 2)^2$

Exercise:

Problem: Second derivative of $y = 4^x + x^2 \sin(x)$

Solution:

$$4^x (\ln 4)^2 + 2 \sin x + 4x \cos x - x^2 \sin x$$

Find the equation of the tangent line to the following equations at the specified point.

Exercise:

Problem: $y = \cos^{-1}(x) + x$ at $x = 0$

Exercise:

Problem: $y = x + e^x - \frac{1}{x}$ at $x = 1$

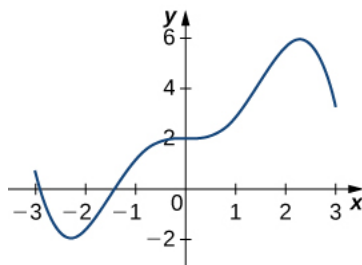
Solution:

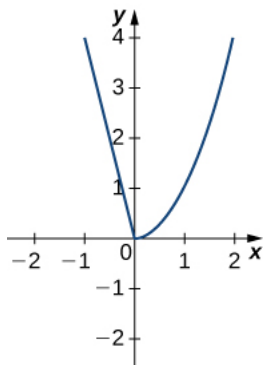
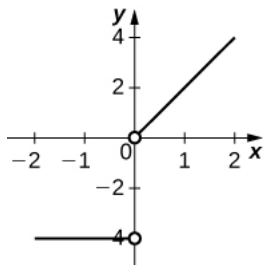
$$T = (2 + e)x - 2$$

Draw the derivative for the following graphs.

Exercise:

Problem:



Exercise:**Problem:****Solution:**

The following questions concern the water level in Ocean City, New Jersey, in January, which can be approximated by $w(t) = 1.9 + 2.9 \cos\left(\frac{\pi}{6}t\right)$, where t is measured in hours after midnight, and the height is measured in feet.

Exercise:

Problem: Find and graph the derivative. What is the physical meaning?

Exercise:

Problem: Find $w'(3)$. What is the physical meaning of this value?

Solution:

$w'(3) = -\frac{2.9\pi}{6}$. At 3 a.m. the tide is decreasing at a rate of 1.514 ft/hr.

The following questions consider the wind speeds of Hurricane Katrina, which affected New Orleans, Louisiana, in August 2005. The data are displayed in a table.

Hours after Midnight, August 26	Wind Speed (mph)
1	45
5	75
11	100
29	115
49	145
58	175
73	155
81	125
85	95
107	35

Wind Speeds of Hurricane KatrinaSource:

http://news.nationalgeographic.com/news/2005/09/0914_050914_katrina_timeline.html.

Exercise:

Problem:

Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

Exercise:

Problem: Estimate the derivative of the wind speed at hour 83. What is the physical meaning?

Solution:

−7.5. The wind speed is decreasing at a rate of 7.5 mph/hr

Glossary

logarithmic differentiation

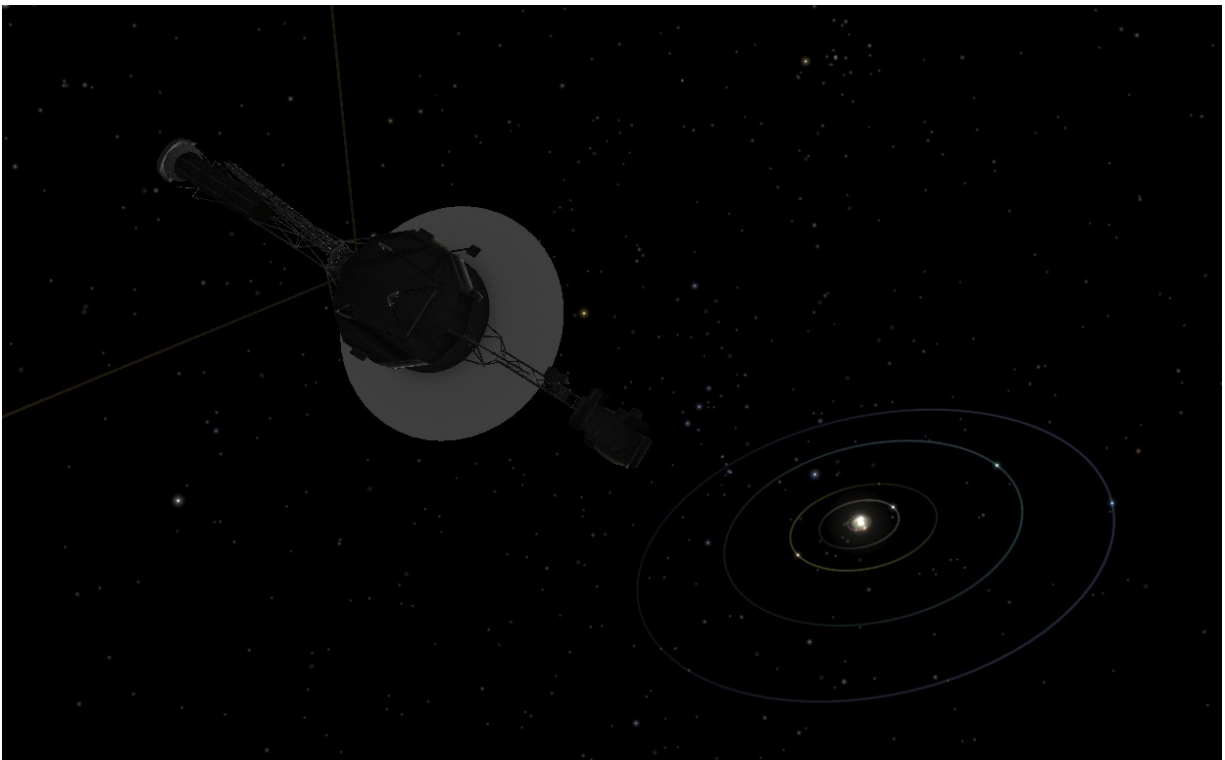
is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

Introduction

class="introduction"

The view of
our solar
system from
NASA's
Voyager 1
spacecraft in
late 2010 is
shown here.

(credit:
modificatio
n of work
by NASA,
Flickr)



Voyager 1 is the most distant man made object, at more than 17.3 billion kilometers (10.7 billion miles) from Earth. The colored lines represent the orbits of the planets; the orbits of the three closest planets (including Earth) are lost in the glare of the sun on this scale. Voyager is so far from Earth that, at this time, a radio signal from Earth takes more than 16 hours to reach the spacecraft at the speed of light.

According to Kepler's laws of planetary motion, the shape of a planet's orbit is elliptical, with the Sun at one focus of the ellipse. We study this idea in more detail in Conic Sections. (See [\[link\]](#).)

In this chapter we also study parametric equations, which give us a convenient way to describe curves, or to study the position of a particle or object in two dimensions as a function of time. We will use parametric equations for describing many topics later in this text.

Parametric Equations

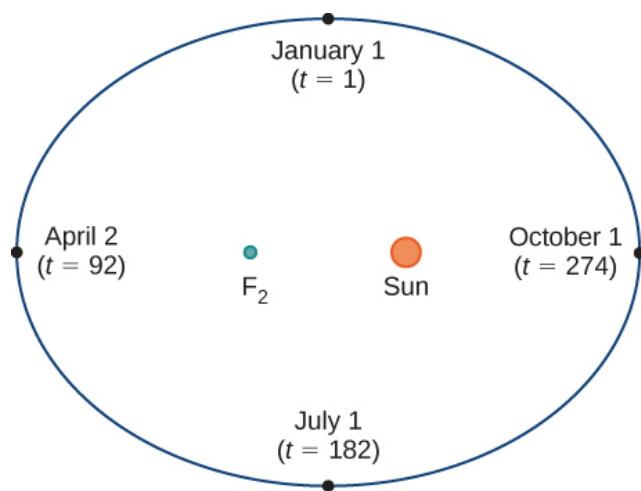
- Plot a curve described by parametric equations.
- Convert the parametric equations of a curve into the form $y = f(x)$.
- Recognize the parametric equations of basic curves, such as a line and a circle.
- Recognize the parametric equations of a cycloid.

In this section we examine parametric equations and their graphs. In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on, and as the parameter increases, the values of x and y trace out a path along a plane curve. For example, if the parameter is t (a common choice), then t might represent time. Then x and y are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path.

Parametric Equations and Their Graphs

Consider the orbit of Earth around the Sun. Our year lasts approximately 365.25 days, but for this discussion we will use 365 days. On January 1 of each year, the physical location of Earth with respect to the Sun is nearly the same, except for leap years, when the lag introduced by the extra $\frac{1}{4}$ day of orbiting time is built into the calendar. We call January 1 “day 1” of the year. Then, for example, day 31 is January 31, day 59 is February 28, and so on.

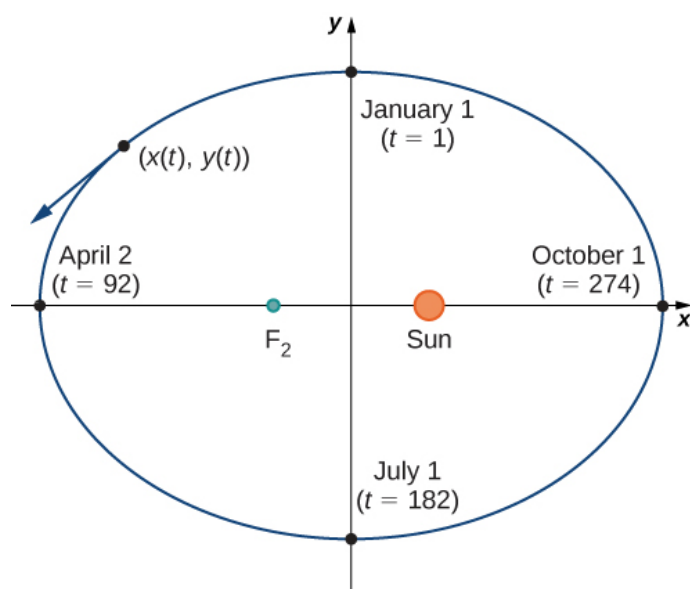
The number of the day in a year can be considered a variable that determines Earth’s position in its orbit. As Earth revolves around the Sun, its physical location changes relative to the Sun. After one full year, we are back where we started, and a new year begins. According to Kepler’s laws of planetary motion, the shape of the orbit is elliptical, with the Sun at one focus of the ellipse. We study this idea in more detail in [Conic Sections](#).



Earth’s orbit around the Sun in one year.

[\[link\]](#) depicts Earth’s orbit around the Sun during one year. The point labeled F_2 is one of the foci of the ellipse; the other focus is occupied by the Sun. If we superimpose coordinate axes over this graph, then we can assign ordered pairs to each point on the ellipse ([\[link\]](#)). Then each x value on the graph is a value of position

as a function of time, and each y value is also a value of position as a function of time. Therefore, each point on the graph corresponds to a value of Earth's position as a function of time.



Coordinate axes superimposed on the orbit of Earth.

We can determine the functions for $x(t)$ and $y(t)$, thereby parameterizing the orbit of Earth around the Sun. The variable t is called an independent parameter and, in this context, represents time relative to the beginning of each year.

A curve in the (x, y) plane can be represented parametrically. The equations that are used to define the curve are called **parametric equations**.

Note:

Definition

If x and y are continuous functions of t on an interval I , then the equations

Equation:

$$x = x(t) \text{ and } y = y(t)$$

are called parametric equations and t is called the **parameter**. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is called a **parametric curve** or *plane curve*, and is denoted by C .

Notice in this definition that x and y are used in two ways. The first is as functions of the independent variable t . As t varies over the interval I , the functions $x(t)$ and $y(t)$ generate a set of ordered pairs (x, y) . This set of ordered pairs generates the graph of the parametric equations. In this second usage, to designate the ordered

pairs, x and y are variables. It is important to distinguish the variables x and y from the functions $x(t)$ and $y(t)$.

Example:

Exercise:

Problem:

Graphing a Parametrically Defined Curve

Sketch the curves described by the following parametric equations:

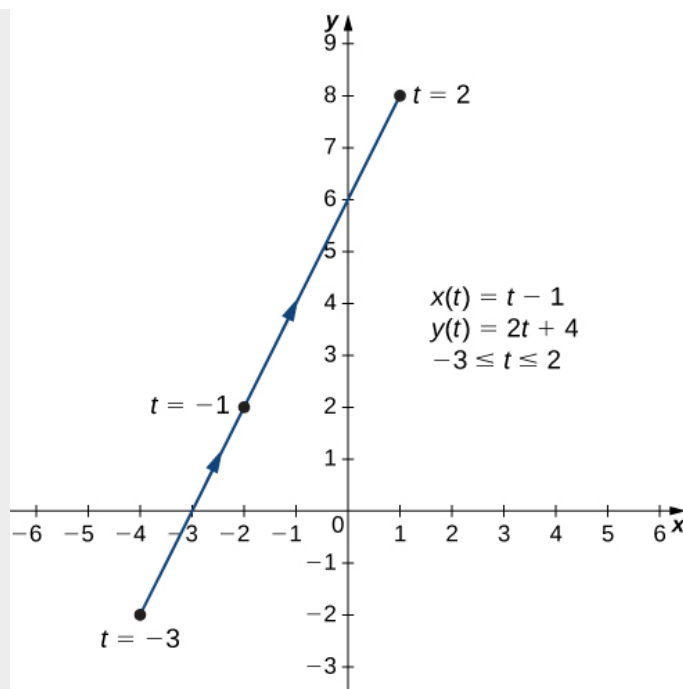
- a. $x(t) = t - 1, \quad y(t) = 2t + 4, \quad -3 \leq t \leq 2$
- b. $x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad -2 \leq t \leq 3$
- c. $x(t) = 4 \cos t, \quad y(t) = 4 \sin t, \quad 0 \leq t \leq 2\pi$

Solution:

- a. To create a graph of this curve, first set up a table of values. Since the independent variable in both $x(t)$ and $y(t)$ is t , let t appear in the first column. Then $x(t)$ and $y(t)$ will appear in the second and third columns of the table.

t	$x(t)$	$y(t)$
-3	-4	-2
-2	-3	0
-1	-2	2
0	-1	4
1	0	6
2	1	8

The second and third columns in this table provide a set of points to be plotted. The graph of these points appears in [\[link\]](#). The arrows on the graph indicate the **orientation** of the graph, that is, the direction that a point moves on the graph as t varies from -3 to 2 .

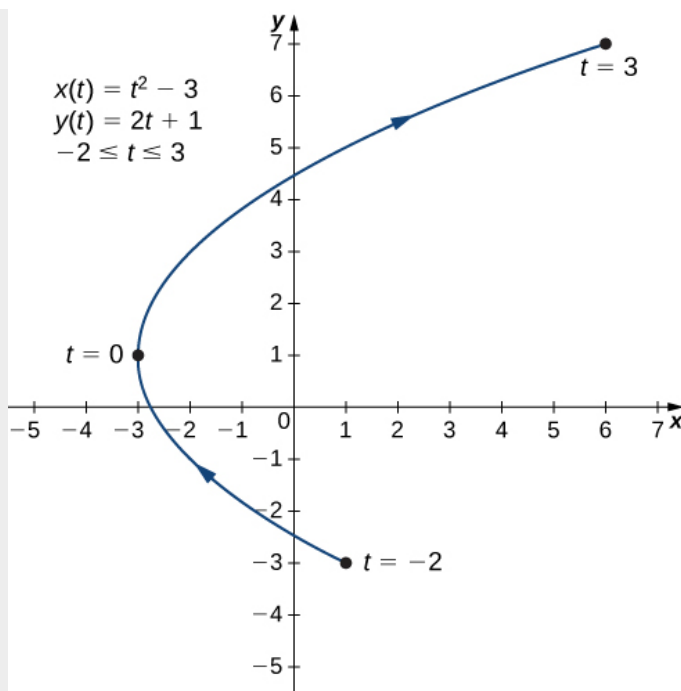


Graph of the plane curve described by the parametric equations in part a.

b. To create a graph of this curve, again set up a table of values.

t	$x(t)$	$y(t)$
-2	1	-3
-1	-2	-1
0	-3	1
1	-2	3
2	1	5
3	6	7

The second and third columns in this table give a set of points to be plotted ([link](#)). The first point on the graph (corresponding to $t = -2$) has coordinates $(1, -3)$, and the last point (corresponding to $t = 3$) has coordinates $(6, 7)$. As t progresses from -2 to 3 , the point on the curve travels along a parabola. The direction the point moves is again called the orientation and is indicated on the graph.

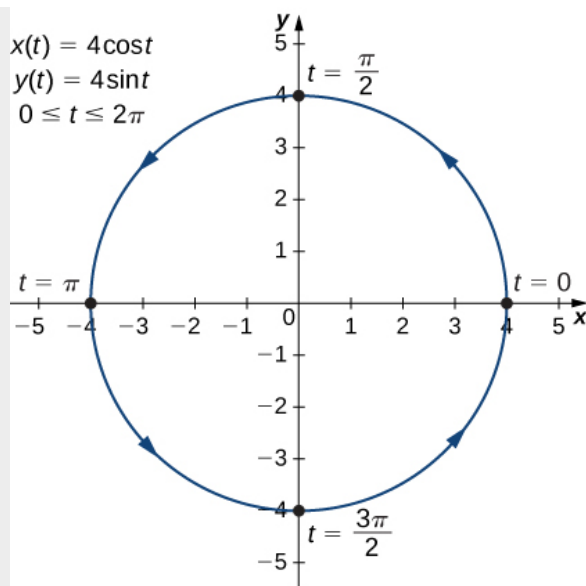


Graph of the plane curve described by the parametric equations in part b.

c. In this case, use multiples of $\pi/6$ for t and create another table of values:

t	$x(t)$	$y(t)$		t	$x(t)$	$y(t)$
0	4	0		$\frac{7\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2
$\frac{\pi}{6}$	$2\sqrt{3} \approx 3.5$	2		$\frac{4\pi}{3}$	-2	$-2\sqrt{3} \approx -3.5$
$\frac{\pi}{3}$	2	$2\sqrt{3} \approx 3.5$		$\frac{3\pi}{2}$	0	-4
$\frac{\pi}{2}$	0	4		$\frac{5\pi}{3}$	2	$-2\sqrt{3} \approx -3.5$
$\frac{2\pi}{3}$	-2	$2\sqrt{3} \approx 3.5$		$\frac{11\pi}{6}$	$2\sqrt{3} \approx 3.5$	2
$\frac{5\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2		2π	4	0
π	-4	0				

The graph of this plane curve appears in the following graph.



Graph of the plane curve described by the parametric equations in part c.

This is the graph of a circle with radius 4 centered at the origin, with a counterclockwise orientation. The starting point and ending points of the curve both have coordinates $(4, 0)$.

Note:

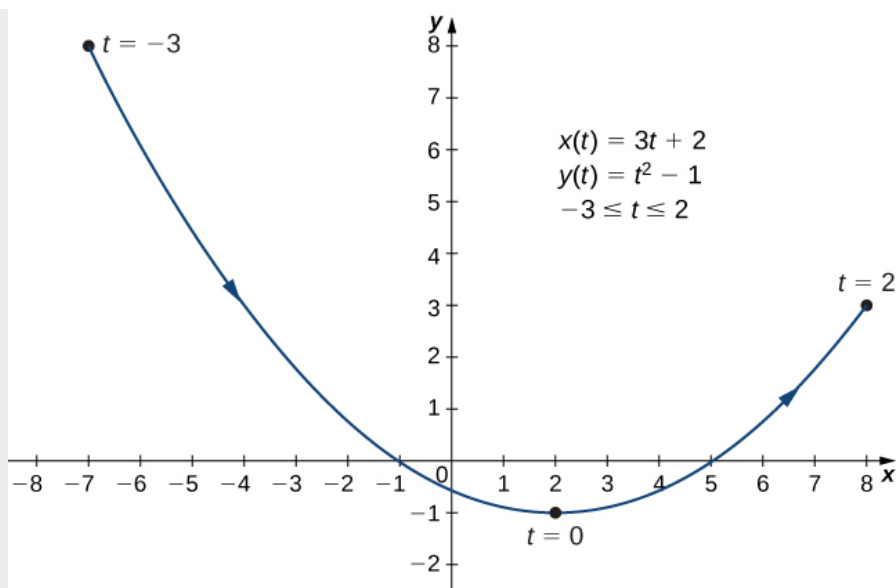
Exercise:

Problem: Sketch the curve described by the parametric equations

Equation:

$$x(t) = 3t + 2, \quad y(t) = t^2 - 1, \quad -3 \leq t \leq 2.$$

Solution:



Hint

Make a table of values for $x(t)$ and $y(t)$ using t values from -3 to 2 .

Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables x and y . Then we can apply any previous knowledge of equations of curves in the plane to identify the curve. For example, the equations describing the plane curve in [\[link\]](#)b. are

Equation:

$$x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad -2 \leq t \leq 3.$$

Solving the second equation for t gives

Equation:

$$t = \frac{y - 1}{2}.$$

This can be substituted into the first equation:

Equation:

$$x = \left(\frac{y - 1}{2} \right)^2 - 3 = \frac{y^2 - 2y + 1}{4} - 3 = \frac{y^2 - 2y - 11}{4}.$$

This equation describes x as a function of y . These steps give an example of *eliminating the parameter*. The graph of this function is a parabola opening to the right. Recall that the plane curve started at $(1, -3)$ and ended at $(6, 7)$. These terminations were due to the restriction on the parameter t .

Example:

Exercise:

Problem:

Eliminating the Parameter

Eliminate the parameter for each of the plane curves described by the following parametric equations and describe the resulting graph.

a. $x(t) = \sqrt{2t+4}$, $y(t) = 2t+1$, $-2 \leq t \leq 6$

b. $x(t) = 4 \cos t$, $y(t) = 3 \sin t$, $0 \leq t \leq 2\pi$

Solution:

- a. To eliminate the parameter, we can solve either of the equations for t . For example, solving the first equation for t gives

Equation:

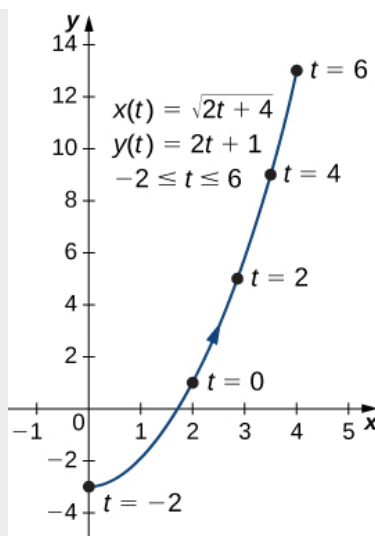
$$\begin{aligned}x &= \sqrt{2t+4} \\x^2 &= 2t+4 \\x^2-4 &= 2t \\t &= \frac{x^2-4}{2}.\end{aligned}$$

Note that when we square both sides it is important to observe that $x \geq 0$. Substituting $t = \frac{x^2-4}{2}$ this into $y(t)$ yields

Equation:

$$\begin{aligned}y(t) &= 2t+1 \\y &= 2\left(\frac{x^2-4}{2}\right)+1 \\y &= x^2-4+1 \\y &= x^2-3.\end{aligned}$$

This is the equation of a parabola opening upward. There is, however, a domain restriction because of the limits on the parameter t . When $t = -2$, $x = \sqrt{2(-2)+4} = 0$, and when $t = 6$, $x = \sqrt{2(6)+4} = 4$. The graph of this plane curve follows.



Graph of the plane curve described by the parametric equations in part a.

- b. Sometimes it is necessary to be a bit creative in eliminating the parameter. The parametric equations for this example are

Equation:

$$x(t) = 4 \cos t \text{ and } y(t) = 3 \sin t.$$

Solving either equation for t directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by 4 and the second equation by 3 (and suppressing the t) gives us

Equation:

$$\cos t = \frac{x}{4} \text{ and } \sin t = \frac{y}{3}.$$

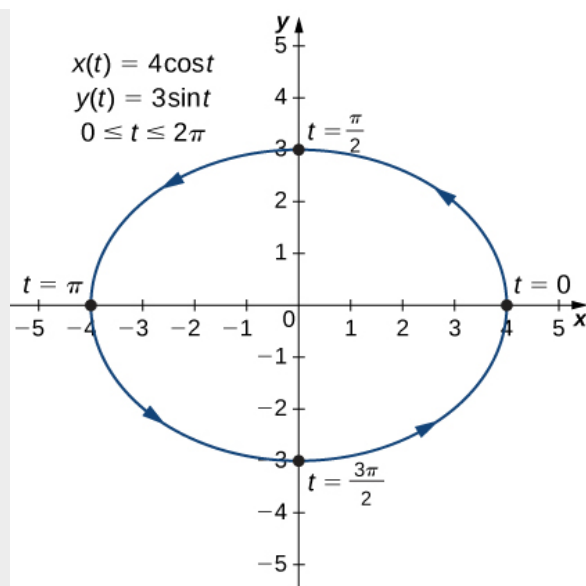
Now use the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ and replace the expressions for $\sin t$ and $\cos t$ with the equivalent expressions in terms of x and y . This gives

Equation:

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

This is the equation of a horizontal ellipse centered at the origin, with semimajor axis 4 and semiminor axis 3 as shown in the following graph.



Graph of the plane curve described by the parametric equations in part b.

As t progresses from 0 to 2π , a point on the curve traverses the ellipse once, in a counterclockwise direction. Recall from the section opener that the orbit of Earth around the Sun is also elliptical. This is a perfect example of using parameterized curves to model a real-world phenomenon.

Note:

Exercise:

Problem:

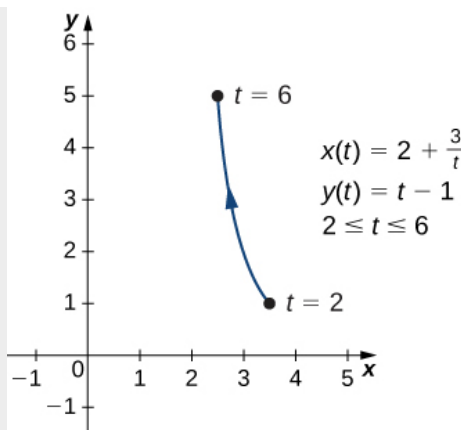
Eliminate the parameter for the plane curve defined by the following parametric equations and describe the resulting graph.

Equation:

$$x(t) = 2 + \frac{3}{t}, \quad y(t) = t - 1, \quad 2 \leq t \leq 6$$

Solution:

$x = 2 + \frac{3}{y+1}$, or $y = -1 + \frac{3}{x-2}$. This equation describes a portion of a rectangular hyperbola centered at $(2, -1)$.



Hint

Solve one of the equations for t and substitute into the other equation.

So far we have seen the method of eliminating the parameter, assuming we know a set of parametric equations that describe a plane curve. What if we would like to start with the equation of a curve and determine a pair of parametric equations for that curve? This is certainly possible, and in fact it is possible to do so in many different ways for a given curve. The process is known as **parameterization of a curve**.

Example:

Exercise:

Problem:

Parameterizing a Curve

Find two different pairs of parametric equations to represent the graph of $y = 2x^2 - 3$.

Solution:

First, it is always possible to parameterize a curve by defining $x(t) = t$, then replacing x with t in the equation for $y(t)$. This gives the parameterization

Equation:

$$x(t) = t, \quad y(t) = 2t^2 - 3.$$

Since there is no restriction on the domain in the original graph, there is no restriction on the values of t .

We have complete freedom in the choice for the second parameterization. For example, we can choose $x(t) = 3t - 2$. The only thing we need to check is that there are no restrictions imposed on x ; that is, the range of $x(t)$ is all real numbers. This is the case for $x(t) = 3t - 2$. Now since $y = 2x^2 - 3$, we can substitute $x(t) = 3t - 2$ for x . This gives

Equation:

$$\begin{aligned}
 y(t) &= 2(3t - 2)^2 - 2 \\
 &= 2(9t^2 - 12t + 4) - 2 \\
 &= 18t^2 - 24t + 8 - 2 \\
 &= 18t^2 - 24t + 6.
 \end{aligned}$$

Therefore, a second parameterization of the curve can be written as

Equation:

$$x(t) = 3t - 2 \text{ and } y(t) = 18t^2 - 24t + 6.$$

Note:

Exercise:

Problem: Find two different sets of parametric equations to represent the graph of $y = x^2 + 2x$.

Solution:

One possibility is $x(t) = t$, $y(t) = t^2 + 2t$. Another possibility is $x(t) = 2t - 3$, $y(t) = (2t - 3)^2 + 2(2t - 3) = 4t^2 - 8t + 3$.

There are, in fact, an infinite number of possibilities.

Hint

Follow the steps in [\[link\]](#). Remember we have freedom in choosing the parameterization for $x(t)$.

Cycloids and Other Parametric Curves

Imagine going on a bicycle ride through the country. The tires stay in contact with the road and rotate in a predictable pattern. Now suppose a very determined ant is tired after a long day and wants to get home. So he hangs onto the side of the tire and gets a free ride. The path that this ant travels down a straight road is called a **cycloid** ([\[link\]](#)). A cycloid generated by a circle (or bicycle wheel) of radius a is given by the parametric equations

Equation:

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

To see why this is true, consider the path that the center of the wheel takes. The center moves along the x -axis at a constant height equal to the radius of the wheel. If the radius is a , then the coordinates of the center can be given by the equations

Equation:

$$x(t) = at, \quad y(t) = a$$

for any value of t . Next, consider the ant, which rotates around the center along a circular path. If the bicycle is moving from left to right then the wheels are rotating in a clockwise direction. A possible parameterization of

the circular motion of the ant (relative to the center of the wheel) is given by

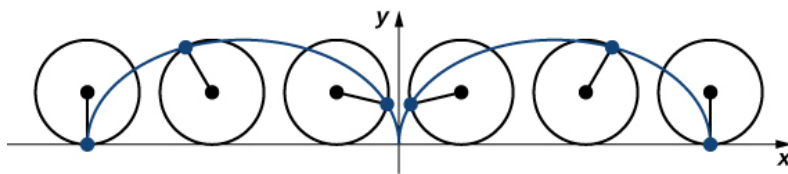
Equation:

$$x(t) = -a \sin t, \quad y(t) = -a \cos t.$$

(The negative sign is needed to reverse the orientation of the curve. If the negative sign were not there, we would have to imagine the wheel rotating counterclockwise.) Adding these equations together gives the equations for the cycloid.

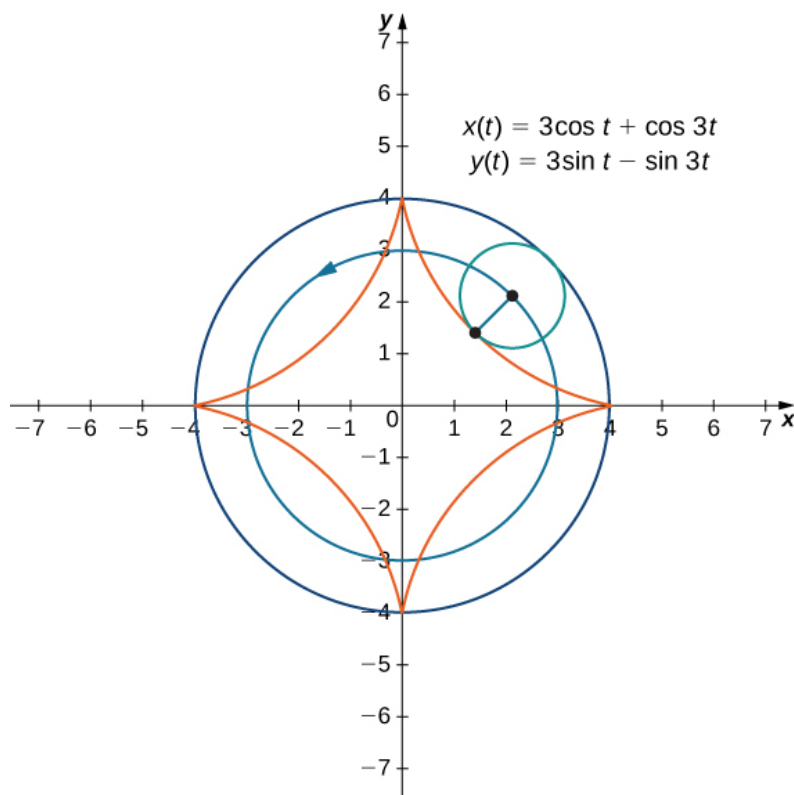
Equation:

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$



A wheel traveling along a road without slipping; the point on the edge of the wheel traces out a cycloid.

Now suppose that the bicycle wheel doesn't travel along a straight road but instead moves along the inside of a larger wheel, as in [\[link\]](#). In this graph, the green circle is traveling around the blue circle in a counterclockwise direction. A point on the edge of the green circle traces out the red graph, which is called a hypocycloid.



Graph of the hypocycloid described by the parametric equations shown.

The general parametric equations for a hypocycloid are

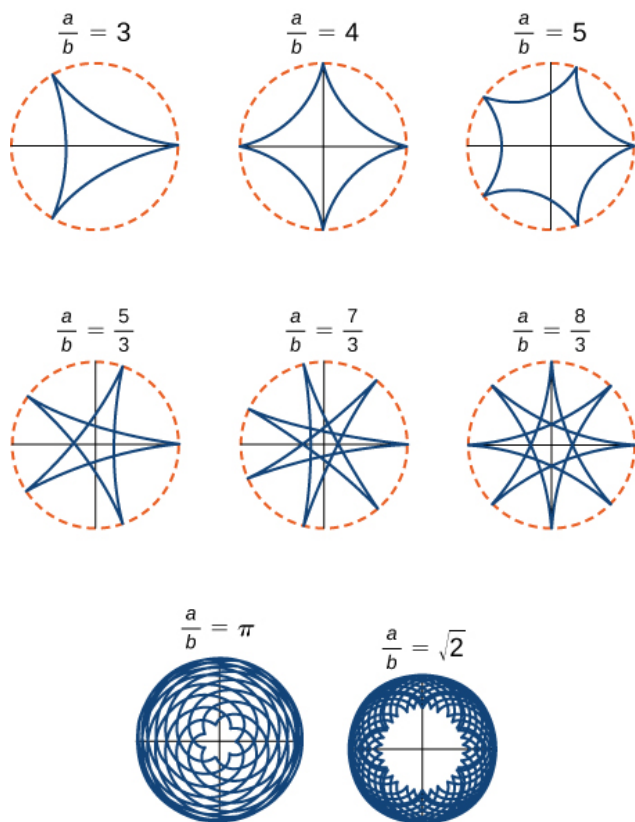
Equation:

$$x(t) = (a - b) \cos t + b \cos \left(\frac{a-b}{b} \right) t$$

$$y(t) = (a - b) \sin t - b \sin \left(\frac{a-b}{b} \right) t.$$

These equations are a bit more complicated, but the derivation is somewhat similar to the equations for the cycloid. In this case we assume the radius of the larger circle is a and the radius of the smaller circle is b . Then the center of the wheel travels along a circle of radius $a - b$. This fact explains the first term in each equation above. The period of the second trigonometric function in both $x(t)$ and $y(t)$ is equal to $\frac{2\pi b}{a-b}$.

The ratio $\frac{a}{b}$ is related to the number of **cusps** on the graph (cusps are the corners or pointed ends of the graph), as illustrated in [\[link\]](#). This ratio can lead to some very interesting graphs, depending on whether or not the ratio is rational. [\[link\]](#) corresponds to $a = 4$ and $b = 1$. The result is a hypocycloid with four cusps. [\[link\]](#) shows some other possibilities. The last two hypocycloids have irrational values for $\frac{a}{b}$. In these cases the hypocycloids have an infinite number of cusps, so they never return to their starting point. These are examples of what are known as *space-filling curves*.



Graph of various hypocycloids corresponding to different values of a/b .

Note:

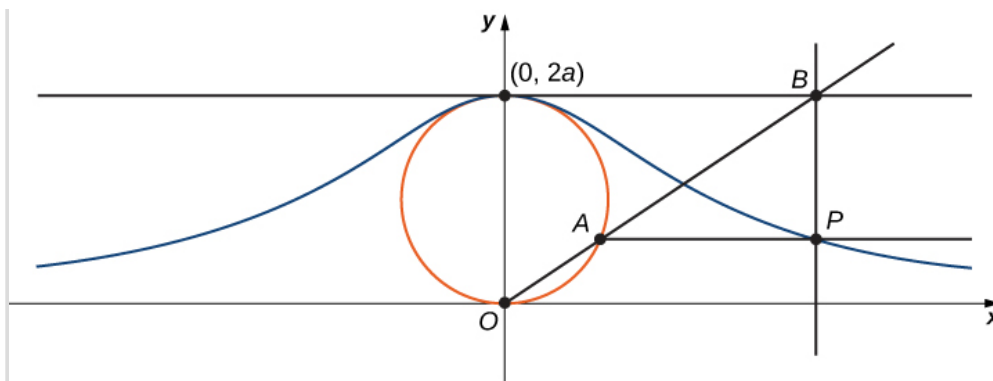
The Witch of Agnesi

Many plane curves in mathematics are named after the people who first investigated them, like the folium of Descartes or the spiral of Archimedes. However, perhaps the strangest name for a curve is the witch of Agnesi. Why a witch?

Maria Gaetana Agnesi (1718–1799) was one of the few recognized women mathematicians of eighteenth-century Italy. She wrote a popular book on analytic geometry, published in 1748, which included an interesting curve that had been studied by Fermat in 1630. The mathematician Guido Grandi showed in 1703 how to construct this curve, which he later called the “versoria,” a Latin term for a rope used in sailing. Agnesi used the Italian term for this rope, “versiera,” but in Latin, this same word means a “female goblin.” When Agnesi’s book was translated into English in 1801, the translator used the term “witch” for the curve, instead of rope. The name “witch of Agnesi” has stuck ever since.

The witch of Agnesi is a curve defined as follows: Start with a circle of radius a so that the points $(0, 0)$ and $(0, 2a)$ are points on the circle ([link](#)). Let O denote the origin. Choose any other point A on the circle, and draw the secant line OA . Let B denote the point at which the line OA intersects the horizontal line through $(0, 2a)$. The vertical line through B intersects the horizontal line through A at the point P . As the point A varies, the path that the point P travels is the witch of Agnesi curve for the given circle.

Witch of Agnesi curves have applications in physics, including modeling water waves and distributions of spectral lines. In probability theory, the curve describes the probability density function of the Cauchy distribution. In this project you will parameterize these curves.



As the point A moves around the circle, the point P traces out the witch of Agnesi curve for the given circle.

1. On the figure, label the following points, lengths, and angle:

- C is the point on the x -axis with the same x -coordinate as A .
- x is the x -coordinate of P , and y is the y -coordinate of P .
- E is the point $(0, a)$.
- F is the point on the line segment OA such that the line segment EF is perpendicular to the line segment OA .
- b is the distance from O to F .
- c is the distance from F to A .
- d is the distance from O to B .
- θ is the measure of angle $\angle COA$.

The goal of this project is to parameterize the witch using θ as a parameter. To do this, write equations for x and y in terms of only θ .

- Show that $d = \frac{2a}{\sin \theta}$.
- Note that $x = d \cos \theta$. Show that $x = 2a \cot \theta$. When you do this, you will have parameterized the x -coordinate of the curve with respect to θ . If you can get a similar equation for y , you will have parameterized the curve.
- In terms of θ , what is the angle $\angle EOA$?
- Show that $b + c = 2a \cos \left(\frac{\pi}{2} - \theta \right)$.
- Show that $y = 2a \cos \left(\frac{\pi}{2} - \theta \right) \sin \theta$.
- Show that $y = 2a \sin^2 \theta$. You have now parameterized the y -coordinate of the curve with respect to θ .
- Conclude that a parameterization of the given witch curve is

Equation:

$$x = 2a \cot \theta, y = 2a \sin^2 \theta, -\infty < \theta < \infty.$$

- Use your parameterization to show that the given witch curve is the graph of the function $f(x) = \frac{8a^3}{x^2 + 4a^2}$.

Note:

Travels with My Ant: The Curtate and Prolate Cycloids

Earlier in this section, we looked at the parametric equations for a cycloid, which is the path a point on the edge of a wheel traces as the wheel rolls along a straight path. In this project we look at two different variations of the cycloid, called the curtate and prolate cycloids.

First, let's revisit the derivation of the parametric equations for a cycloid. Recall that we considered a tenacious ant trying to get home by hanging onto the edge of a bicycle tire. We have assumed the ant climbed onto the tire at the very edge, where the tire touches the ground. As the wheel rolls, the ant moves with the edge of the tire ([link](#)).

As we have discussed, we have a lot of flexibility when parameterizing a curve. In this case we let our parameter t represent the angle the tire has rotated through. Looking at [link](#), we see that after the tire has rotated through an angle of t , the position of the center of the wheel, $C = (x_C, y_C)$, is given by

Equation:

$$x_C = at \text{ and } y_C = a.$$

Furthermore, letting $A = (x_A, y_A)$ denote the position of the ant, we note that

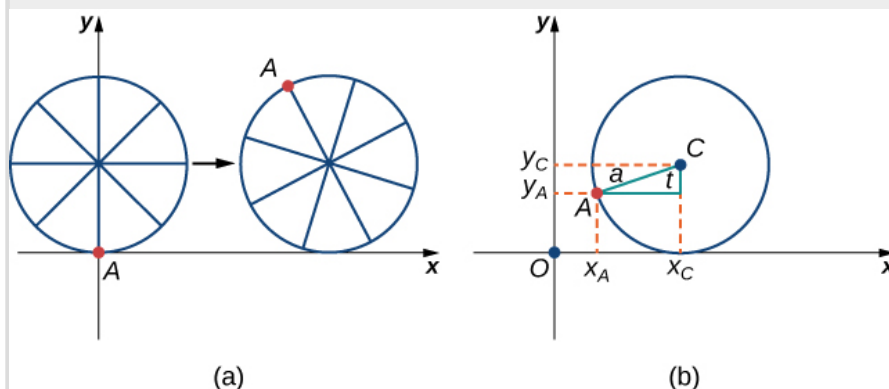
Equation:

$$x_C - x_A = a \sin t \text{ and } y_C - y_A = a \cos t.$$

Then

Equation:

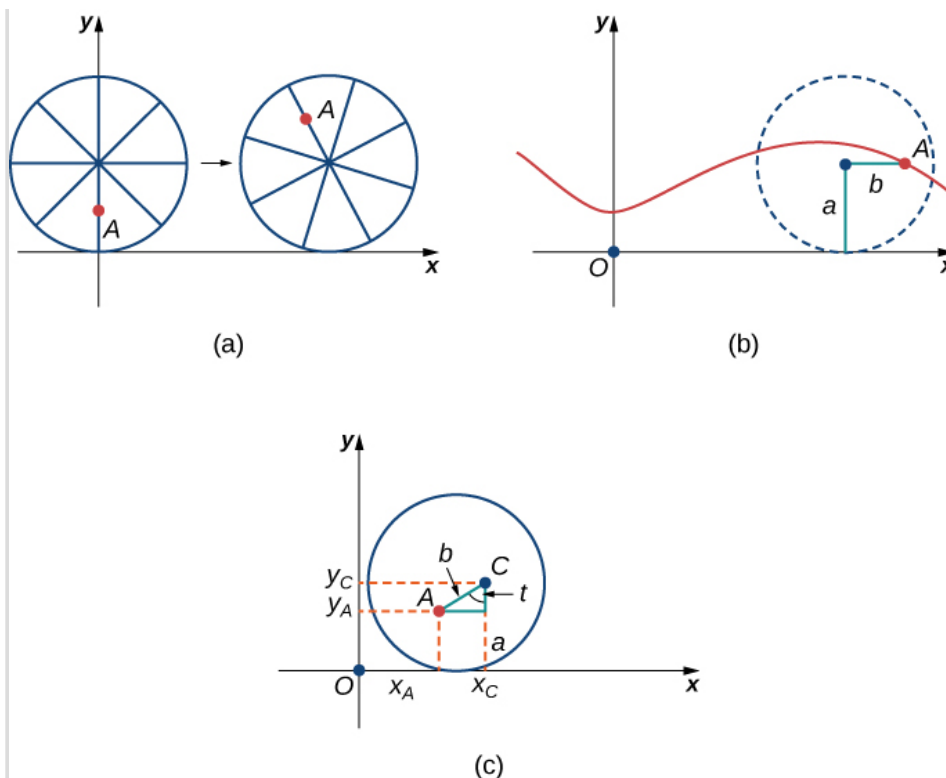
$$\begin{aligned} x_A &= x_C - a \sin t = at - a \sin t = a(t - \sin t) \\ y_A &= y_C - a \cos t = a - a \cos t = a(1 - \cos t). \end{aligned}$$



(a) The ant clings to the edge of the bicycle tire as the tire rolls along the ground. (b) Using geometry to determine the position of the ant after the tire has rotated through an angle of t .

Note that these are the same parametric representations we had before, but we have now assigned a physical meaning to the parametric variable t .

After a while the ant is getting dizzy from going round and round on the edge of the tire. So he climbs up one of the spokes toward the center of the wheel. By climbing toward the center of the wheel, the ant has changed his path of motion. The new path has less up-and-down motion and is called a curtate cycloid ([link](#)). As shown in the figure, we let b denote the distance along the spoke from the center of the wheel to the ant. As before, we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant.



(a) The ant climbs up one of the spokes toward the center of the wheel. (b) The ant's path of motion after he climbs closer to the center of the wheel. This is called a curtate cycloid. (c) The new setup, now that the ant has moved closer to the center of the wheel.

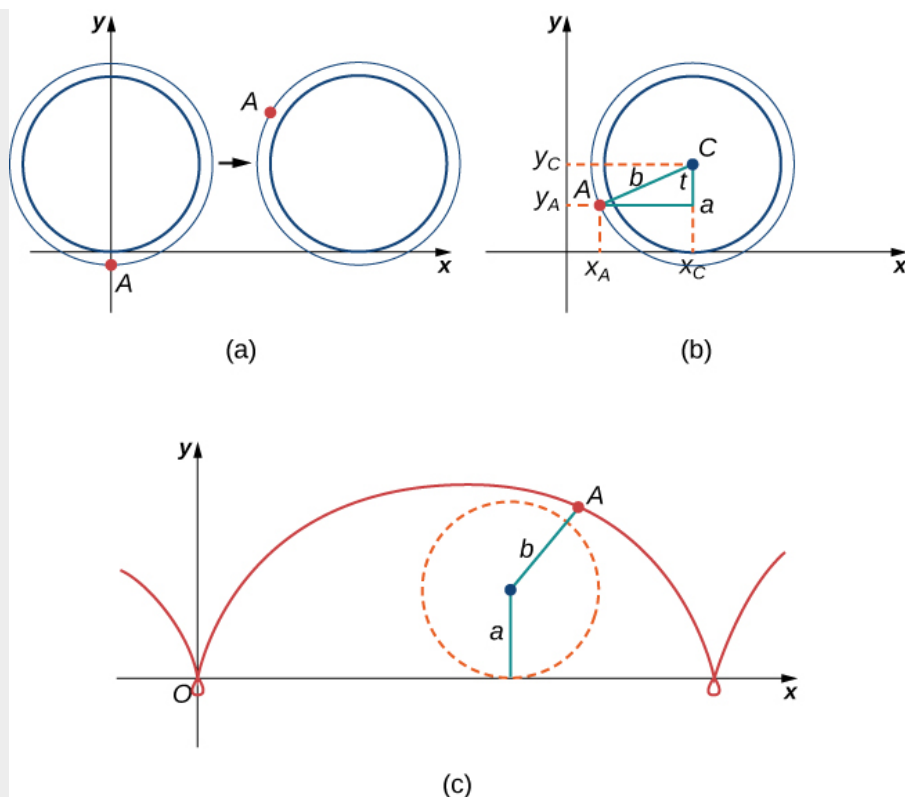
1. What is the position of the center of the wheel after the tire has rotated through an angle of t ?
2. Use geometry to find expressions for $x_C - x_A$ and for $y_C - y_A$.
3. On the basis of your answers to parts 1 and 2, what are the parametric equations representing the curtate cycloid?

Once the ant's head clears, he realizes that the bicyclist has made a turn, and is now traveling away from his home. So he drops off the bicycle tire and looks around. Fortunately, there is a set of train tracks nearby, headed back in the right direction. So the ant heads over to the train tracks to wait. After a while, a train goes by, heading in the right direction, and he manages to jump up and just catch the edge of the train wheel (without getting squished!).

The ant is still worried about getting dizzy, but the train wheel is slippery and has no spokes to climb, so he decides to just hang on to the edge of the wheel and hope for the best. Now, train wheels have a flange to keep the wheel running on the tracks. So, in this case, since the ant is hanging on to the very edge of the flange, the distance from the center of the wheel to the ant is actually greater than the radius of the wheel ([link](#)).

The setup here is essentially the same as when the ant climbed up the spoke on the bicycle wheel. We let b denote the distance from the center of the wheel to the ant, and we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant ([link](#)).

When the distance from the center of the wheel to the ant is greater than the radius of the wheel, his path of motion is called a prolate cycloid. A graph of a prolate cycloid is shown in the figure.



(a) The ant is hanging onto the flange of the train wheel. (b) The new setup, now that the ant has jumped onto the train wheel. (c) The ant travels along a prolate cycloid.

4. Using the same approach you used in parts 1– 3, find the parametric equations for the path of motion of the ant.
5. What do you notice about your answer to part 3 and your answer to part 4?
Notice that the ant is actually traveling backward at times (the “loops” in the graph), even though the train continues to move forward. He is probably going to be *really* dizzy by the time he gets home!

Key Concepts

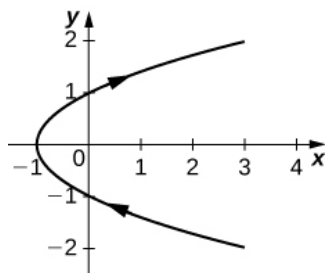
- Parametric equations provide a convenient way to describe a curve. A parameter can represent time or some other meaningful quantity.
- It is often possible to eliminate the parameter in a parameterized curve to obtain a function or relation describing that curve.
- There is always more than one way to parameterize a curve.
- Parametric equations can describe complicated curves that are difficult or perhaps impossible to describe using rectangular coordinates.

For the following exercises, sketch the curves below by eliminating the parameter t . Give the orientation of the curve.

Exercise:

Problem: $x = t^2 + 2t, y = t + 1$

Solution:



orientation: bottom to top

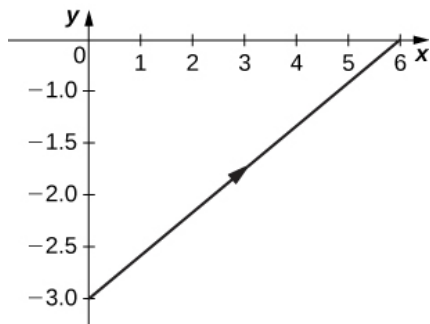
Exercise:

Problem: $x = \cos(t), y = \sin(t), (0, 2\pi]$

Exercise:

Problem: $x = 2t + 4, y = t - 1$

Solution:



orientation: left to right

Exercise:

Problem: $x = 3 - t, y = 2t - 3, 1.5 \leq t \leq 3$

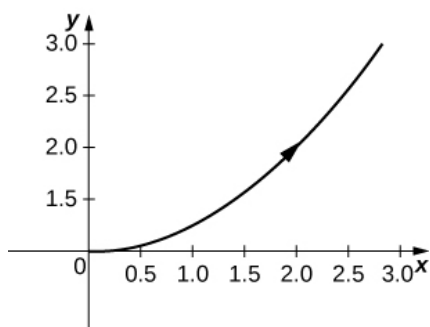
For the following exercises, eliminate the parameter and sketch the graphs.

Exercise:

Problem: $x = 2t^2, y = t^4 + 1$

Solution:

$$y = \frac{x^2}{4} + 1$$



For the following exercises, use technology (CAS or calculator) to sketch the parametric equations.

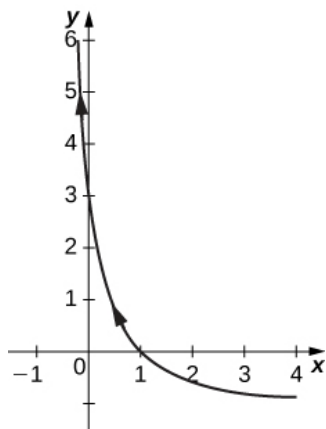
Exercise:

Problem: [T] $x = t^2 + t, \quad y = t^2 - 1$

Exercise:

Problem: [T] $x = e^{-t}, \quad y = e^{2t} - 1$

Solution:



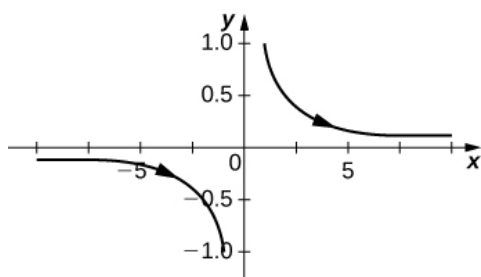
Exercise:

Problem: [T] $x = 3 \cos t, \quad y = 4 \sin t$

Exercise:

Problem: [T] $x = \sec t, \quad y = \cos t$

Solution:



For the following exercises, sketch the parametric equations by eliminating the parameter. Indicate any asymptotes of the graph.

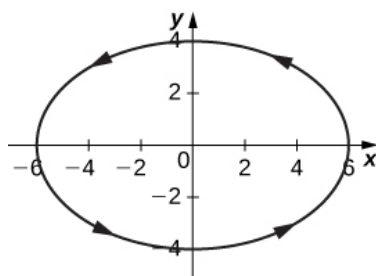
Exercise:

Problem: $x = e^t$, $y = e^{2t} + 1$

Exercise:

Problem: $x = 6 \sin(2\theta)$, $y = 4 \cos(2\theta)$

Solution:



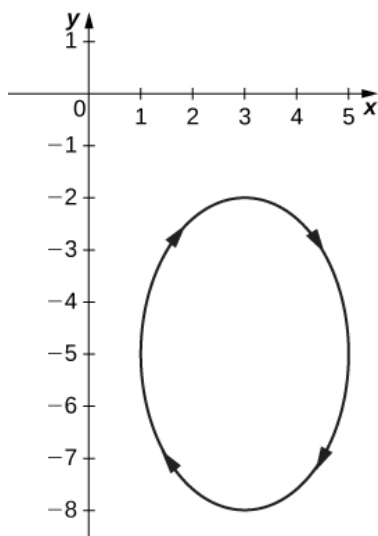
Exercise:

Problem: $x = \cos \theta$, $y = 2 \sin(2\theta)$

Exercise:

Problem: $x = 3 - 2 \cos \theta$, $y = -5 + 3 \sin \theta$

Solution:



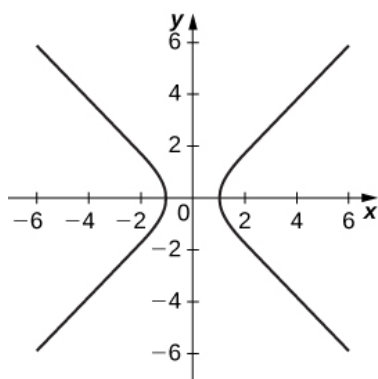
Exercise:

Problem: $x = 4 + 2 \cos \theta$, $y = -1 + \sin \theta$

Exercise:

Problem: $x = \sec t$, $y = \tan t$

Solution:



Asymptotes are $y = x$ and $y = -x$

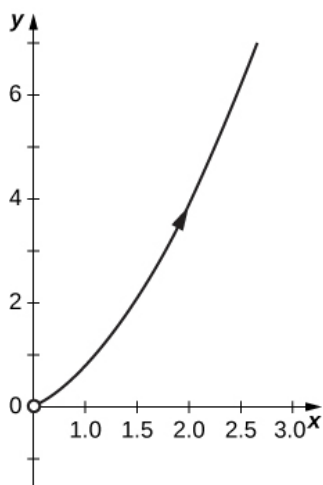
Exercise:

Problem: $x = \ln(2t)$, $y = t^2$

Exercise:

Problem: $x = e^t$, $y = e^{2t}$

Solution:



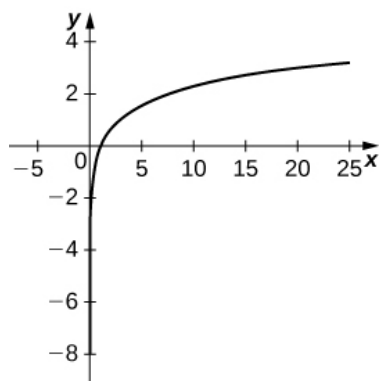
Exercise:

Problem: $x = e^{-2t}$, $y = e^{3t}$

Exercise:

Problem: $x = t^3$, $y = 3 \ln t$

Solution:



Exercise:

Problem: $x = 4 \sec \theta$, $y = 3 \tan \theta$

For the following exercises, convert the parametric equations of a curve into rectangular form. No sketch is necessary. State the domain of the rectangular form.

Exercise:

Problem: $x = t^2 - 1$, $y = \frac{t}{2}$

Solution:

$$x = 4y^2 - 1; \text{ domain: } x \in [1, \infty).$$

Exercise:

$$\textbf{Problem: } x = \frac{1}{\sqrt{t+1}}, \quad y = \frac{t}{1+t}, t > -1$$

Exercise:

$$\textbf{Problem: } x = 4 \cos \theta, y = 3 \sin \theta, t \in (0, 2\pi]$$

Solution:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1; \text{ domain } x \in [-4, 4].$$

Exercise:

$$\textbf{Problem: } x = \cosh t, \quad y = \sinh t$$

Exercise:

$$\textbf{Problem: } x = 2t - 3, \quad y = 6t - 7$$

Solution:

$$y = 3x + 2; \text{ domain: all real numbers.}$$

Exercise:

$$\textbf{Problem: } x = t^2, \quad y = t^3$$

Exercise:

$$\textbf{Problem: } x = 1 + \cos t, \quad y = 3 - \sin t$$

Solution:

$$(x - 1)^2 + (y - 3)^2 = 1; \text{ domain: } x \in [0, 2].$$

Exercise:

$$\textbf{Problem: } x = \sqrt{t}, \quad y = 2t + 4$$

Exercise:

$$\textbf{Problem: } x = \sec t, \quad y = \tan t, \pi \leq t < \frac{3\pi}{2}$$

Solution:

$$y = \sqrt{x^2 - 1}; \text{ domain: } x \in [-1, 1].$$

Exercise:

$$\textbf{Problem: } x = 2 \cosh t, \quad y = 4 \sinh t$$

Exercise:

Problem: $x = \cos(2t), \quad y = \sin t$

Solution:

$$y^2 = \frac{1-x}{2}; \text{ domain: } x \in [2, \infty) \cup (-\infty, -2].$$

Exercise:

Problem: $x = 4t + 3, y = 16t^2 - 9$

Exercise:

Problem: $x = t^2, \quad y = 2 \ln t, t \geq 1$

Solution:

$$y = \ln x; \text{ domain: } x \in (0, \infty).$$

Exercise:

Problem: $x = t^3, \quad y = 3 \ln t, t \geq 1$

Exercise:

Problem: $x = t^n, \quad y = n \ln t, t \geq 1$, where n is a natural number

Solution:

$$y = \ln x; \text{ domain: } x \in (0, \infty).$$

Exercise:

Problem: $\begin{matrix} x = \ln(5t) \\ y = \ln(t^2) \end{matrix}$ where $1 \leq t \leq e$

Exercise:

Problem: $\begin{matrix} x = 2 \sin(8t) \\ y = 2 \cos(8t) \end{matrix}$

Solution:

$$x^2 + y^2 = 4; \text{ domain: } x \in [-2, 2].$$

Exercise:

Problem: $\begin{matrix} x = \tan t \\ y = \sec^2 t - 1 \end{matrix}$

For the following exercises, the pairs of parametric equations represent lines, parabolas, circles, ellipses, or hyperbolas. Name the type of basic curve that each pair of equations represents.

Exercise:

Problem: $x = 3t + 4$
 $y = 5t - 2$

Solution:

line

Exercise:

Problem: $x - 4 = 5t$
 $y + 2 = t$

Exercise:

Problem: $x = 2t + 1$
 $y = t^2 - 3$

Solution:

parabola

Exercise:

Problem: $x = 3 \cos t$
 $y = 3 \sin t$

Exercise:

Problem: $x = 2 \cos(3t)$
 $y = 2 \sin(3t)$

Solution:

circle

Exercise:

Problem: $x = \cosh t$
 $y = \sinh t$

Exercise:

Problem: $x = 3 \cos t$
 $y = 4 \sin t$

Solution:

ellipse

Exercise:

Problem: $x = 2 \cos(3t)$
 $y = 5 \sin(3t)$

Exercise:

Problem: $x = 3 \cosh(4t)$
 $y = 4 \sinh(4t)$

Solution:

hyperbola

Exercise:

Problem: $x = 2 \cosh t$
 $y = 2 \sinh t$

Exercise:

Problem: Show that $x = h + r \cos \theta$
 $y = k + r \sin \theta$ represents the equation of a circle.

Exercise:

Problem:

Use the equations in the preceding problem to find a set of parametric equations for a circle whose radius is 5 and whose center is $(-2, 3)$.

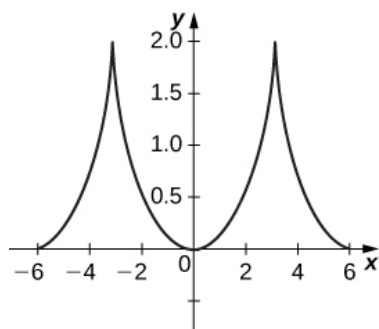
For the following exercises, use a graphing utility to graph the curve represented by the parametric equations and identify the curve from its equation.

Exercise:

Problem: [T] $x = \theta + \sin \theta$
 $y = 1 - \cos \theta$

Solution:

The equations represent a cycloid.

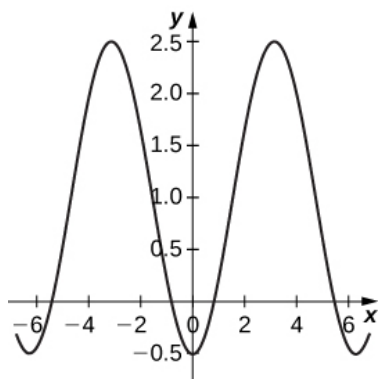


Exercise:

Problem: [T] $x = 2t - 2 \sin t$
 $y = 2 - 2 \cos t$

Exercise:

Problem: [T]
$$\begin{aligned} x &= t - 0.5 \sin t \\ y &= 1 - 1.5 \cos t \end{aligned}$$

Solution:**Exercise:****Problem:**

An airplane traveling horizontally at 100 m/s over flat ground at an elevation of 4000 meters must drop an emergency package on a target on the ground. The trajectory of the package is given by $x = 100t$, $y = -4.9t^2 + 4000$, $t \geq 0$ where the origin is the point on the ground directly beneath the plane at the moment of release. How many horizontal meters before the target should the package be released in order to hit the target?

Exercise:**Problem:**

The trajectory of a bullet is given by $x = v_0 (\cos \alpha) t$, $y = v_0 (\sin \alpha) t - \frac{1}{2}gt^2$ where $v_0 = 500$ m/s, $g = 9.8 = 9.8 \text{ m/s}^2$, and $\alpha = 30$ degrees. When will the bullet hit the ground? How far from the gun will the bullet hit the ground?

Solution:

22,092 meters at approximately 51 seconds.

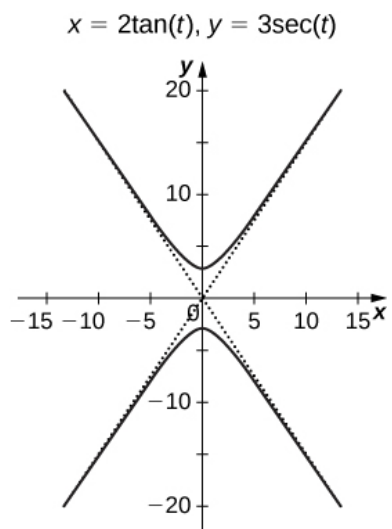
Exercise:

Problem: [T] Use technology to sketch the curve represented by $x = \sin(4t)$, $y = \sin(3t)$, $0 \leq t \leq 2\pi$.

Exercise:

Problem: [T] Use technology to sketch $x = 2 \tan(t)$, $y = 3 \sec(t)$, $-\pi < t < \pi$.

Solution:



Exercise:

Problem:

Sketch the curve known as an *epitrochoid*, which gives the path of a point on a circle of radius b as it rolls on the outside of a circle of radius a . The equations are

$$x = (a + b)\cos t - c \cdot \cos \left[\frac{(a+b)t}{b} \right]$$

$$y = (a + b)\sin t - c \cdot \sin \left[\frac{(a+b)t}{b} \right].$$

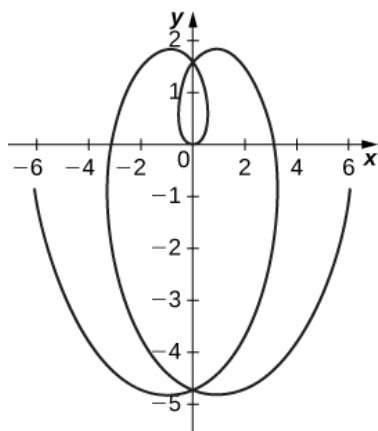
Let $a = 1, b = 2, c = 1$.

Exercise:

Problem:

[T] Use technology to sketch the spiral curve given by $x = t \cos(t), y = t \sin(t)$ from $-2\pi \leq t \leq 2\pi$.

Solution:

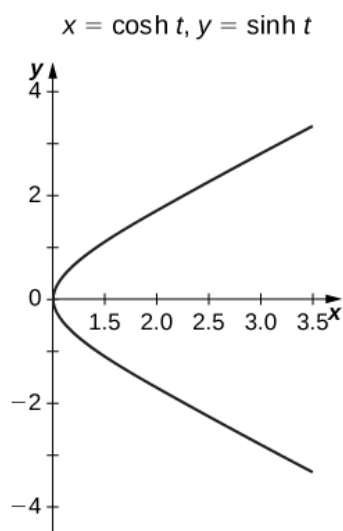


Exercise:**Problem:**

[T] Use technology to graph the curve given by the parametric equations $x = 2 \cot(t)$, $y = 1 - \cos(2t)$, $-\pi/2 \leq t \leq \pi/2$. This curve is known as the witch of Agnesi.

Exercise:

Problem: [T] Sketch the curve given by parametric equations $x = \cosh(t)$, $y = \sinh(t)$, where $-2 \leq t \leq 2$.

Solution:**Glossary****cycloid**

the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage

cusp

a pointed end or part where two curves meet

orientation

the direction that a point moves on a graph as the parameter increases

parameter

an independent variable that both x and y depend on in a parametric curve; usually represented by the variable t

parametric curve

the graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \leq t \leq b$ combined with the equations

parametric equations

the equations $x = x(t)$ and $y = y(t)$ that define a parametric curve

parameterization of a curve

rewriting the equation of a curve defined by a function $y = f(x)$ as parametric equations

Calculus of Parametric Curves

- Determine derivatives and equations of tangents for parametric curves.
- Find the area under a parametric curve.
- Use the equation for arc length of a parametric curve.
- Apply the formula for surface area to a volume generated by a parametric curve.

Now that we have introduced the concept of a parameterized curve, our next step is to learn how to work with this concept in the context of calculus. For example, if we know a parameterization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?

Another scenario: Suppose we would like to represent the location of a baseball after the ball leaves a pitcher's hand. If the position of the baseball is represented by the plane curve $(x(t), y(t))$, then we should be able to use calculus to find the speed of the ball at any given time. Furthermore, we should be able to calculate just how far that ball has traveled as a function of time.

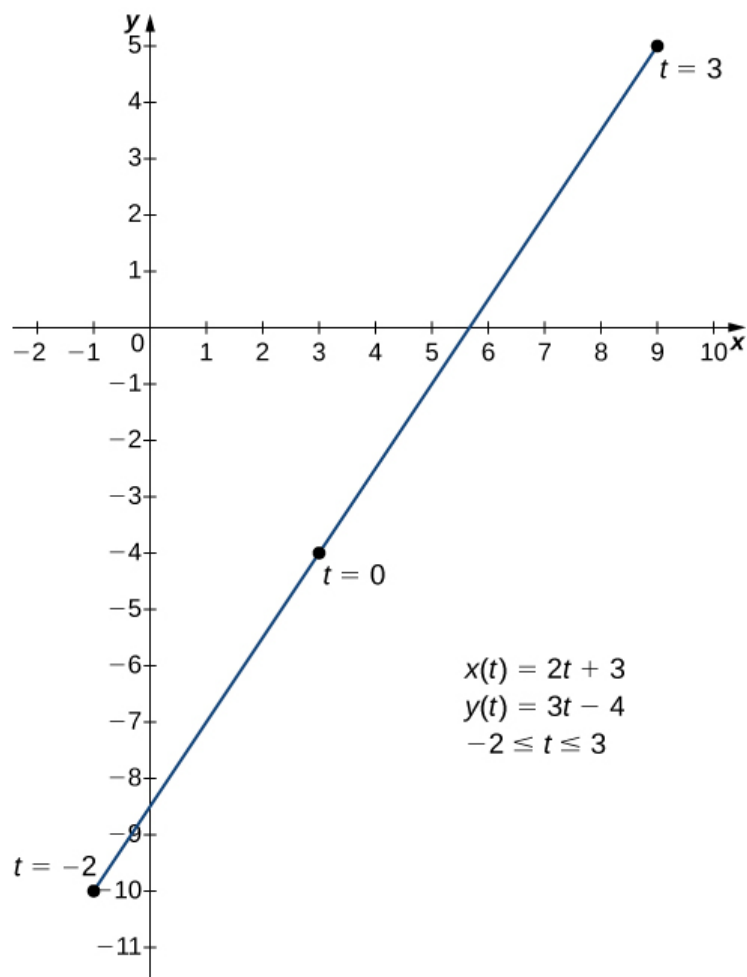
Derivatives of Parametric Equations

We start by asking how to calculate the slope of a line tangent to a parametric curve at a point. Consider the plane curve defined by the parametric equations

Equation:

$$x(t) = 2t + 3, \quad y(t) = 3t - 4, \quad -2 \leq t \leq 3.$$

The graph of this curve appears in [\[link\]](#). It is a line segment starting at $(-1, -10)$ and ending at $(9, 5)$.



Graph of the line segment described by the given parametric equations.

We can eliminate the parameter by first solving the equation $x(t) = 2t + 3$ for t :

Equation:

$$\begin{aligned}
 x(t) &= 2t + 3 \\
 x - 3 &= 2t \\
 t &= \frac{x-3}{2}.
 \end{aligned}$$

Substituting this into $y(t)$, we obtain

Equation:

$$\begin{aligned}
 y(t) &= 3t - 4 \\
 y &= 3\left(\frac{x-3}{2}\right) - 4 \\
 y &= \frac{3x}{2} - \frac{9}{2} - 4 \\
 y &= \frac{3x}{2} - \frac{17}{2}.
 \end{aligned}$$

The slope of this line is given by $\frac{dy}{dx} = \frac{3}{2}$. Next we calculate $x'(t)$ and $y'(t)$. This gives $x'(t) = 2$ and $y'(t) = 3$. Notice that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}$. This is no coincidence, as outlined in the following theorem.

Note:

Derivative of Parametric Equations

Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

Equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}.$$

Proof

This theorem can be proven using the Chain Rule. In particular, assume that the parameter t can be eliminated, yielding a differentiable function $y = F(x)$. Then $y(t) = F(x(t))$. Differentiating both sides of this equation using the Chain Rule yields

Equation:

$$y'(t) = F'(x(t))x'(t),$$

so

Equation:

$$F'(x(t)) = \frac{y'(t)}{x'(t)}.$$

But $F'(x(t)) = \frac{dy}{dx}$, which proves the theorem.

□

[\[link\]](#) can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y = f(x)$ is any point $x = x_0$ such that either $f'(x_0) = 0$

or $f'(x_0)$ does not exist. [\[link\]](#) gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y = f(x)$ or not.

Example:

Exercise:

Problem:

Finding the Derivative of a Parametric Curve

Calculate the derivative $\frac{dy}{dx}$ for each of the following parametrically defined plane curves, and locate any critical points on their respective graphs.

- a. $x(t) = t^2 - 3$, $y(t) = 2t - 1$, $-3 \leq t \leq 4$
- b. $x(t) = 2t + 1$, $y(t) = t^3 - 3t + 4$, $-2 \leq t \leq 5$
- c. $x(t) = 5 \cos t$, $y(t) = 5 \sin t$, $0 \leq t \leq 2\pi$

Solution:

- a. To apply [\[link\]](#), first calculate $x'(t)$ and $y'(t)$:

Equation:

$$x'(t) = 2t$$

$$y'(t) = 2.$$

Next substitute these into the equation:

Equation:

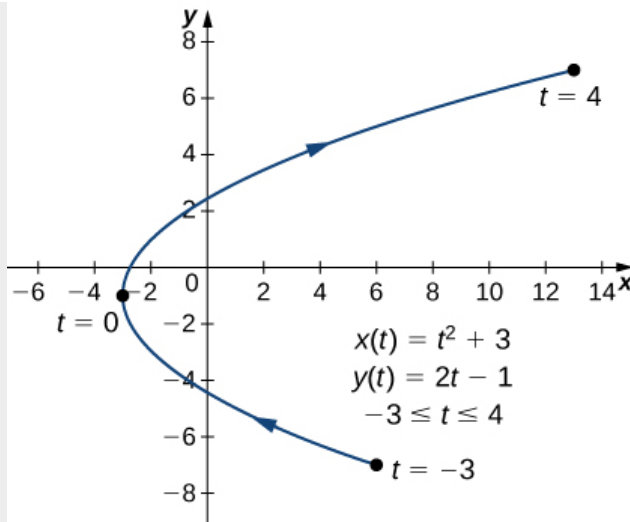
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{2}{2t}$$

$$\frac{dy}{dx} = \frac{1}{t}.$$

This derivative is undefined when $t = 0$. Calculating $x(0)$ and $y(0)$ gives

$x(0) = (0)^2 - 3 = -3$ and $y(0) = 2(0) - 1 = -1$, which corresponds to the point $(-3, -1)$ on the graph. The graph of this curve is a parabola opening to the right, and the point $(-3, -1)$ is its vertex as shown.



Graph of the parabola described by parametric equations in part a.

b. To apply [link](#), first calculate $x'(t)$ and $y'(t)$:

Equation:

$$x'(t) = 2$$

$$y'(t) = 3t^2 - 3.$$

Next substitute these into the equation:

Equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2}.$$

This derivative is zero when $t = \pm 1$. When $t = -1$ we have

Equation:

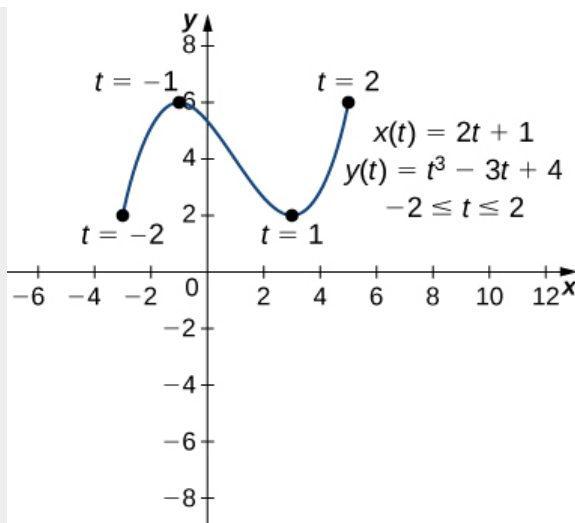
$$x(-1) = 2(-1) + 1 = -1 \text{ and } y(-1) = (-1)^3 - 3(-1) + 4 = -1 + 3 + 4 = 6,$$

which corresponds to the point $(-1, 6)$ on the graph. When $t = 1$ we have

Equation:

$$x(1) = 2(1) + 1 = 3 \text{ and } y(1) = (1)^3 - 3(1) + 4 = 1 - 3 + 4 = 2,$$

which corresponds to the point $(3, 2)$ on the graph. The point $(3, 2)$ is a relative minimum and the point $(-1, 6)$ is a relative maximum, as seen in the following graph.



Graph of the curve described by parametric equations in part b.

c. To apply [\[link\]](#), first calculate $x'(t)$ and $y'(t)$:

Equation:

$$x'(t) = -5 \sin t$$

$$y'(t) = 5 \cos t.$$

Next substitute these into the equation:

Equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{5 \cos t}{-5 \sin t}$$

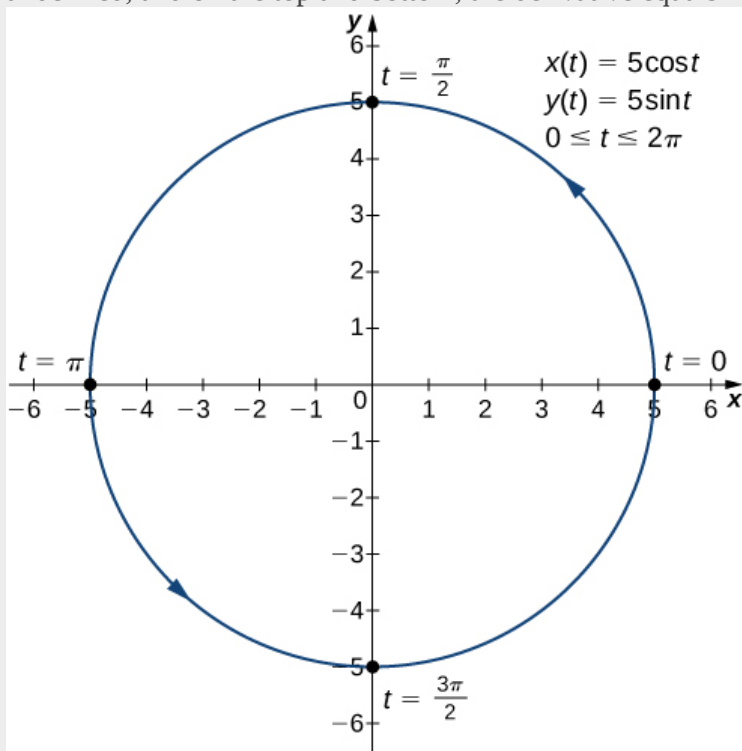
$$\frac{dy}{dx} = -\cot t.$$

This derivative is zero when $\cos t = 0$ and is undefined when $\sin t = 0$. This gives $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π as critical points for t . Substituting each of these into $x(t)$ and $y(t)$, we obtain

t	$x(t)$	$y(t)$
0	5	0

t	$x(t)$	$y(t)$
$\frac{\pi}{2}$	0	5
π	-5	0
$\frac{3\pi}{2}$	0	-5
2π	5	0

These points correspond to the sides, top, and bottom of the circle that is represented by the parametric equations ([link](#)). On the left and right edges of the circle, the derivative is undefined, and on the top and bottom, the derivative equals zero.



Graph of the curve described by parametric equations in part c.

Note:

Exercise:

Problem: Calculate the derivative dy/dx for the plane curve defined by the equations
Equation:

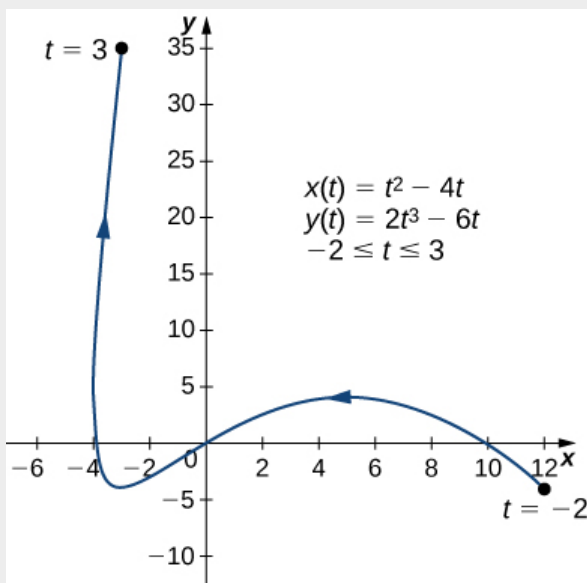
$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \leq t \leq 3$$

and locate any critical points on its graph.

Solution:

$$x'(t) = 2t - 4 \text{ and } y'(t) = 6t^2 - 6, \text{ so } \frac{dy}{dx} = \frac{6t^2 - 6}{2t - 4} = \frac{3t^2 - 3}{t - 2}.$$

This expression is undefined when $t = 2$ and equal to zero when $t = \pm 1$.



Hint

Calculate $x'(t)$ and $y'(t)$ and use [\[link\]](#).

Example:

Exercise:

Problem:

Finding a Tangent Line

Find the equation of the tangent line to the curve defined by the equations

Equation:

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad -3 \leq t \leq 4 \text{ when } t = 2.$$

Solution:

First find the slope of the tangent line using [\[link\]](#), which means calculating $x'(t)$ and $y'(t)$:

Equation:

$$\begin{aligned}x'(t) &= 2t \\ y'(t) &= 2.\end{aligned}$$

Next substitute these into the equation:

Equation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ \frac{dy}{dx} &= \frac{2}{2t} \\ \frac{dy}{dx} &= \frac{1}{t}.\end{aligned}$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives

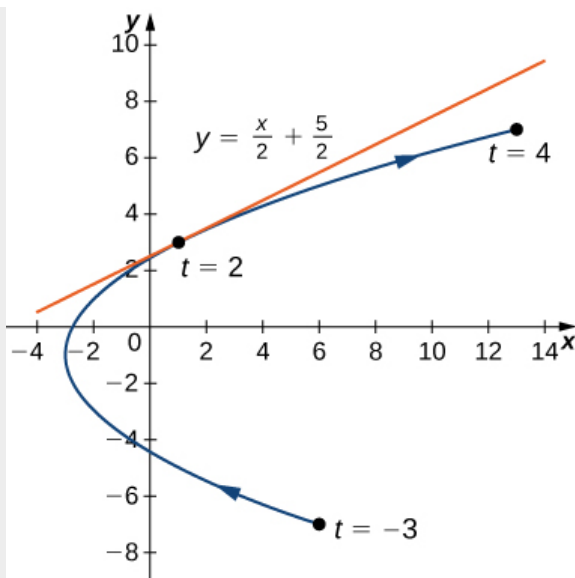
Equation:

$$x(2) = (2)^2 - 3 = 1 \text{ and } y(2) = 2(2) - 1 = 3,$$

which corresponds to the point $(1, 3)$ on the graph ([\[link\]](#)). Now use the point-slope form of the equation of a line to find the equation of the tangent line:

Equation:

$$\begin{aligned}y - y_0 &= m(x - x_0) \\ y - 3 &= \frac{1}{2}(x - 1) \\ y - 3 &= \frac{1}{2}x - \frac{1}{2} \\ y &= \frac{1}{2}x + \frac{5}{2}.\end{aligned}$$



Tangent line to the parabola described by the given parametric equations when $t = 2$.

Note:

Exercise:

Problem: Find the equation of the tangent line to the curve defined by the equations

Equation:

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \leq t \leq 3 \text{ when } t = 5.$$

Solution:

The equation of the tangent line is $y = 24x + 100$.

Hint

Calculate $x'(t)$ and $y'(t)$ and use [\[link\]](#).

Second-Order Derivatives

Our next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f(x)$ is defined to be the derivative of the first derivative; that is,

Equation:

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{dy}{dx} \right].$$

Since $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, we can replace the y on both sides of this equation with $\frac{dy}{dx}$. This gives us

Equation:

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{(d/dt)(dy/dx)}{dx/dt}.$$

If we know dy/dx as a function of t , then this formula is straightforward to apply.

Example:

Exercise:

Problem:

Finding a Second Derivative

Calculate the second derivative d^2y/dx^2 for the plane curve defined by the parametric equations $x(t) = t^2 - 3$, $y(t) = 2t - 1$, $-3 \leq t \leq 4$.

Solution:

From [\[link\]](#) we know that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$. Using [\[link\]](#), we obtain

Equation:

$$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt} = \frac{(d/dt)(1/t)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3}.$$

Note:

Exercise:

Problem: Calculate the second derivative d^2y/dx^2 for the plane curve defined by the equations

Equation:

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad -2 \leq t \leq 3$$

and locate any critical points on its graph.

Solution:

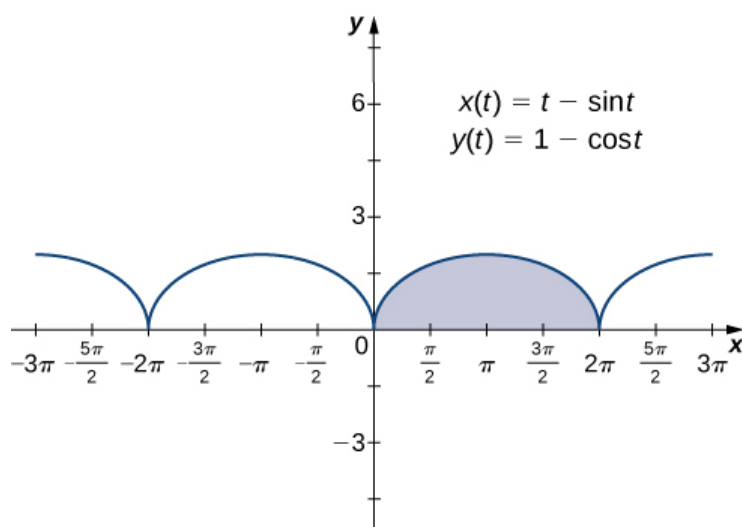
$$\frac{d^2y}{dx^2} = \frac{3t^2 - 12t + 3}{2(t-2)^3}. \text{ Critical points } (5, 4), (-3, -4), \text{ and } (-4, 6).$$

Hint

Start with the solution from the previous checkpoint, and use [\[link\]](#).

Integrals Involving Parametric Equations

Now that we have seen how to calculate the derivative of a plane curve, the next question is this: How do we find the area under a curve defined parametrically? Recall the cycloid defined by the equations $x(t) = t - \sin t$, $y(t) = 1 - \cos t$. Suppose we want to find the area of the shaded region in the following graph.



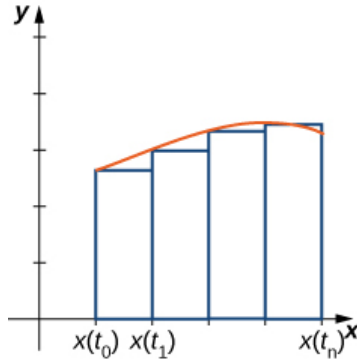
Graph of a cycloid with the arch over $[0, 2\pi]$ highlighted.

To derive a formula for the area under the curve defined by the functions

Equation:

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b,$$

we assume that $x(t)$ is differentiable and start with an equal partition of the interval $a \leq t \leq b$. Suppose $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ and consider the following graph.



Approximating the area
under a parametrically
defined curve.

We use rectangles to approximate the area under the curve. The height of a typical rectangle in this parametrization is $y\left(x\left(\bar{t}_i\right)\right)$ for some value \bar{t}_i in the i th subinterval, and the width can be calculated as $x\left(t_i\right)-x\left(t_{i-1}\right)$. Thus the area of the i th rectangle is given by

Equation:

$$A_i = y\left(x\left(\bar{t}_i\right)\right)\left(x\left(t_i\right)-x\left(t_{i-1}\right)\right).$$

Then a Riemann sum for the area is

Equation:

$$A_n = \sum_{i=1}^n y\left(x\left(\bar{t}_i\right)\right)\left(x\left(t_i\right)-x\left(t_{i-1}\right)\right).$$

Multiplying and dividing each area by $t_i - t_{i-1}$ gives

Equation:

$$A_n = \sum_{i=1}^n y\left(x\left(\bar{t}_i\right)\right)\left(\frac{x\left(t_i\right)-x\left(t_{i-1}\right)}{t_i-t_{i-1}}\right)\left(t_i-t_{i-1}\right) = \sum_{i=1}^n y\left(x\left(\bar{t}_i\right)\right)\left(\frac{x\left(t_i\right)-x\left(t_{i-1}\right)}{\Delta t}\right)\Delta t.$$

Taking the limit as n approaches infinity gives

Equation:

$$A = \lim_{n \rightarrow \infty} A_n = \int_a^b y(t)x'(t) dt.$$

This leads to the following theorem.

Note:**Area under a Parametric Curve**

Consider the non-self-intersecting plane curve defined by the parametric equations

Equation:

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b$$

and assume that $x(t)$ is differentiable. The area under this curve is given by

Equation:

$$A = \int_a^b y(t)x'(t) dt.$$

Example:**Exercise:****Problem:****Finding the Area under a Parametric Curve**

Find the area under the curve of the cycloid defined by the equations

Equation:

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad 0 \leq t \leq 2\pi.$$

Solution:

Using [\[link\]](#), we have

Equation:

$$\begin{aligned} A &= \int_a^b y(t)x'(t) dt \\ &= \int_0^{2\pi} (1 - \cos t)(1 - \cos t) dt \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t) dt \\ &= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2} \right) dt \\ &= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{\cos 2t}{2} \right) dt \\ &= \left. \frac{3t}{2} - 2\sin t + \frac{\sin 2t}{4} \right|_0^{2\pi} \\ &= 3\pi. \end{aligned}$$

Note:

Exercise:

Problem: Find the area under the curve of the hypocycloid defined by the equations

Equation:

$$x(t) = 3 \cos t + \cos 3t, \quad y(t) = 3 \sin t - \sin 3t, \quad 0 \leq t \leq \pi.$$

Solution:

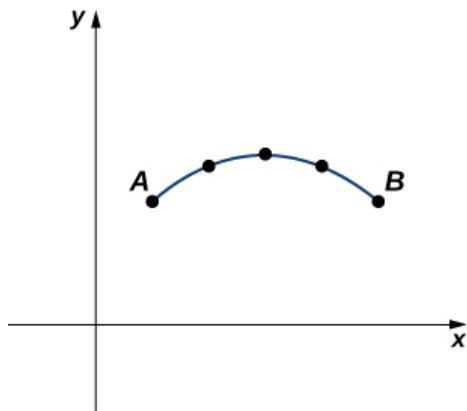
$A = 3\pi$ (Note that the integral formula actually yields a negative answer. This is due to the fact that $x(t)$ is a decreasing function over the interval $[0, 2\pi]$; that is, the curve is traced from right to left.)

Hint

Use [\[link\]](#), along with the identities $\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$ and $\sin^2 t = \frac{1 - \cos 2t}{2}$.

Arc Length of a Parametric Curve

In addition to finding the area under a parametric curve, we sometimes need to find the arc length of a parametric curve. In the case of a line segment, arc length is the same as the distance between the endpoints. If a particle travels from point A to point B along a curve, then the distance that particle travels is the arc length. To develop a formula for arc length, we start with an approximation by line segments as shown in the following graph.



Approximation of a curve by
line segments.

Given a plane curve defined by the functions $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, we start by partitioning the interval $[a, b]$ into n equal subintervals: $t_0 = a < t_1 < t_2 < \dots < t_n = b$. The width of each subinterval is given by $\Delta t = (b - a)/n$. We can calculate the length of each line segment:

Equation:

$$d_1 = \sqrt{(x(t_1) - x(t_0))^2 + (y(t_1) - y(t_0))^2}$$

$$d_2 = \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2} \text{ etc.}$$

Then add these up. We let s denote the exact arc length and s_n denote the approximation by n line segments:

Equation:

$$s \approx \sum_{k=1}^n s_k = \sum_{k=1}^n \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2}.$$

If we assume that $x(t)$ and $y(t)$ are differentiable functions of t , then the Mean Value Theorem ([Introduction to the Applications of Derivatives](#)) applies, so in each subinterval $[t_{k-1}, t_k]$ there exist \hat{t}_k and \tilde{t}_k such that

Equation:

$$x(t_k) - x(t_{k-1}) = x'(\hat{t}_k)(t_k - t_{k-1}) = x'(\hat{t}_k)\Delta t$$

$$y(t_k) - y(t_{k-1}) = y'(\tilde{t}_k)(t_k - t_{k-1}) = y'(\tilde{t}_k)\Delta t.$$

Therefore [\[link\]](#) becomes

Equation:

$$\begin{aligned} s &\approx \sum_{k=1}^n s_k \\ &= \sum_{k=1}^n \sqrt{(x'(\hat{t}_k)\Delta t)^2 + (y'(\tilde{t}_k)\Delta t)^2} \\ &= \sum_{k=1}^n \sqrt{(x'(\hat{t}_k))^2(\Delta t)^2 + (y'(\tilde{t}_k))^2(\Delta t)^2} \\ &= \left(\sum_{k=1}^n \sqrt{(x'(\hat{t}_k))^2 + (y'(\tilde{t}_k))^2} \right) \Delta t. \end{aligned}$$

This is a Riemann sum that approximates the arc length over a partition of the interval $[a, b]$. If we further assume that the derivatives are continuous and let the number of points in the partition increase without bound, the approximation approaches the exact arc length. This gives

Equation:

$$\begin{aligned}s &= \lim_{n \rightarrow \infty} \sum_{k=1}^n s_k \\&= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \sqrt{\left(x'(\hat{t}_k)\right)^2 + \left(y'(\tilde{t}_k)\right)^2} \right) \Delta t \\&= \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} dt.\end{aligned}$$

When taking the limit, the values of \hat{t}_k and \tilde{t}_k are both contained within the same ever-shrinking interval of width Δt , so they must converge to the same value.

We can summarize this method in the following theorem.

Note:

Arc Length of a Parametric Curve

Consider the plane curve defined by the parametric equations

Equation:

$$x = x(t), \quad y = y(t), \quad t_1 \leq t \leq t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of t . Then the arc length of this curve is given by

Equation:

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

At this point a side derivation leads to a previous formula for arc length. In particular, suppose the parameter can be eliminated, leading to a function $y = F(x)$. Then $y(t) = F(x(t))$ and the Chain Rule gives $y'(t) = F'(x(t))x'(t)$. Substituting this into [\[link\]](#) gives

Equation:

$$\begin{aligned}
s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(F'(x)\frac{dx}{dt}\right)^2} dt \\
&= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 (1 + (F'(x))^2)} dt \\
&= \int_{t_1}^{t_2} x'(t) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dt.
\end{aligned}$$

Here we have assumed that $x'(t) > 0$, which is a reasonable assumption. The Chain Rule gives $dx = x'(t) dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula

Equation:

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

which is the formula for arc length obtained in the [Introduction to the Applications of Integration](#).

Example:

Exercise:

Problem:

Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations

Equation:

$$x(t) = 3 \cos t, \quad y(t) = 3 \sin t, \quad 0 \leq t \leq \pi.$$

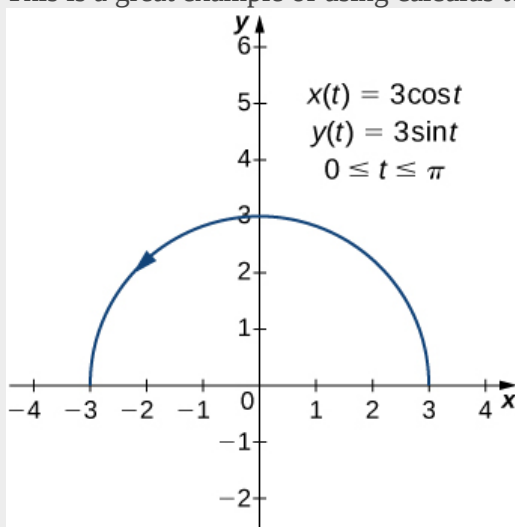
Solution:

The values $t = 0$ to $t = \pi$ trace out the red curve in [\[link\]](#). To determine its length, use [\[link\]](#):

Equation:

$$\begin{aligned}
 s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= \int_0^\pi \sqrt{(-3 \sin t)^2 + (3 \cos t)^2} dt \\
 &= \int_0^\pi \sqrt{9 \sin^2 t + 9 \cos^2 t} dt \\
 &= \int_0^\pi \sqrt{9 (\sin^2 t + \cos^2 t)} dt \\
 &= \int_0^\pi 3 dt = 3t \Big|_0^\pi = 3\pi.
 \end{aligned}$$

Note that the formula for the arc length of a semicircle is πr and the radius of this circle is 3. This is a great example of using calculus to derive a known formula of a geometric quantity.



The arc length of the semicircle is equal to its radius times π .

Note:

Exercise:

Problem: Find the arc length of the curve defined by the equations

Equation:

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad 1 \leq t \leq 3.$$

Solution:

$$s = 2 \left(10^{3/2} - 2^{3/2} \right) \approx 57.589$$

Hint

Use [\[link\]](#).

We now return to the problem posed at the beginning of the section about a baseball leaving a pitcher's hand. Ignoring the effect of air resistance (unless it is a curve ball!), the ball travels a parabolic path. Assuming the pitcher's hand is at the origin and the ball travels left to right in the direction of the positive x -axis, the parametric equations for this curve can be written as

Equation:

$$x(t) = 140t, \quad y(t) = -16t^2 + 2t$$

where t represents time. We first calculate the distance the ball travels as a function of time. This distance is represented by the arc length. We can modify the arc length formula slightly. First rewrite the functions $x(t)$ and $y(t)$ using v as an independent variable, so as to eliminate any confusion with the parameter t :

Equation:

$$x(v) = 140v, \quad y(v) = -16v^2 + 2v.$$

Then we write the arc length formula as follows:

Equation:

$$\begin{aligned} s(t) &= \int_0^t \sqrt{\left(\frac{dx}{dv}\right)^2 + \left(\frac{dy}{dv}\right)^2} dv \\ &= \int_0^t \sqrt{140^2 + (-32v + 2)^2} dv. \end{aligned}$$

The variable v acts as a dummy variable that disappears after integration, leaving the arc length as a function of time t . To integrate this expression we can use a formula from [Appendix A](#).

Equation:

$$\int \sqrt{a^2 + u^2} du = \frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln \left| u + \sqrt{a^2 + u^2} \right| + C.$$

We set $a = 140$ and $u = -32v + 2$. This gives $du = -32dv$, so $dv = -\frac{1}{32}du$. Therefore

Equation:

$$\begin{aligned}
\int \sqrt{140^2 + (-32v + 2)^2} dv &= -\frac{1}{32} \int \sqrt{a^2 + u^2} du \\
&= -\frac{1}{32} \left[\frac{(-32v+2)}{2} \sqrt{140^2 + (-32v + 2)^2} \right. \\
&\quad \left. + \frac{140^2}{2} \ln \left| (-32v + 2) + \sqrt{140^2 + (-32v + 2)^2} \right| \right] + C
\end{aligned}$$

and

Equation:

$$\begin{aligned}
s(t) &= -\frac{1}{32} \left[\frac{(-32t+2)}{2} \sqrt{140^2 + (-32t + 2)^2} + \frac{140^2}{2} \ln \left| (-32t + 2) + \sqrt{140^2 + (-32t + 2)^2} \right| \right] \\
&\quad + \frac{1}{32} \left[\sqrt{140^2 + 2^2} + \frac{140^2}{2} \ln \left| 2 + \sqrt{140^2 + 2^2} \right| \right] \\
&= \left(\frac{t}{2} - \frac{1}{32} \right) \sqrt{1024t^2 - 128t + 19604} - \frac{1225}{4} \ln \left| (-32t + 2) + \sqrt{1024t^2 - 128t + 19604} \right| \\
&\quad + \frac{\sqrt{19604}}{32} + \frac{1225}{4} \ln \left(2 + \sqrt{19604} \right).
\end{aligned}$$

This function represents the distance traveled by the ball as a function of time. To calculate the speed, take the derivative of this function with respect to t . While this may seem like a daunting task, it is possible to obtain the answer directly from the Fundamental Theorem of Calculus:

Equation:

$$\frac{d}{dx} \int_a^x f(u) du = f(x).$$

Therefore

Equation:

$$\begin{aligned}
s'(t) &= \frac{d}{dt} [s(t)] \\
&= \frac{d}{dt} \left[\int_0^t \sqrt{140^2 + (-32v + 2)^2} dv \right] \\
&= \sqrt{140^2 + (-32t + 2)^2} \\
&= \sqrt{1024t^2 - 128t + 19604} \\
&= 2\sqrt{256t^2 - 32t + 4901}.
\end{aligned}$$

One third of a second after the ball leaves the pitcher's hand, the distance it travels is equal to

Equation:

$$\begin{aligned}
s\left(\frac{1}{3}\right) &= \left(\frac{1/3}{2} - \frac{1}{32}\right) \sqrt{1024\left(\frac{1}{3}\right)^2 - 128\left(\frac{1}{3}\right) + 19604} \\
&\quad - \frac{1225}{4} \ln \left| \left(-32\left(\frac{1}{3}\right) + 2\right) + \sqrt{1024\left(\frac{1}{3}\right)^2 - 128\left(\frac{1}{3}\right) + 19604} \right| \\
&\quad + \frac{\sqrt{19604}}{32} + \frac{1225}{4} \ln \left(2 + \sqrt{19604}\right) \\
&\approx 46.69 \text{ feet.}
\end{aligned}$$

This value is just over three quarters of the way to home plate. The speed of the ball is

Equation:

$$s'\left(\frac{1}{3}\right) = 2\sqrt{256\left(\frac{1}{3}\right)^2 - 16\left(\frac{1}{3}\right) + 4901} \approx 140.34 \text{ ft/s.}$$

This speed translates to approximately 95 mph—a major-league fastball.

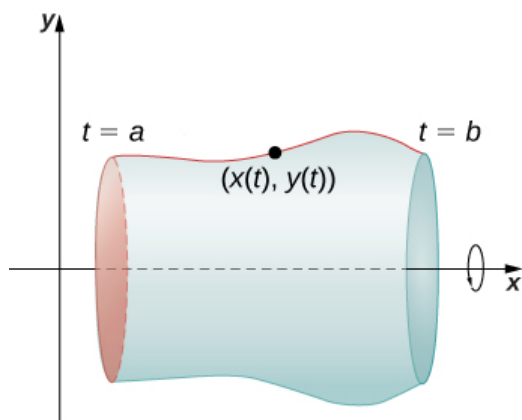
Surface Area Generated by a Parametric Curve

Recall the problem of finding the surface area of a volume of revolution. In [Curve Length and Surface Area](#), we derived a formula for finding the surface area of a volume generated by a function $y = f(x)$ from $x = a$ to $x = b$, revolved around the x -axis:

Equation:

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx.$$

We now consider a volume of revolution generated by revolving a parametrically defined curve $x = x(t), y = y(t), a \leq t \leq b$ around the x -axis as shown in the following figure.



A surface of revolution generated by a parametrically defined curve.

The analogous formula for a parametrically defined curve is

Equation:

$$S = 2\pi \int_a^b y(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

provided that $y(t)$ is not negative on $[a, b]$.

Example:

Exercise:

Problem:

Finding Surface Area

Find the surface area of a sphere of radius r centered at the origin.

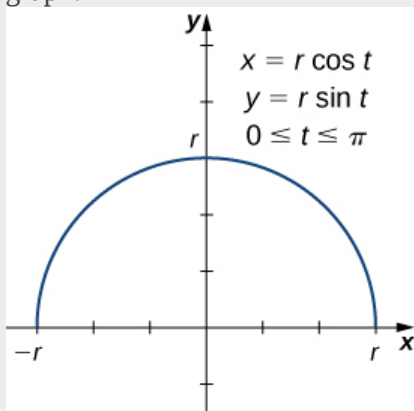
Solution:

We start with the curve defined by the equations

Equation:

$$x(t) = r \cos t, \quad y(t) = r \sin t, \quad 0 \leq t \leq \pi.$$

This generates an upper semicircle of radius r centered at the origin as shown in the following graph.



A semicircle generated by
parametric equations.

When this curve is revolved around the x-axis, it generates a sphere of radius r . To calculate the surface area of the sphere, we use [\[link\]](#):

Equation:

$$\begin{aligned}
 S &= 2\pi \int_a^b y(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt \\
 &= 2\pi \int_0^\pi r \sin t \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt \\
 &= 2\pi \int_0^\pi r \sin t \sqrt{r^2 \sin^2 t + r^2 \cos^2 t} dt \\
 &= 2\pi \int_0^\pi r \sin t \sqrt{r^2 (\sin^2 t + \cos^2 t)} dt \\
 &= 2\pi \int_0^\pi r^2 \sin t dt \\
 &= 2\pi r^2 (-\cos t|_0^\pi) \\
 &= 2\pi r^2 (-\cos \pi + \cos 0) \\
 &= 4\pi r^2.
 \end{aligned}$$

This is, in fact, the formula for the surface area of a sphere.

Note:

Exercise:

Problem: Find the surface area generated when the plane curve defined by the equations

Equation:

$$x(t) = t^3, \quad y(t) = t^2, \quad 0 \leq t \leq 1$$

is revolved around the x-axis.

Solution:

$$A = \frac{\pi(494\sqrt{13}+128)}{1215}$$

Hint

Use [\[link\]](#). When evaluating the integral, use a u -substitution.

Key Concepts

- The derivative of the parametrically defined curve $x = x(t)$ and $y = y(t)$ can be calculated using the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$. Using the derivative, we can find the equation of a tangent line to a parametric curve.

- The area between a parametric curve and the x-axis can be determined by using the formula

$$A = \int_{t_1}^{t_2} y(t)x'(t) dt.$$

- The arc length of a parametric curve can be calculated by using the formula

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

- The surface area of a volume of revolution revolved around the x-axis is given by

$$S = 2\pi \int_a^b y(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt. \text{ If the curve is revolved around the y-axis, then the formula is } S = 2\pi \int_a^b x(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt.$$

Key Equations

- **Derivative of parametric equations**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

- **Second-order derivative of parametric equations**

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{(d/dt)(dy/dx)}{dx/dt}$$

- **Area under a parametric curve**

$$A = \int_a^b y(t)x'(t) dt$$

- **Arc length of a parametric curve**

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

- **Surface area generated by a parametric curve**

$$S = 2\pi \int_a^b y(t) \sqrt{(x'(t))^2 + (y'(t))^2} dt$$

For the following exercises, each set of parametric equations represents a line. Without eliminating the parameter, find the slope of each line.

Exercise:

Problem: $x = 3 + t, \quad y = 1 - t$

Exercise:

Problem: $x = 8 + 2t, \quad y = 1$

Solution:

Exercise:

Problem: $x = 4 - 3t, \quad y = -2 + 6t$

Exercise:

Problem: $x = -5t + 7, \quad y = 3t - 1$

Solution:

$$\frac{-3}{5}$$

For the following exercises, determine the slope of the tangent line, then find the equation of the tangent line at the given value of the parameter.

Exercise:

Problem: $x = 3 \sin t, \quad y = 3 \cos t, \quad t = \frac{\pi}{4}$

Exercise:

Problem: $x = \cos t, \quad y = 8 \sin t, t = \frac{\pi}{2}$

Solution:

Slope = 0; $y = 8$.

Exercise:

Problem: $x = 2t, \quad y = t^3, \quad t = -1$

Exercise:

Problem: $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t = 1$

Solution:

Slope is undefined; $x = 2$.

Exercise:

Problem: $x = \sqrt{t}, \quad y = 2t, \quad t = 4$

For the following exercises, find all points on the curve that have the given slope.

Exercise:

Problem: $x = 4 \cos t, \quad y = 4 \sin t, \text{ slope} = 0.5$

Solution:

$$t = \arctan(-2); \left(\frac{4}{\sqrt{5}}, \frac{-8}{\sqrt{5}} \right).$$

Exercise:

Problem: $x = 2 \cos t, \quad y = 8 \sin t, \quad \text{slope} = -1$

Exercise:

Problem: $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad \text{slope} = 1$

Solution:

No points possible; undefined expression.

Exercise:

Problem: $x = 2 + \sqrt{t}, \quad y = 2 - 4t, \quad \text{slope} = 0$

For the following exercises, write the equation of the tangent line in Cartesian coordinates for the given parameter t .

Exercise:

Problem: $x = e^{\sqrt{t}}, \quad y = 1 - \ln t^2, \quad t = 1$

Solution:

$$y = -\left(\frac{2}{e}\right)x + 3$$

Exercise:

Problem: $x = t \ln t, \quad y = \sin^2 t, t = \frac{\pi}{4}$

Exercise:

Problem: $x = e^t, \quad y = (t - 1)^2, \quad \text{at}(1, 1)$

Solution:

$$y = 2x - 7$$

Exercise:

Problem:

For $x = \sin(2t), y = 2 \sin t$ where $0 \leq t < 2\pi$. Find all values of t at which a horizontal tangent line exists.

Exercise:

Problem:

For $x = \sin(2t)$, $y = 2 \sin t$ where $0 \leq t < 2\pi$. Find all values of t at which a vertical tangent line exists.

Solution:

$$\frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$

Exercise:

Problem: Find all points on the curve $x = 4 \cos(t)$, $y = 4 \sin(t)$ that have the slope of $\frac{1}{2}$.

Exercise:

Problem: Find $\frac{dy}{dx}$ for $x = \sin(t)$, $y = \cos(t)$.

Solution:

$$\frac{dy}{dx} = -\tan(t)$$

Exercise:

Problem: Find the equation of the tangent line to $x = \sin(t)$, $y = \cos(t)$ at $t = \frac{\pi}{4}$.

Exercise:

Problem: For the curve $x = 4t$, $y = 3t - 2$, find the slope and concavity of the curve at $t = 3$.

Solution:

$\frac{dy}{dx} = \frac{3}{4}$ and $\frac{d^2y}{dx^2} = 0$, so the curve is neither concave up nor concave down at $t = 3$. Therefore the graph is linear and has a constant slope but no concavity.

Exercise:**Problem:**

For the parametric curve whose equation is $x = 4 \cos \theta$, $y = 4 \sin \theta$, find the slope and concavity of the curve at $\theta = \frac{\pi}{4}$.

Exercise:**Problem:**

Find the slope and concavity for the curve whose equation is $x = 2 + \sec \theta$, $y = 1 + 2 \tan \theta$ at $\theta = \frac{\pi}{6}$.

Solution:

$$\frac{dy}{dx} = 4, \frac{d^2y}{dx^2} = -6\sqrt{3}; \text{ the curve is concave down at } \theta = \frac{\pi}{6}.$$

Exercise:**Problem:**

Find all points on the curve $x = t + 4, y = t^3 - 3t$ at which there are vertical and horizontal tangents.

Exercise:**Problem:**

Find all points on the curve $x = \sec \theta, y = \tan \theta$ at which horizontal and vertical tangents exist.

Solution:

No horizontal tangents. Vertical tangents at $(1, 0), (-1, 0)$.

For the following exercises, find d^2y/dx^2 .

Exercise:

Problem: $x = t^4 - 1, y = t - t^2$

Exercise:

Problem: $x = \sin(\pi t), y = \cos(\pi t)$

Solution:

$-\sec^3(\pi t)$

Exercise:

Problem: $x = e^{-t}, y = te^{2t}$

For the following exercises, find points on the curve at which tangent line is horizontal or vertical.

Exercise:

Problem: $x = t(t^2 - 3), y = 3(t^2 - 3)$

Solution:

Horizontal $(0, -9)$; vertical $(\pm 2, -6)$.

Exercise:

Problem: $x = \frac{3t}{1+t^3}, y = \frac{3t^2}{1+t^3}$

For the following exercises, find dy/dx at the value of the parameter.

Exercise:

Problem: $x = \cos t, \quad y = \sin t, \quad t = \frac{3\pi}{4}$

Solution:

1

Exercise:

Problem: $x = \sqrt{t}, \quad y = 2t + 4, \quad t = 9$

Exercise:

Problem: $x = 4 \cos(2\pi s), \quad y = 3 \sin(2\pi s), \quad s = -\frac{1}{4}$

Solution:

0

For the following exercises, find d^2y/dx^2 at the given point without eliminating the parameter.

Exercise:

Problem: $x = \frac{1}{2}t^2, \quad y = \frac{1}{3}t^3, \quad t = 2$

Exercise:

Problem: $x = \sqrt{t}, \quad y = 2t + 4, \quad t = 1$

Solution:

4

Exercise:

Problem:

Find t intervals on which the curve $x = 3t^2, y = t^3 - t$ is concave up as well as concave down.

Exercise:

Problem: Determine the concavity of the curve $x = 2t + \ln t, y = 2t - \ln t$.

Solution:

Concave up on $t > 0$.

Exercise:

Problem:

Sketch and find the area under one arch of the cycloid $x = r(\theta - \sin \theta), y = r(1 - \cos \theta)$.

Exercise:**Problem:**

Find the area bounded by the curve $x = \cos t, y = e^t, 0 \leq t \leq \frac{\pi}{2}$ and the lines $y = 1$ and $x = 0$.

Solution:

1

Exercise:

Problem: Find the area enclosed by the ellipse $x = a \cos \theta, y = b \sin \theta, 0 \leq \theta < 2\pi$.

Exercise:**Problem:**

Find the area of the region bounded by $x = 2 \sin^2 \theta, y = 2 \sin^2 \theta \tan \theta$, for $0 \leq \theta \leq \frac{\pi}{2}$.

Solution:

$$\frac{3\pi}{2}$$

For the following exercises, find the area of the regions bounded by the parametric curves and the indicated values of the parameter.

Exercise:

Problem: $x = 2 \cot \theta, y = 2 \sin^2 \theta, 0 \leq \theta \leq \pi$

Exercise:

Problem: [T] $x = 2a \cos t - a \cos(2t), y = 2a \sin t - a \sin(2t), 0 \leq t < 2\pi$

Solution:

$$6\pi a^2$$

Exercise:

Problem: [T] $x = a \sin(2t), y = b \sin(t), 0 \leq t < 2\pi$ (the “hourglass”)

Exercise:

Problem: [T] $x = 2a \cos t - a \sin(2t), y = b \sin t, 0 \leq t < 2\pi$ (the “teardrop”)

Solution:

$$2\pi ab$$

For the following exercises, find the arc length of the curve on the indicated interval of the parameter.

Exercise:

Problem: $x = 4t + 3, \quad y = 3t - 2, \quad 0 \leq t \leq 2$

Exercise:

Problem: $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad 0 \leq t \leq 1$

Solution:

$$\frac{1}{3} \left(2\sqrt{2} - 1 \right)$$

Exercise:

Problem: $x = \cos(2t), \quad y = \sin(2t), \quad 0 \leq t \leq \frac{\pi}{2}$

Exercise:

Problem: $x = 1 + t^2, \quad y = (1 + t)^3, \quad 0 \leq t \leq 1$

Solution:

$$7.075$$

Exercise:

Problem:

$x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \leq t \leq \frac{\pi}{2}$ (express answer as a decimal rounded to three places)

Exercise:

Problem: $x = a \cos^3 \theta, y = a \sin^3 \theta$ on the interval $[0, 2\pi)$ (the hypocycloid)

Solution:

$$6a$$

Exercise:

Problem: Find the length of one arch of the cycloid $x = 4(t - \sin t), y = 4(1 - \cos t)$.

Exercise:

Problem:

Find the distance traveled by a particle with position (x, y) as t varies in the given time interval:
 $x = \sin^2 t, \quad y = \cos^2 t, \quad 0 \leq t \leq 3\pi$.

Solution:

$$6\sqrt{2}$$

Exercise:

Problem: Find the length of one arch of the cycloid $x = \theta - \sin \theta, y = 1 - \cos \theta$.

Exercise:

Problem:

Show that the total length of the ellipse $x = 4 \sin \theta, y = 3 \cos \theta$ is

$$L = 16 \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta, \text{ where } e = \frac{c}{a} \text{ and } c = \sqrt{a^2 - b^2}.$$

Exercise:

Problem: Find the length of the curve $x = e^t - t, y = 4e^{t/2}, -8 \leq t \leq 3$.

For the following exercises, find the area of the surface obtained by rotating the given curve about the x-axis.

Exercise:

Problem: $x = t^3, y = t^2, 0 \leq t \leq 1$

Solution:

$$\frac{2\pi(247\sqrt{13}+64)}{1215}$$

Exercise:

Problem: $x = a \cos^3 \theta, y = a \sin^3 \theta, 0 \leq \theta \leq \frac{\pi}{2}$

Exercise:

Problem:

[T] Use a CAS to find the area of the surface generated by rotating

$x = t + t^3, y = t - \frac{1}{t^2}, 1 \leq t \leq 2$ about the x-axis. (Answer to three decimal places.)

Solution:

$$59.101$$

Exercise:

Problem:

Find the surface area obtained by rotating $x = 3t^2, y = 2t^3, 0 \leq t \leq 5$ about the y-axis.

Exercise:

Problem:

Find the area of the surface generated by revolving $x = t^2, y = 2t, 0 \leq t \leq 4$ about the x -axis.

Solution:

$$\frac{8\pi}{3} (17\sqrt{17} - 1)$$

Exercise:**Problem:**

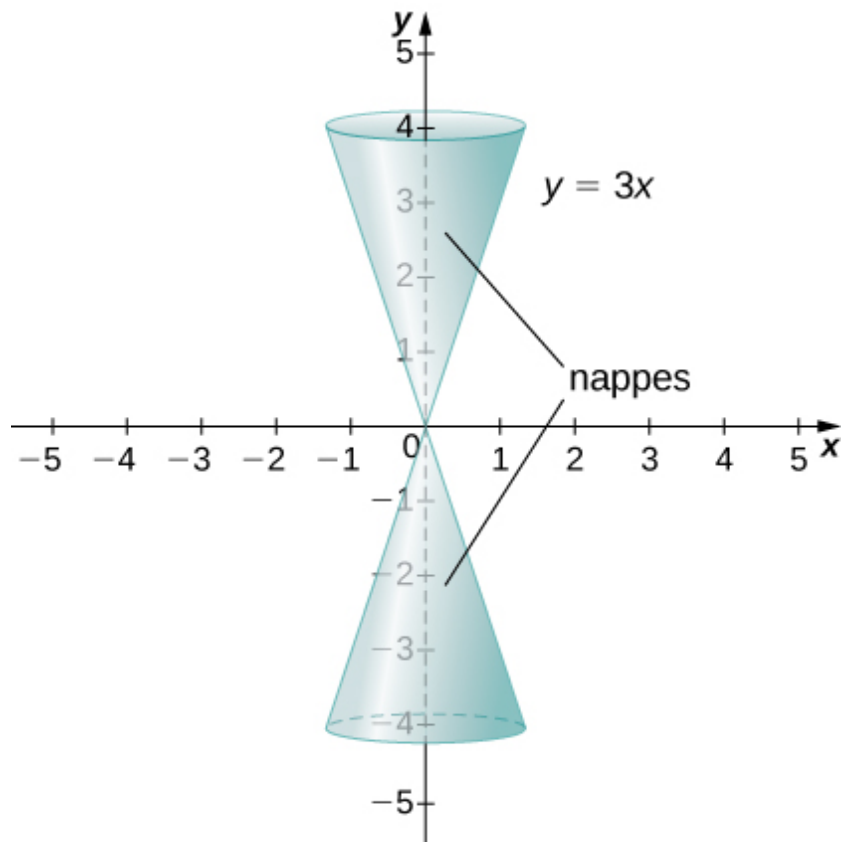
Find the surface area generated by revolving $x = t^2, y = 2t^2, 0 \leq t \leq 1$ about the y -axis.

Conic Sections

- Identify the equation of a parabola in standard form with given focus and directrix.
- Identify the equation of an ellipse in standard form with given foci.
- Identify the equation of a hyperbola in standard form with given foci.
- Recognize a parabola, ellipse, or hyperbola from its eccentricity value.
- Write the polar equation of a conic section with eccentricity e .
- Identify when a general equation of degree two is a parabola, ellipse, or hyperbola.

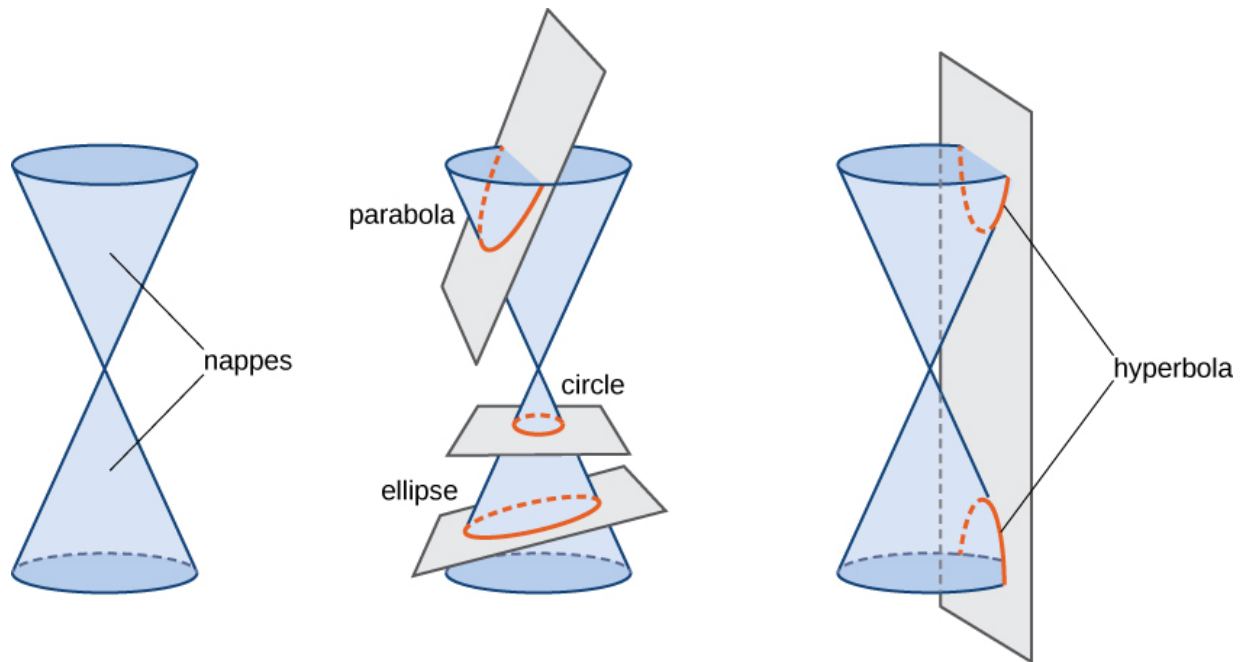
Conic sections have been studied since the time of the ancient Greeks, and were considered to be an important mathematical concept. As early as 320 BCE, such Greek mathematicians as Menaechmus, Appollonius, and Archimedes were fascinated by these curves. Appollonius wrote an entire eight-volume treatise on conic sections in which he was, for example, able to derive a specific method for identifying a conic section through the use of geometry. Since then, important applications of conic sections have arisen (for example, in astronomy), and the properties of conic sections are used in radio telescopes, satellite dish receivers, and even architecture. In this section we discuss the three basic conic sections, some of their properties, and their equations.

Conic sections get their name because they can be generated by intersecting a plane with a cone. A cone has two identically shaped parts called **nappes**. One nappe is what most people mean by “cone,” having the shape of a party hat. A right circular cone can be generated by revolving a line passing through the origin around the y -axis as shown.



A cone generated by revolving the line $y = 3x$ around the y -axis.

Conic sections are generated by the intersection of a plane with a cone ([\[link\]](#)). If the plane is parallel to the axis of revolution (the y -axis), then the **conic section** is a hyperbola. If the plane is parallel to the generating line, the conic section is a parabola. If the plane is perpendicular to the axis of revolution, the conic section is a circle. If the plane intersects one nappe at an angle to the axis (other than 90°), then the conic section is an ellipse.



The four conic sections. Each conic is determined by the angle the plane makes with the axis of the cone.

Parabolas

A parabola is generated when a plane intersects a cone parallel to the generating line. In this case, the plane intersects only one of the nappes. A parabola can also be defined in terms of distances.

Note:

Definition

A parabola is the set of all points whose distance from a fixed point, called the **focus**, is equal to the distance from a fixed line, called the **directrix**. The point halfway between the focus and the directrix is called the **vertex** of the parabola.

A graph of a typical parabola appears in [\[link\]](#). Using this diagram in conjunction with the distance formula, we can derive an equation for a parabola. Recall the distance formula: Given point P with coordinates (x_1, y_1) and point Q with coordinates (x_2, y_2) , the distance between them is given by the formula

Equation:

$$d(P, Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Then from the definition of a parabola and [\[link\]](#), we get

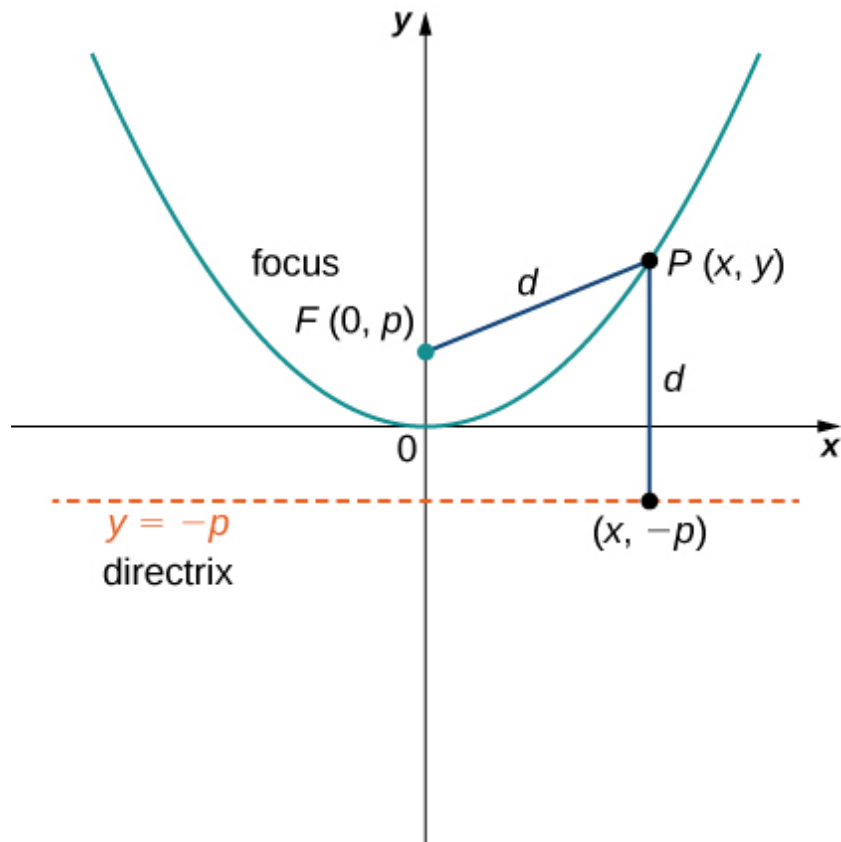
Equation:

$$\begin{aligned} d(F, P) &= d(P, Q) \\ \sqrt{(0 - x)^2 + (p - y)^2} &= \sqrt{(x - x)^2 + (-p - y)^2}. \end{aligned}$$

Squaring both sides and simplifying yields

Equation:

$$\begin{aligned} x^2 + (p - y)^2 &= 0^2 + (-p - y)^2 \\ x^2 + p^2 - 2py + y^2 &= p^2 + 2py + y^2 \\ x^2 - 2py &= 2py \\ x^2 &= 4py. \end{aligned}$$



A typical parabola in which the distance from the focus to the vertex is represented by the variable p .

Now suppose we want to relocate the vertex. We use the variables (h, k) to denote the coordinates of the vertex. Then if the focus is directly above the vertex, it has coordinates $(h, k + p)$ and the directrix has the equation $y = k - p$. Going through the same derivation yields the formula $(x - h)^2 = 4p(y - k)$. Solving this equation for y leads to the following theorem.

Note:
Equations for Parabolas

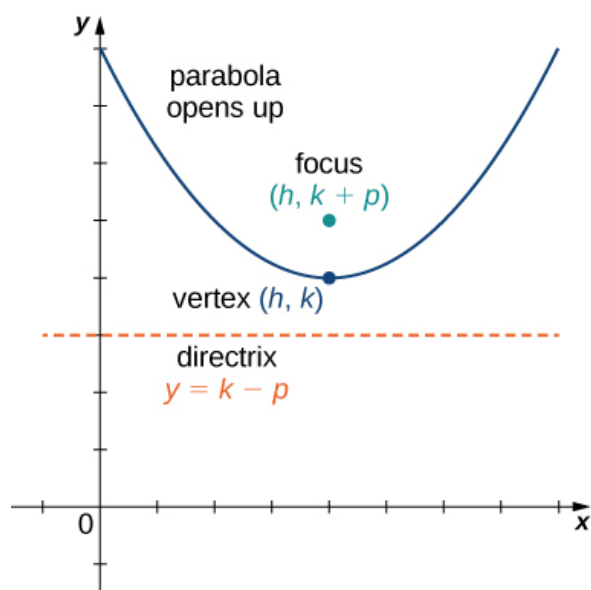
Given a parabola opening upward with vertex located at (h, k) and focus located at $(h, k + p)$, where p is a constant, the equation for the parabola is given by

Equation:

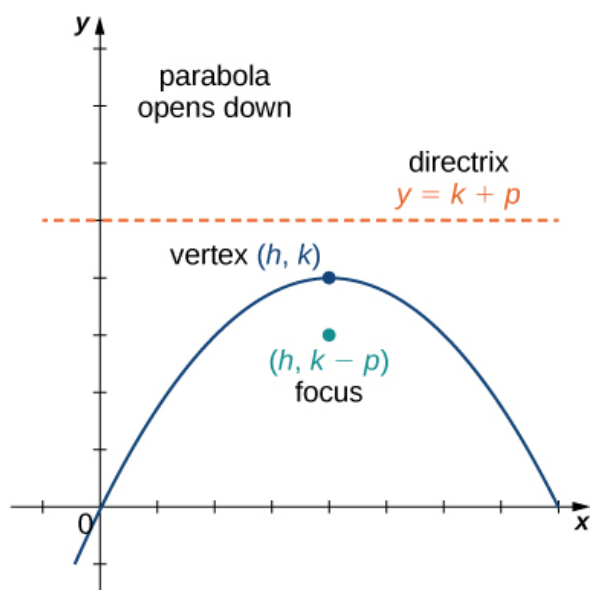
$$y = \frac{1}{4p}(x - h)^2 + k.$$

This is the **standard form** of a parabola.

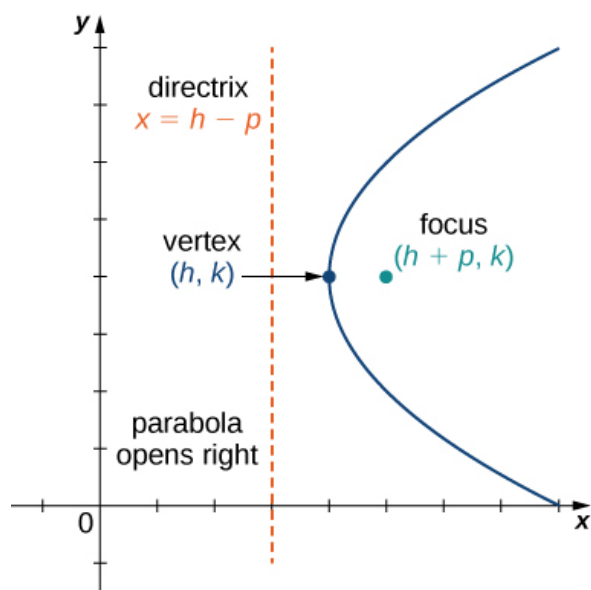
We can also study the cases when the parabola opens down or to the left or the right. The equation for each of these cases can also be written in standard form as shown in the following graphs.



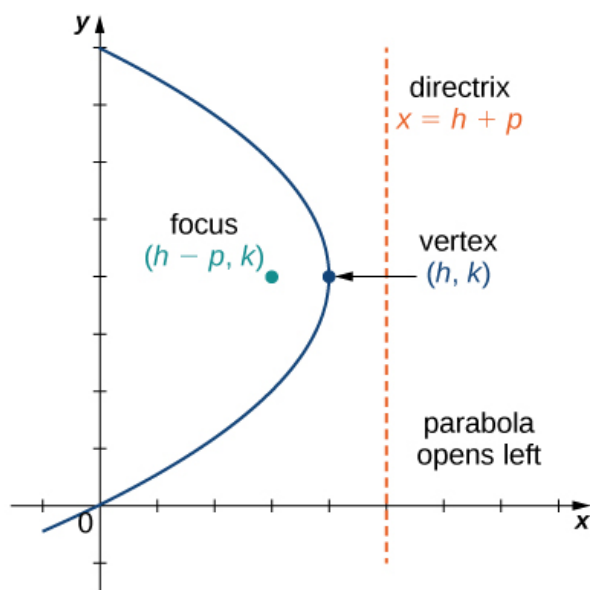
$$y = \frac{1}{4p}(x - h)^2 + k$$



$$y = -\frac{1}{4p}(x - h)^2 + k$$



$$x = \frac{1}{4p}(y - k)^2 + h$$



$$x = -\frac{1}{4p}(y - k)^2 + h$$

Four parabolas, opening in various directions, along with their equations in standard form.

In addition, the equation of a parabola can be written in the **general form**, though in this form the values of h , k , and p are not immediately recognizable. The general form of a parabola is written as

Equation:

$$ax^2 + bx + cy + d = 0 \quad \text{or} \quad ay^2 + bx + cy + d = 0.$$

The first equation represents a parabola that opens either up or down. The second equation represents a parabola that opens either to the left or to the right. To put the equation into standard form, use the method of completing the square.

Example:

Exercise:

Problem:

Converting the Equation of a Parabola from General into Standard Form

Put the equation $x^2 - 4x - 8y + 12 = 0$ into standard form and graph the resulting parabola.

Solution:

Since y is not squared in this equation, we know that the parabola opens either upward or downward. Therefore we need to solve this equation for y , which will put the equation into standard form. To do that, first add $8y$ to both sides of the equation:

Equation:

$$8y = x^2 - 4x + 12.$$

The next step is to complete the square on the right-hand side. Start by grouping the first two terms on the right-hand side using parentheses:

Equation:

$$8y = (x^2 - 4x) + 12.$$

Next determine the constant that, when added inside the parentheses, makes the quantity inside the parentheses a perfect square trinomial. To do this, take half the coefficient of x and square it. This gives $\left(\frac{-4}{2}\right)^2 = 4$. Add 4 inside the parentheses and subtract 4 outside the parentheses, so the value of the equation is not changed:

Equation:

$$8y = (x^2 - 4x + 4) + 12 - 4.$$

Now combine like terms and factor the quantity inside the parentheses:

Equation:

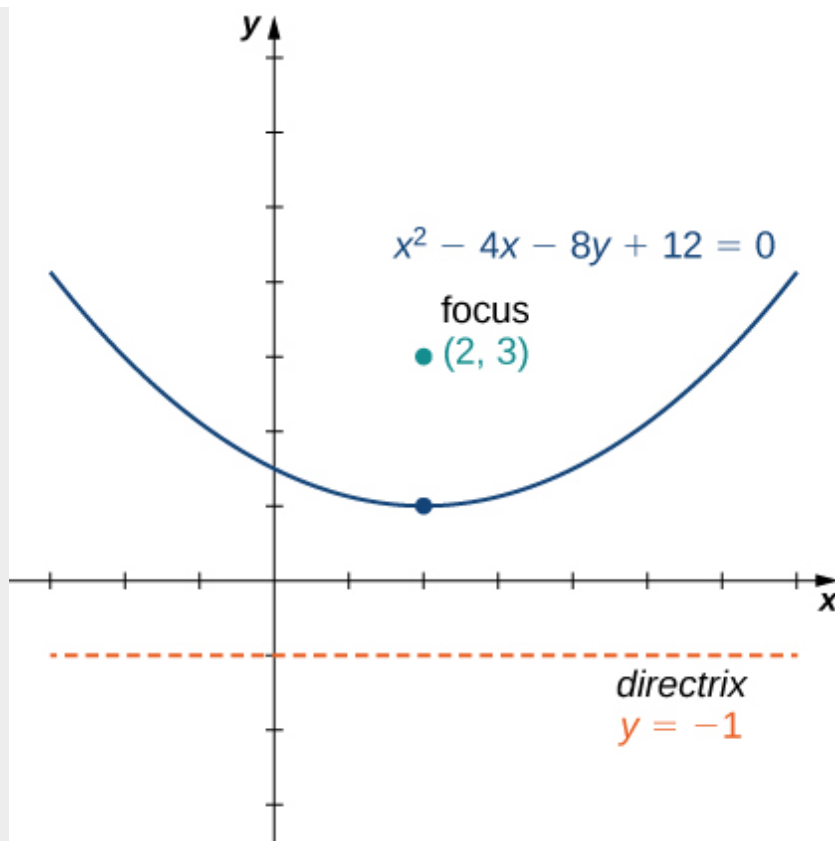
$$8y = (x - 2)^2 + 8.$$

Finally, divide by 8:

Equation:

$$y = \frac{1}{8}(x - 2)^2 + 1.$$

This equation is now in standard form. Comparing this to [\[link\]](#) gives $h = 2$, $k = 1$, and $p = 2$. The parabola opens up, with vertex at $(2, 1)$, focus at $(2, 3)$, and directrix $y = -1$. The graph of this parabola appears as follows.



The parabola in [\[link\]](#).

Note:

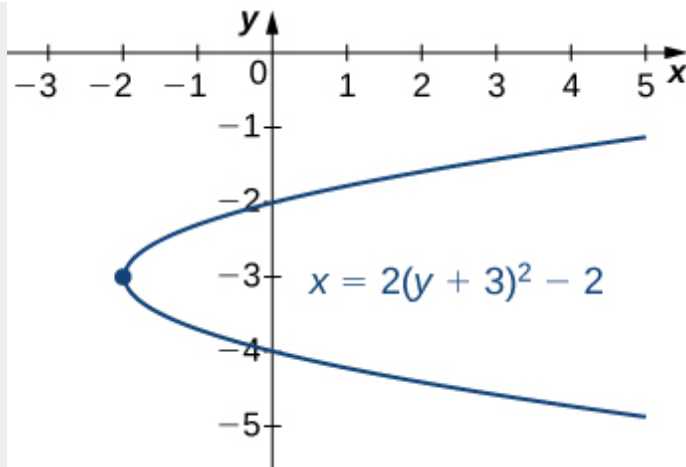
Exercise:

Problem:

Put the equation $2y^2 - x + 12y + 16 = 0$ into standard form and graph the resulting parabola.

Solution:

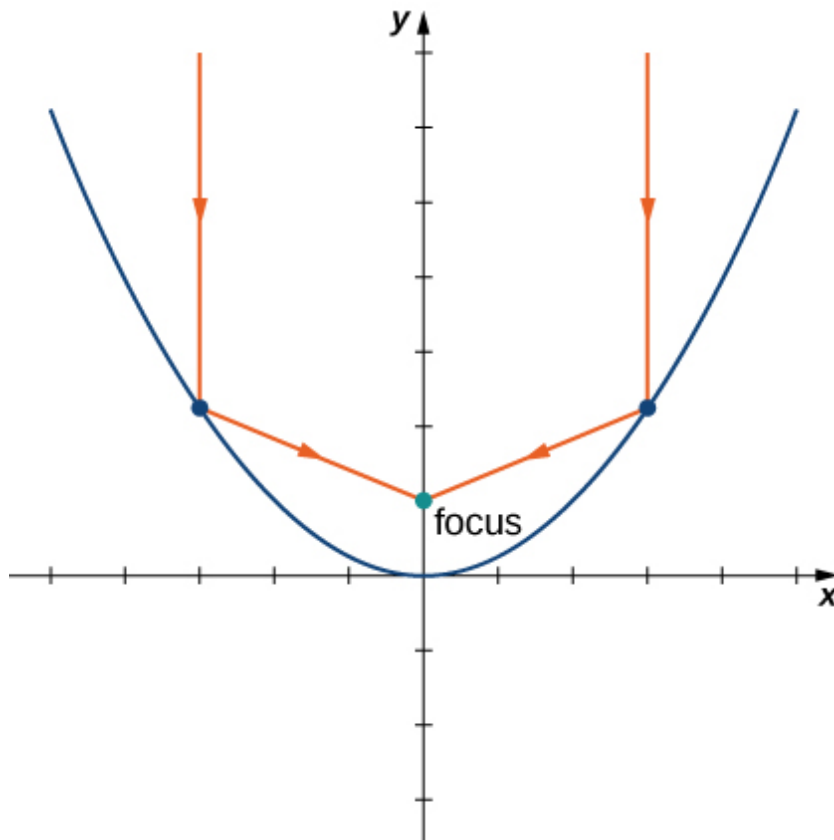
$$x = 2(y + 3)^2 - 2$$



Hint

Solve for x . Check which direction the parabola opens.

The axis of symmetry of a vertical (opening up or down) parabola is a vertical line passing through the vertex. The parabola has an interesting reflective property. Suppose we have a satellite dish with a parabolic cross section. If a beam of electromagnetic waves, such as light or radio waves, comes into the dish in a straight line from a satellite (parallel to the axis of symmetry), then the waves reflect off the dish and collect at the focus of the parabola as shown.



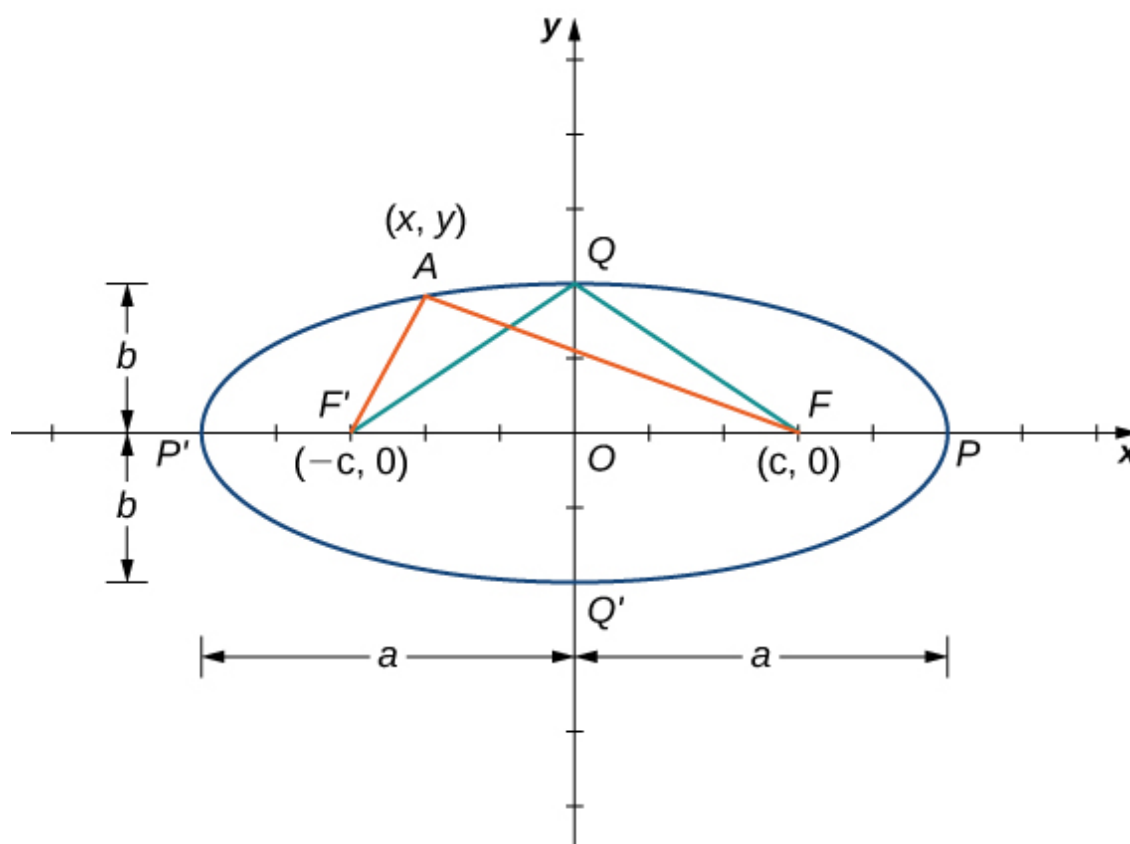
Consider a parabolic dish designed to collect signals from a satellite in space. The dish is aimed directly at the satellite, and a receiver is located at the focus of the parabola. Radio waves coming in from the satellite are reflected off the surface of the parabola to the receiver, which collects and decodes the digital signals. This allows a small receiver to gather signals from a wide angle of sky. Flashlights and headlights in a car work on the same principle, but in reverse: the source of the light (that is, the light bulb) is located at the focus and the reflecting surface on the parabolic mirror focuses the beam straight ahead. This allows a small light bulb to illuminate a wide angle of space in front of the flashlight or car.

Ellipses

An ellipse can also be defined in terms of distances. In the case of an ellipse, there are two foci (plural of focus), and two directrices (plural of directrix). We look at the directrices in more detail later in this section.

Note:**Definition**

An *ellipse* is the set of all points for which the sum of their distances from two fixed points (the foci) is constant.



A typical ellipse in which the sum of the distances from any point on the ellipse to the foci is constant.

A graph of a typical ellipse is shown in [\[link\]](#). In this figure the foci are labeled as F and F' . Both are the same fixed distance from the origin, and this distance is represented by the variable c . Therefore the coordinates of F are $(c, 0)$ and the coordinates of F' are $(-c, 0)$. The points P and P' are located at the ends of the **major axis** of the ellipse, and have coordinates $(a, 0)$ and $(-a, 0)$, respectively. The major axis is always the longest

distance across the ellipse, and can be horizontal or vertical. Thus, the length of the major axis in this ellipse is $2a$. Furthermore, P and P' are called the vertices of the ellipse. The points Q and Q' are located at the ends of the **minor axis** of the ellipse, and have coordinates $(0, b)$ and $(0, -b)$, respectively. The minor axis is the shortest distance across the ellipse. The minor axis is perpendicular to the major axis.

According to the definition of the ellipse, we can choose any point on the ellipse and the sum of the distances from this point to the two foci is constant. Suppose we choose the point P . Since the coordinates of point P are $(a, 0)$, the sum of the distances is

Equation:

$$d(P, F) + d(P, F') = (a - c) + (a + c) = 2a.$$

Therefore the sum of the distances from an arbitrary point A with coordinates (x, y) is also equal to $2a$. Using the distance formula, we get

Equation:

$$\begin{aligned} d(A, F) + d(A, F') &= 2a \\ \sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} &= 2a. \end{aligned}$$

Subtract the second radical from both sides and square both sides:

Equation:

$$\begin{aligned} \sqrt{(x - c)^2 + y^2} &= 2a - \sqrt{(x + c)^2 + y^2} \\ (x - c)^2 + y^2 &= 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2 \\ x^2 - 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \\ -2cx &= 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + 2cx. \end{aligned}$$

Now isolate the radical on the right-hand side and square again:

Equation:

$$\begin{aligned}-2cx &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx \\ 4a\sqrt{(x+c)^2 + y^2} &= 4a^2 + 4cx \\ \sqrt{(x+c)^2 + y^2} &= a + \frac{cx}{a} \\ (x+c)^2 + y^2 &= a^2 + 2cx + \frac{c^2x^2}{a^2} \\ x^2 + 2cx + c^2 + y^2 &= a^2 + 2cx + \frac{c^2x^2}{a^2} \\ x^2 + c^2 + y^2 &= a^2 + \frac{c^2x^2}{a^2}.\end{aligned}$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

Equation:

$$\begin{aligned}x^2 - \frac{c^2x^2}{a^2} + y^2 &= a^2 - c^2 \\ \frac{(a^2-c^2)x^2}{a^2} + y^2 &= a^2 - c^2.\end{aligned}$$

Divide both sides by $a^2 - c^2$. This gives the equation

Equation:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If we refer back to [\[link\]](#), then the length of each of the two green line segments is equal to a . This is true because the sum of the distances from the point Q to the foci F and F' is equal to $2a$, and the lengths of these two line segments are equal. This line segment forms a right triangle with hypotenuse length a and leg lengths b and c . From the Pythagorean theorem, $a^2 + b^2 = c^2$ and $b^2 = a^2 - c^2$. Therefore the equation of the ellipse becomes

Equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Finally, if the center of the ellipse is moved from the origin to a point (h, k) , we have the following standard form of an ellipse.

Note:**Equation of an Ellipse in Standard Form**

Consider the ellipse with center (h, k) , a horizontal major axis with length $2a$, and a vertical minor axis with length $2b$. Then the equation of this ellipse in standard form is

Equation:

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 - b^2$. The equations of the directrices are $x = h \pm \frac{a^2}{c}$.

If the major axis is vertical, then the equation of the ellipse becomes

Equation:

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$

and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 - b^2$. The equations of the directrices in this case are $y = k \pm \frac{a^2}{c}$.

If the major axis is horizontal, then the ellipse is called horizontal, and if the major axis is vertical, then the ellipse is called vertical. The equation of an ellipse is in general form if it is in the form

$Ax^2 + By^2 + Cx + Dy + E = 0$, where A and B are either both positive or both negative. To convert the equation from general to standard form, use the method of completing the square.

Example:

Exercise:

Problem:

Finding the Standard Form of an Ellipse

Put the equation $9x^2 + 4y^2 - 36x + 24y + 36 = 0$ into standard form and graph the resulting ellipse.

Solution:

First subtract 36 from both sides of the equation:

Equation:

$$9x^2 + 4y^2 - 36x + 24y = -36.$$

Next group the x terms together and the y terms together, and factor out the common factor:

Equation:

$$(9x^2 - 36x) + (4y^2 + 24y) = -36$$

$$9(x^2 - 4x) + 4(y^2 + 6y) = -36.$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of x and square it. This gives $\left(\frac{-4}{2}\right)^2 = 4$. In the second set of parentheses, take half the coefficient of y and square it. This gives $\left(\frac{6}{2}\right)^2 = 9$. Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually

adding 36 to the left-hand side. Similarly, we are adding 36 to the second set as well. Therefore the equation becomes

Equation:

$$9(x^2 - 4x + 4) + 4(y^2 + 6y + 9) = -36 + 36 + 36$$

$$9(x^2 - 4x + 4) + 4(y^2 + 6y + 9) = 36.$$

Now factor both sets of parentheses and divide by 36:

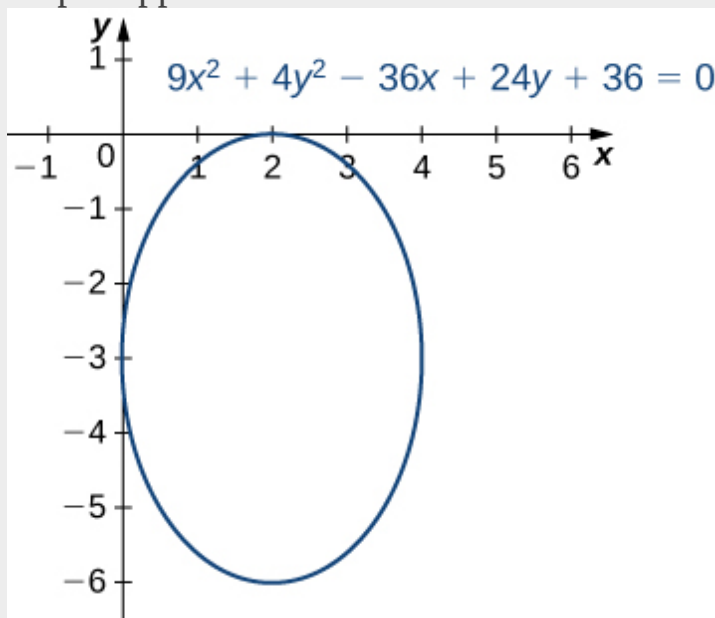
Equation:

$$9(x - 2)^2 + 4(y + 3)^2 = 36$$

$$\frac{9(x-2)^2}{36} + \frac{4(y+3)^2}{36} = 1$$

$$\frac{(x-2)^2}{4} + \frac{(y+3)^2}{9} = 1.$$

The equation is now in standard form. Comparing this to [\[link\]](#) gives $h = 2$, $k = -3$, $a = 3$, and $b = 2$. This is a vertical ellipse with center at $(2, -3)$, major axis 6, and minor axis 4. The graph of this ellipse appears as follows.



The ellipse in [\[link\]](#).

Note:

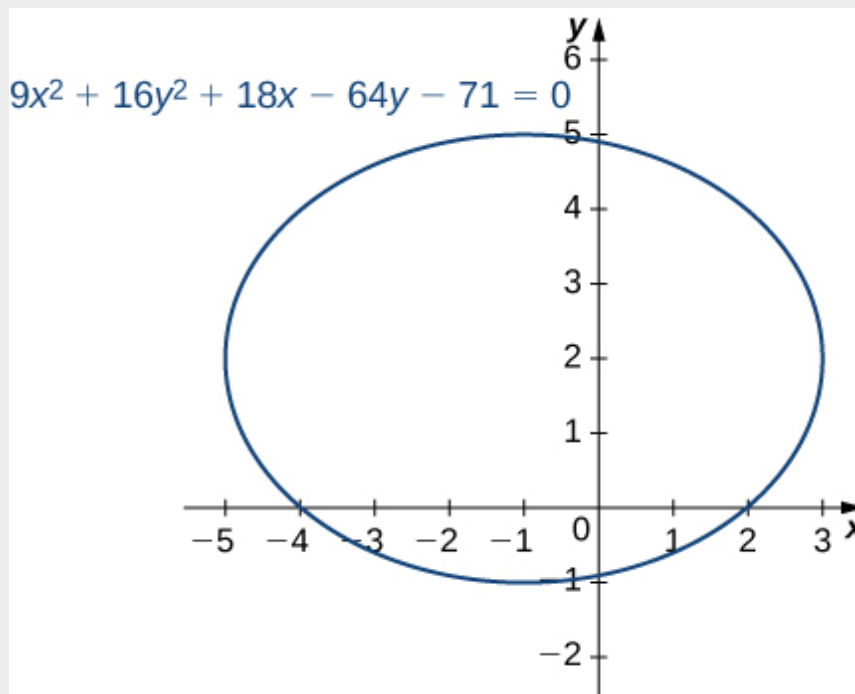
Exercise:

Problem:

Put the equation $9x^2 + 16y^2 + 18x - 64y - 71 = 0$ into standard form and graph the resulting ellipse.

Solution:

$$\frac{(x+1)^2}{16} + \frac{(y-2)^2}{9} = 1$$

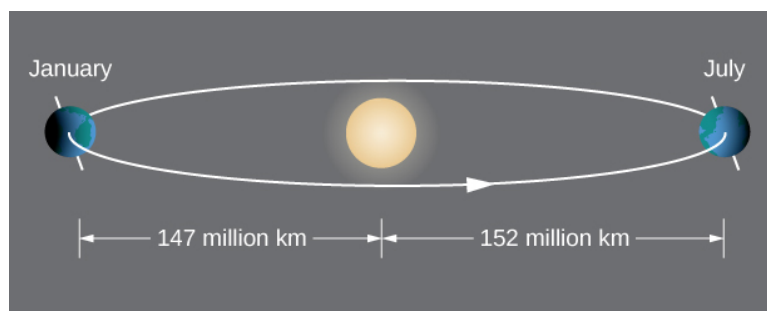


Hint

Move the constant over and complete the square.

According to Kepler's first law of planetary motion, the orbit of a planet around the Sun is an ellipse with the Sun at one of the foci as shown in [\[link\]](#)(a). Because Earth's orbit is an ellipse, the distance from the Sun varies throughout the year. A commonly held misconception is that Earth is closer to the Sun in the summer. In fact, in summer for the northern hemisphere, Earth is farther from the Sun than during winter. The difference in season is caused by the tilt of Earth's axis in the orbital plane. Comets that orbit the Sun, such as Halley's Comet, also have elliptical orbits, as do moons orbiting the planets and satellites orbiting Earth.

Ellipses also have interesting reflective properties: A light ray emanating from one focus passes through the other focus after mirror reflection in the ellipse. The same thing occurs with a sound wave as well. The National Statuary Hall in the U.S. Capitol in Washington, DC, is a famous room in an elliptical shape as shown in [\[link\]](#)(b). This hall served as the meeting place for the U.S. House of Representatives for almost fifty years. The location of the two foci of this semi-elliptical room are clearly identified by marks on the floor, and even if the room is full of visitors, when two people stand on these spots and speak to each other, they can hear each other much more clearly than they can hear someone standing close by. Legend has it that John Quincy Adams had his desk located on one of the foci and was able to eavesdrop on everyone else in the House without ever needing to stand. Although this makes a good story, it is unlikely to be true, because the original ceiling produced so many echoes that the entire room had to be hung with carpets to dampen the noise. The ceiling was rebuilt in 1902 and only then did the now-famous whispering effect emerge. Another famous whispering gallery—the site of many marriage proposals—is in Grand Central Station in New York City.



(a)



(b)

- (a) Earth's orbit around the Sun is an ellipse with the Sun at one focus.
 (b) Statuary Hall in the U.S. Capitol is a whispering gallery with an elliptical cross section.

Hyperbolas

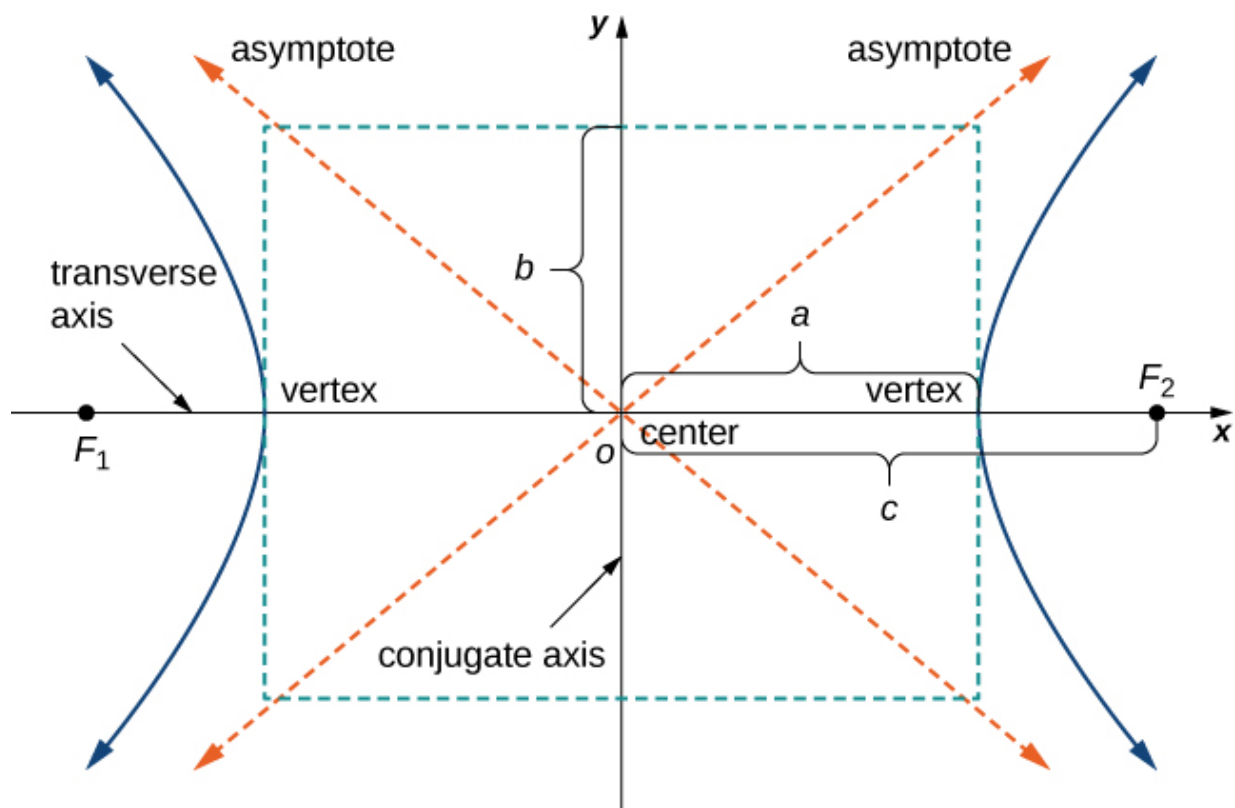
A hyperbola can also be defined in terms of distances. In the case of a hyperbola, there are two foci and two directrices. Hyperbolas also have two asymptotes.

Note:

Definition

A hyperbola is the set of all points where the difference between their distances from two fixed points (the foci) is constant.

A graph of a typical hyperbola appears as follows.



A typical hyperbola in which the difference of the distances from any point on the ellipse to the foci is constant. The transverse axis is also called the major axis, and the conjugate axis is also called the minor axis.

The derivation of the equation of a hyperbola in standard form is virtually identical to that of an ellipse. One slight hitch lies in the definition: The difference between two numbers is always positive. Let P be a point on the hyperbola with coordinates (x, y) . Then the definition of the hyperbola gives $|d(P, F_1) - d(P, F_2)| = \text{constant}$. To simplify the derivation, assume that P is on the right branch of the hyperbola, so the absolute value bars drop. If it is on the left branch, then the subtraction is reversed. The vertex of the right branch has coordinates $(a, 0)$, so

Equation:

$$d(P, F_1) - d(P, F_2) = (c + a) - (c - a) = 2a.$$

This equation is therefore true for any point on the hyperbola. Returning to the coordinates (x, y) for P :

Equation:

$$\begin{aligned}d(P, F_1) - d(P, F_2) &= 2a \\ \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= 2a.\end{aligned}$$

Add the second radical from both sides and square both sides:

Equation:

$$\begin{aligned}\sqrt{(x-c)^2 + y^2} &= 2a + \sqrt{(x+c)^2 + y^2} \\ (x-c)^2 + y^2 &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \\ x^2 - 2cx + c^2 + y^2 &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \\ -2cx &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + 2cx.\end{aligned}$$

Now isolate the radical on the right-hand side and square again:

Equation:

$$\begin{aligned}-2cx &= 4a^2 + 4a\sqrt{(x+c)^2 + y^2} + 2cx \\ 4a\sqrt{(x+c)^2 + y^2} &= -4a^2 - 4cx \\ \sqrt{(x+c)^2 + y^2} &= -a - \frac{cx}{a} \\ (x+c)^2 + y^2 &= a^2 + 2cx + \frac{c^2x^2}{a^2} \\ x^2 + 2cx + c^2 + y^2 &= a^2 + 2cx + \frac{c^2x^2}{a^2} \\ x^2 + c^2 + y^2 &= a^2 + \frac{c^2x^2}{a^2}.\end{aligned}$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

Equation:

$$\begin{aligned}x^2 - \frac{c^2 x^2}{a^2} + y^2 &= a^2 - c^2 \\ \frac{(a^2 - c^2)x^2}{a^2} + y^2 &= a^2 - c^2.\end{aligned}$$

Finally, divide both sides by $a^2 - c^2$. This gives the equation

Equation:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

We now define b so that $b^2 = c^2 - a^2$. This is possible because $c > a$. Therefore the equation of the ellipse becomes

Equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Finally, if the center of the hyperbola is moved from the origin to the point (h, k) , we have the following standard form of a hyperbola.

Note:

Equation of a Hyperbola in Standard Form

Consider the hyperbola with center (h, k) , a horizontal major axis, and a vertical minor axis. Then the equation of this ellipse is

Equation:

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are given by $y = k \pm \frac{b}{a}(x - h)$. The equations of the directrices are

Equation:

$$x = k \pm \frac{a^2}{\sqrt{a^2 + b^2}} = h \pm \frac{a^2}{c}.$$

If the major axis is vertical, then the equation of the hyperbola becomes

Equation:

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are given by $y = k \pm \frac{a}{b}(x - h)$. The equations of the directrices are

Equation:

$$y = k \pm \frac{a^2}{\sqrt{a^2 + b^2}} = k \pm \frac{a^2}{c}.$$

If the major axis (transverse axis) is horizontal, then the hyperbola is called horizontal, and if the major axis is vertical then the hyperbola is called vertical. The equation of a hyperbola is in general form if it is in the form $Ax^2 + By^2 + Cx + Dy + E = 0$, where A and B have opposite signs. In order to convert the equation from general to standard form, use the method of completing the square.

Example:

Exercise:

Problem:

Finding the Standard Form of a Hyperbola

Put the equation $9x^2 - 16y^2 + 36x + 32y - 124 = 0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Solution:

First add 124 to both sides of the equation:

Equation:

$$9x^2 - 16y^2 + 36x + 32y = 124.$$

Next group the x terms together and the y terms together, then factor out the common factors:

Equation:

$$\begin{aligned}(9x^2 + 36x) - (16y^2 - 32y) &= 124 \\ 9(x^2 + 4x) - 16(y^2 - 2y) &= 124.\end{aligned}$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of x and square it. This gives $\left(\frac{4}{2}\right)^2 = 4$. In the second set of parentheses, take half the coefficient of y and square it.

This gives $\left(\frac{-2}{2}\right)^2 = 1$. Add these inside each pair of parentheses.

Since the first set of parentheses has a 9 in front, we are actually adding 36 to the left-hand side. Similarly, we are subtracting 16 from the second set of parentheses. Therefore the equation becomes

Equation:

$$\begin{aligned}9(x^2 + 4x + 4) - 16(y^2 - 2y + 1) &= 124 + 36 - 16 \\ 9(x^2 + 4x + 4) - 16(y^2 - 2y + 1) &= 144.\end{aligned}$$

Next factor both sets of parentheses and divide by 144:

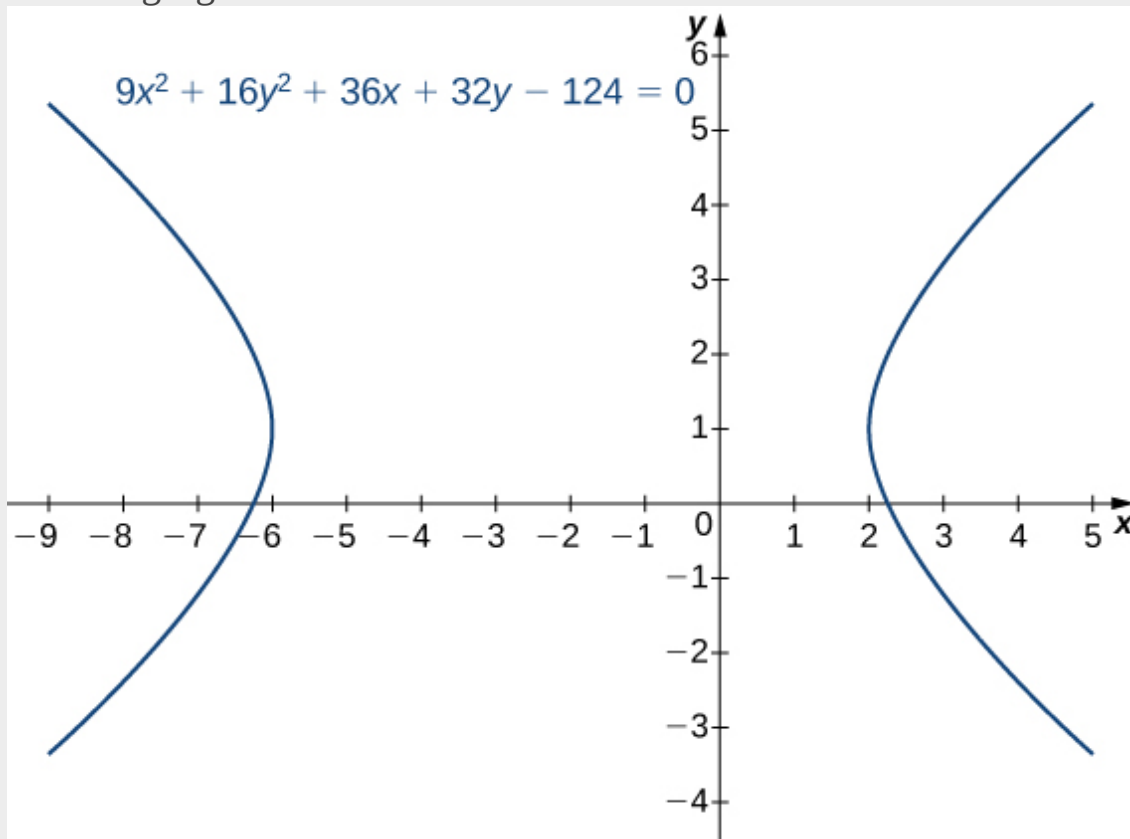
Equation:

$$9(x + 2)^2 - 16(y - 1)^2 = 144$$

$$\frac{9(x+2)^2}{144} - \frac{16(y-1)^2}{144} = 1$$

$$\frac{(x+2)^2}{16} - \frac{(y-1)^2}{9} = 1.$$

The equation is now in standard form. Comparing this to [\[link\]](#) gives $h = -2$, $k = 1$, $a = 4$, and $b = 3$. This is a horizontal hyperbola with center at $(-2, 1)$ and asymptotes given by the equations $y = 1 \pm \frac{3}{4}(x + 2)$. The graph of this hyperbola appears in the following figure.



Graph of the hyperbola in [\[link\]](#).

Note:

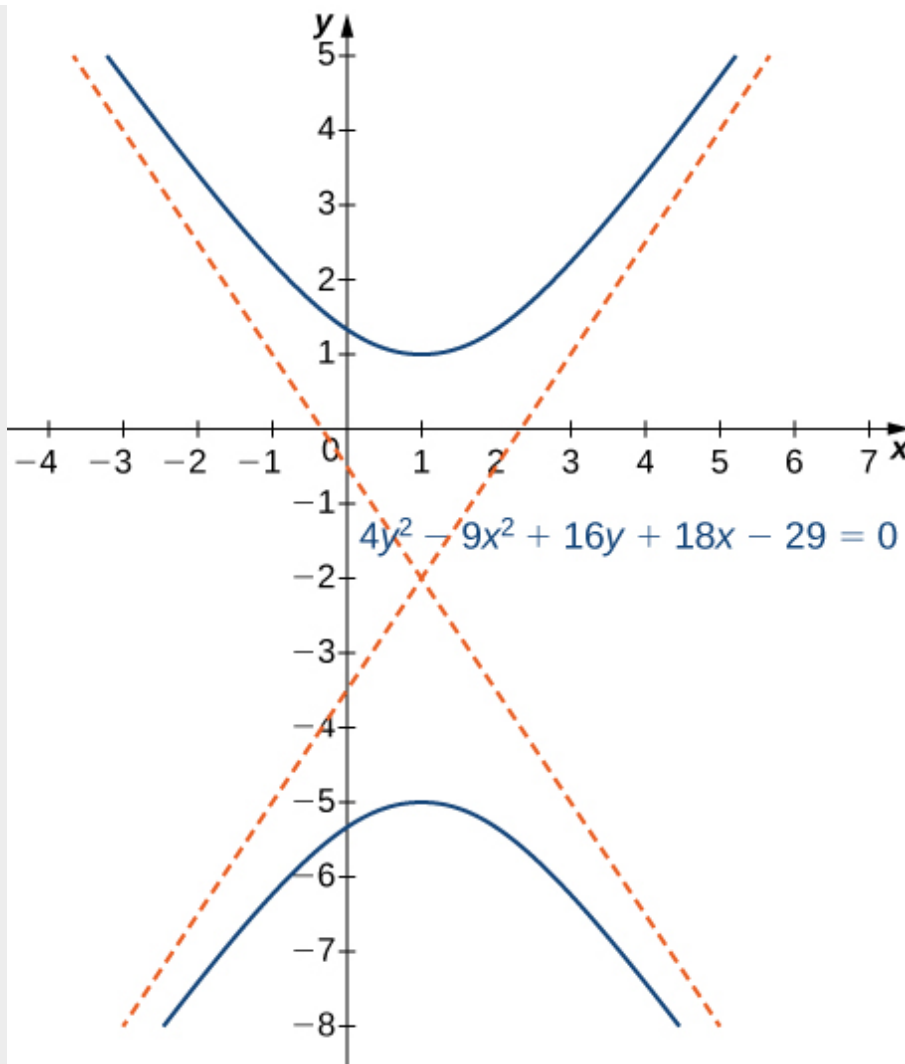
Exercise:

Problem:

Put the equation $4y^2 - 9x^2 + 16y + 18x - 29 = 0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Solution:

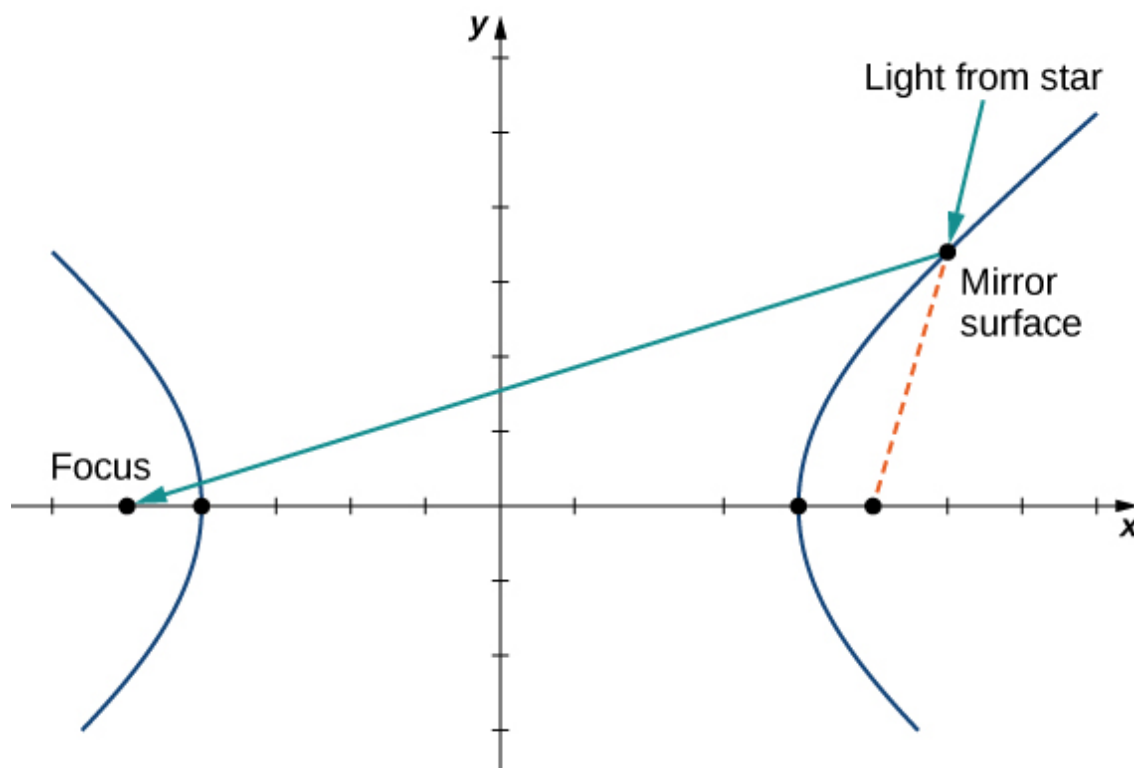
$\frac{(y+2)^2}{9} - \frac{(x-1)^2}{4} = 1$. This is a vertical hyperbola. Asymptotes
 $y = -2 \pm \frac{3}{2}(x - 1)$.



Hint

Move the constant over and complete the square. Check which direction the hyperbola opens.

Hyperbolas also have interesting reflective properties. A ray directed toward one focus of a hyperbola is reflected by a hyperbolic mirror toward the other focus. This concept is illustrated in the following figure.



A hyperbolic mirror used to collect light from distant stars.

This property of the hyperbola has important applications. It is used in radio direction finding (since the difference in signals from two towers is constant along hyperbolas), and in the construction of mirrors inside telescopes (to reflect light coming from the parabolic mirror to the eyepiece). Another interesting fact about hyperbolas is that for a comet entering the solar system, if the speed is great enough to escape the Sun's gravitational pull, then the path that the comet takes as it passes through the solar system is hyperbolic.

Eccentricity and Directrix

An alternative way to describe a conic section involves the directrices, the foci, and a new property called eccentricity. We will see that the value of the eccentricity of a conic section can uniquely define that conic.

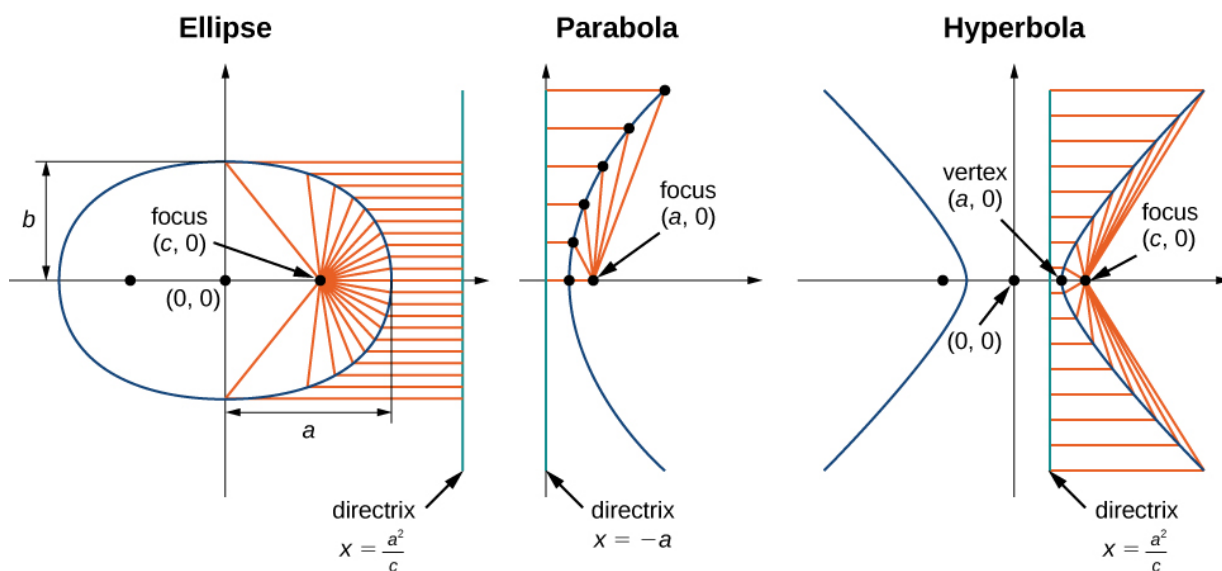
Note:**Definition**

The **eccentricity** e of a conic section is defined to be the distance from any point on the conic section to its focus, divided by the perpendicular distance from that point to the nearest directrix. This value is constant for any conic section, and can define the conic section as well:

1. If $e = 1$, the conic is a parabola.
2. If $e < 1$, it is an ellipse.
3. If $e > 1$, it is a hyperbola.

The eccentricity of a circle is zero. The directrix of a conic section is the line that, together with the point known as the focus, serves to define a conic section. Hyperbolas and noncircular ellipses have two foci and two associated directrices. Parabolas have one focus and one directrix.

The three conic sections with their directrices appear in the following figure.



The three conic sections with their foci and directrices.

Recall from the definition of a parabola that the distance from any point on the parabola to the focus is equal to the distance from that same point to the directrix. Therefore, by definition, the eccentricity of a parabola must be 1. The equations of the directrices of a horizontal ellipse are $x = \pm \frac{a^2}{c}$. The right vertex of the ellipse is located at $(a, 0)$ and the right focus is $(c, 0)$. Therefore the distance from the vertex to the focus is $a - c$ and the distance from the vertex to the right directrix is $\frac{a^2}{c} - a$. This gives the eccentricity as

Equation:

$$e = \frac{a - c}{\frac{a^2}{c} - a} = \frac{c(a - c)}{a^2 - ac} = \frac{c(a - c)}{a(a - c)} = \frac{c}{a}.$$

Since $c < a$, this step proves that the eccentricity of an ellipse is less than 1. The directrices of a horizontal hyperbola are also located at $x = \pm \frac{a^2}{c}$, and a similar calculation shows that the eccentricity of a hyperbola is also $e = \frac{c}{a}$. However in this case we have $c > a$, so the eccentricity of a hyperbola is greater than 1.

Example:

Exercise:

Problem:

Determining Eccentricity of a Conic Section

Determine the eccentricity of the ellipse described by the equation

Equation:

$$\frac{(x - 3)^2}{16} + \frac{(y + 2)^2}{25} = 1.$$

Solution:

From the equation we see that $a = 5$ and $b = 4$. The value of c can be calculated using the equation $a^2 = b^2 + c^2$ for an ellipse. Substituting the values of a and b and solving for c gives $c = 3$. Therefore the eccentricity of the ellipse is $e = \frac{c}{a} = \frac{3}{5} = 0.6$.

Note:

Exercise:

Problem:

Determine the eccentricity of the hyperbola described by the equation

Equation:

$$\frac{(y - 3)^2}{49} - \frac{(x + 2)^2}{25} = 1.$$

Solution:

$$e = \frac{c}{a} = \frac{\sqrt{74}}{7} \approx 1.229$$

Hint

First find the values of a and b , then determine c using the equation $c^2 = a^2 + b^2$.

Polar Equations of Conic Sections

Sometimes it is useful to write or identify the equation of a conic section in polar form. To do this, we need the concept of the focal parameter. The **focal parameter** of a conic section p is defined as the distance from a focus to the nearest directrix. The following table gives the focal parameters for the different types of conics, where a is the length of the semi-major axis (i.e., half the length of the major axis), c is the distance from the origin to the

focus, and e is the eccentricity. In the case of a parabola, a represents the distance from the vertex to the focus.

Conic	e	p
Ellipse	$0 < e < 1$	$\frac{a^2 - c^2}{c} = \frac{a(1 - e^2)}{e}$
Parabola	$e = 1$	$2a$
Hyperbola	$e > 1$	$\frac{c^2 - a^2}{c} = \frac{a(e^2 - 1)}{e}$

Eccentricities and Focal Parameters of the Conic Sections

Using the definitions of the focal parameter and eccentricity of the conic section, we can derive an equation for any conic section in polar coordinates. In particular, we assume that one of the foci of a given conic section lies at the pole. Then using the definition of the various conic sections in terms of distances, it is possible to prove the following theorem.

Note:

Polar Equation of Conic Sections

The polar equation of a conic section with focal parameter p is given by

Equation:

$$r = \frac{ep}{1 \pm e \cos \theta} \text{ or } r = \frac{ep}{1 \pm e \sin \theta}.$$

In the equation on the left, the major axis of the conic section is horizontal, and in the equation on the right, the major axis is vertical. To work with a conic section written in polar form, first make the constant term in the denominator equal to 1. This can be done by dividing both the numerator and the denominator of the fraction by the constant that appears in front of the plus or minus in the denominator. Then the coefficient of the sine or cosine in the denominator is the eccentricity. This value identifies the conic. If cosine appears in the denominator, then the conic is horizontal. If sine appears, then the conic is vertical. If both appear then the axes are rotated. The center of the conic is not necessarily at the origin. The center is at the origin only if the conic is a circle (i.e., $e = 0$).

Example:**Exercise:****Problem:****Graphing a Conic Section in Polar Coordinates**

Identify and create a graph of the conic section described by the equation

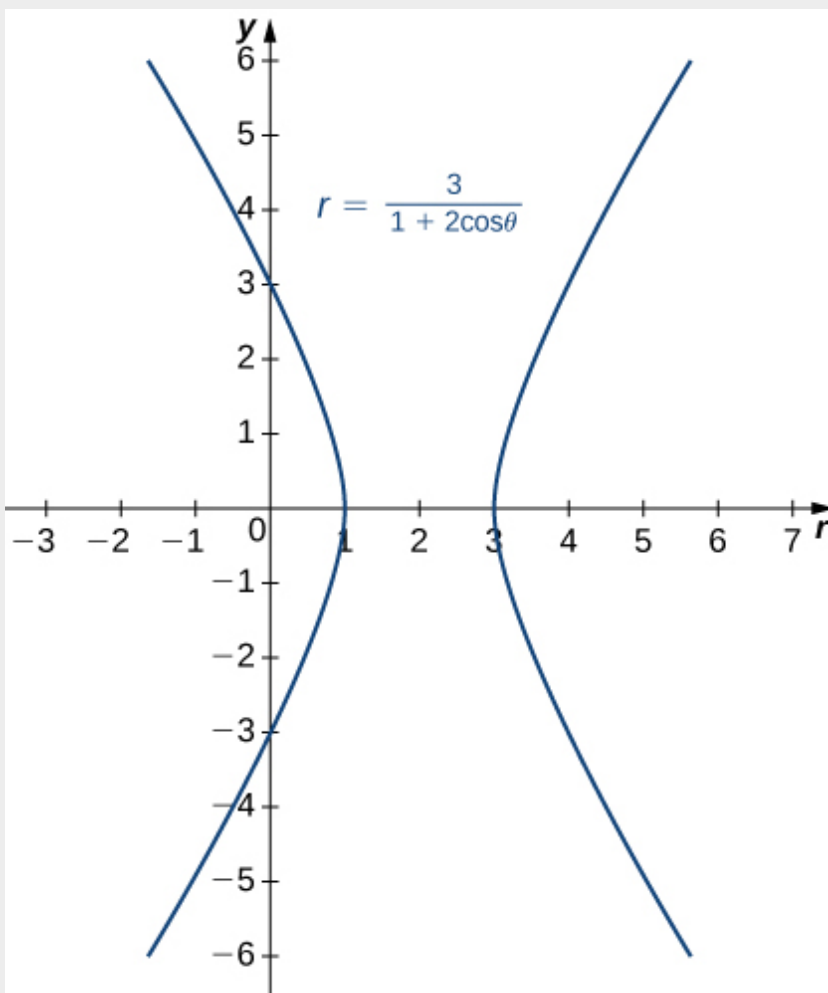
Equation:

$$r = \frac{3}{1 + 2 \cos \theta}.$$

Solution:

The constant term in the denominator is 1, so the eccentricity of the conic is 2. This is a hyperbola. The focal parameter p can be calculated by using the equation $ep = 3$. Since $e = 2$, this gives $p = \frac{3}{2}$. The cosine function appears in the denominator, so the hyperbola is horizontal. Pick a few values for θ and create a table of values. Then we can graph the hyperbola ([link](#)).

θ	r	θ	r
0	1	π	-3
$\frac{\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$	$\frac{5\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$
$\frac{\pi}{2}$	3	$\frac{3\pi}{2}$	3
$\frac{3\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$	$\frac{7\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$



Graph of the hyperbola described in [\[link\]](#).

Note:

Exercise:

Problem:

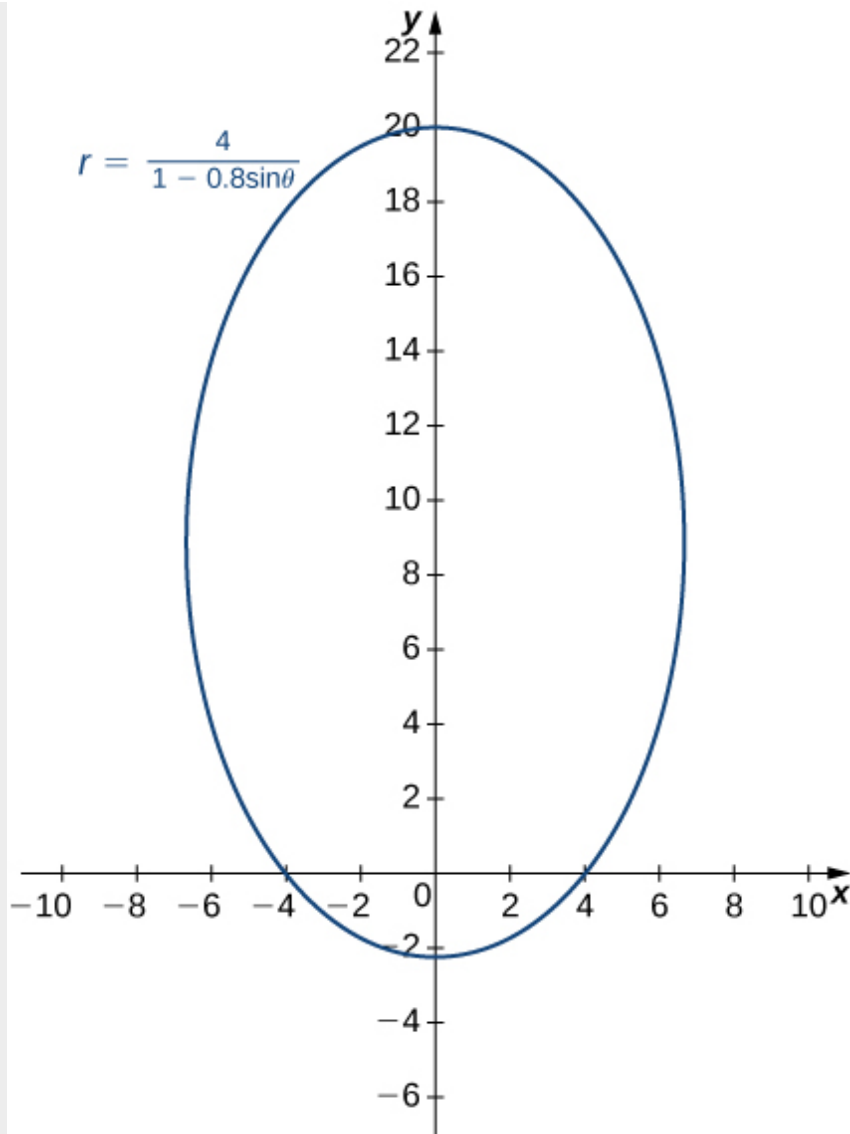
Identify and create a graph of the conic section described by the equation

Equation:

$$r = \frac{4}{1 - 0.8 \sin \theta}.$$

Solution:

Here $e = 0.8$ and $p = 5$. This conic section is an ellipse.



Hint

First find the values of e and p , and then create a table of values.

General Equations of Degree Two

A general equation of degree two can be written in the form

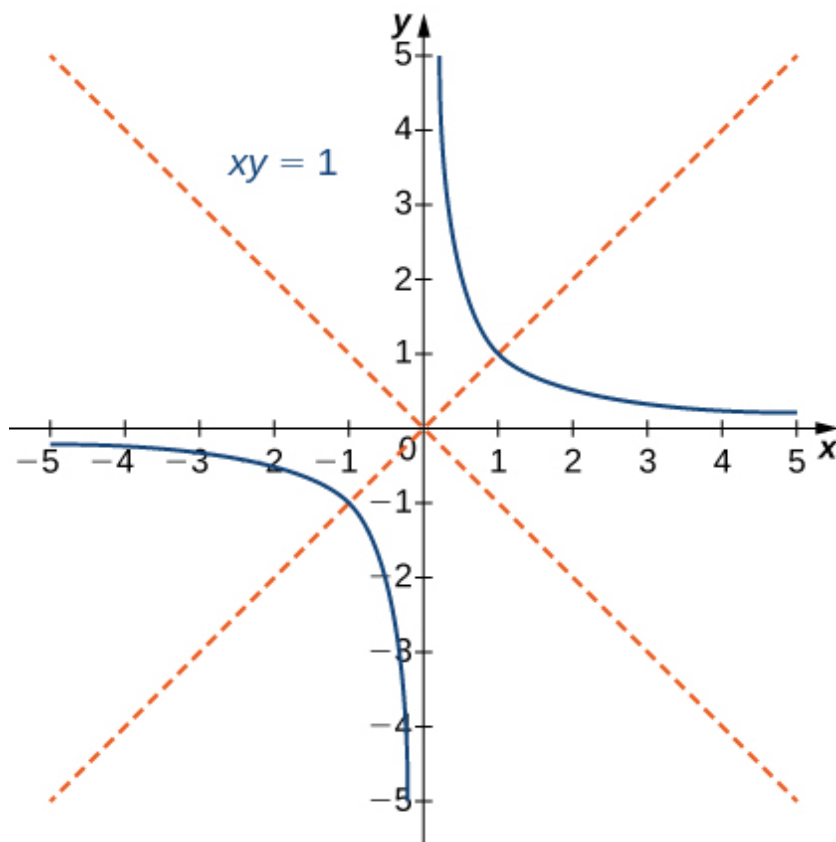
Equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The graph of an equation of this form is a conic section. If $B \neq 0$ then the coordinate axes are rotated. To identify the conic section, we use the **discriminant** of the conic section $4AC - B^2$. One of the following cases must be true:

1. $4AC - B^2 > 0$. If so, the graph is an ellipse.
2. $4AC - B^2 = 0$. If so, the graph is a parabola.
3. $4AC - B^2 < 0$. If so, the graph is a hyperbola.

The simplest example of a second-degree equation involving a cross term is $xy = 1$. This equation can be solved for y to obtain $y = \frac{1}{x}$. The graph of this function is called a *rectangular hyperbola* as shown.



Graph of the equation $xy = 1$; The red lines indicate the rotated axes.

The asymptotes of this hyperbola are the x and y coordinate axes. To determine the angle θ of rotation of the conic section, we use the formula $\cot 2\theta = \frac{A-C}{B}$. In this case $A = C = 0$ and $B = 1$, so $\cot 2\theta = (0 - 0)/1 = 0$ and $\theta = 45^\circ$. The method for graphing a conic section with rotated axes involves determining the coefficients of the conic in the rotated coordinate system. The new coefficients are labeled A', B', C', D', E' , and F' , and are given by the formulas

Equation:

$$A' = A \cos^2 \theta + B \cos \theta \sin \theta + C \sin^2 \theta$$

$$B' = 0$$

$$C' = A \sin^2 \theta - B \sin \theta \cos \theta + C \cos^2 \theta$$

$$D' = D \cos \theta + E \sin \theta$$

$$E' = -D \sin \theta + E \cos \theta$$

$$F' = F.$$

The procedure for graphing a rotated conic is the following:

1. Identify the conic section using the discriminant $4AC - B^2$.
2. Determine θ using the formula $\cot 2\theta = \frac{A-C}{B}$.
3. Calculate A', B', C', D', E' , and F' .
4. Rewrite the original equation using A', B', C', D', E' , and F' .
5. Draw a graph using the rotated equation.

Example:

Exercise:

Problem:

Identifying a Rotated Conic

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

Equation:

$$13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0.$$

Solution:

In this equation, $A = 13$, $B = -6\sqrt{3}$, $C = 7$, $D = 0$, $E = 0$, and $F = -256$. The discriminant of this equation is

$$4AC - B^2 = 4(13)(7) - (-6\sqrt{3})^2 = 364 - 108 = 256.$$

Therefore this conic is an ellipse. To calculate the angle of rotation of the axes, use $\cot 2\theta = \frac{A-C}{B}$. This gives

Equation:

$$\begin{aligned}\cot 2\theta &= \frac{A-C}{B} \\ &= \frac{13-7}{-6\sqrt{3}} \\ &= -\frac{\sqrt{3}}{3}.\end{aligned}$$

Therefore $2\theta = 120^\circ$ and $\theta = 60^\circ$, which is the angle of the rotation of the axes.

To determine the rotated coefficients, use the formulas given above:

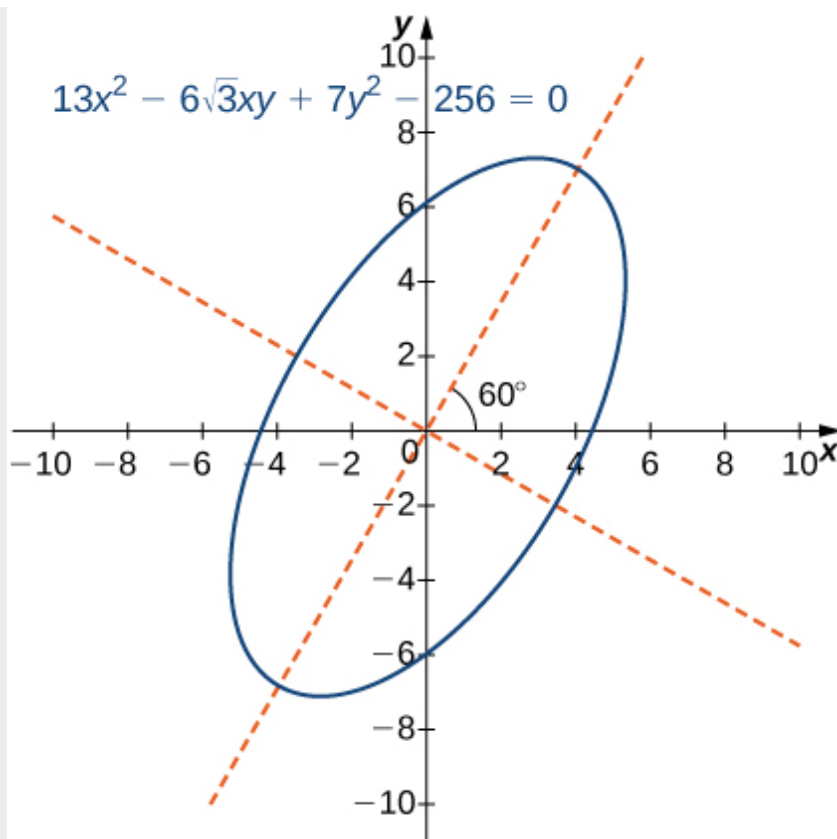
Equation:

$$\begin{aligned}
A' &= A \cos^2 \theta + B \cos \theta \sin \theta + C \sin^2 \theta \\
&= 13 \cos^2 60 + (-6\sqrt{3}) \cos 60 \sin 60 + 7 \sin^2 60 \\
&= 13 \left(\frac{1}{2}\right)^2 - 6\sqrt{3} \left(\frac{1}{2}\right) \left(\frac{\sqrt{3}}{2}\right) + 7 \left(\frac{\sqrt{3}}{2}\right)^2 \\
&= 4, \\
B' &= 0, \\
C' &= A \sin^2 \theta - B \sin \theta \cos \theta + C \cos^2 \theta \\
&= 13 \sin^2 60 + (-6\sqrt{3}) \sin 60 \cos 60 = 7 \cos^2 60 \\
&= \left(\frac{\sqrt{3}}{2}\right)^2 + 6\sqrt{3} \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{2}\right) + 7 \left(\frac{1}{2}\right)^2 \\
&= 16, \\
D' &= D \cos \theta + E \sin \theta \\
&= (0) \cos 60 + (0) \sin 60 \\
&= 0, \\
E' &= -D \sin \theta + E \cos \theta \\
&= -(0) \sin 60 + (0) \cos 60 \\
&= 0, \\
F' &= F \\
&= -256.
\end{aligned}$$

The equation of the conic in the rotated coordinate system becomes
Equation:

$$\begin{aligned}
4(x')^2 + 16(y')^2 &= 256 \\
\frac{(x')^2}{64} + \frac{(y')^2}{16} &= 1.
\end{aligned}$$

A graph of this conic section appears as follows.



Graph of the ellipse described by the equation $13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0$. The axes are rotated 60° . The red dashed lines indicate the rotated axes.

Note:

Exercise:

Problem:

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

Equation:

$$3x^2 + 5xy - 2y^2 - 125 = 0.$$

Solution:

The conic is a hyperbola and the angle of rotation of the axes is $\theta = 22.5^\circ$.

Hint

Follow steps 1 and 2 of the five-step method outlined above.

Key Concepts

- The equation of a vertical parabola in standard form with given focus and directrix is $y = \frac{1}{4p}(x - h)^2 + k$ where p is the distance from the vertex to the focus and (h, k) are the coordinates of the vertex.
- The equation of a horizontal ellipse in standard form is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where the center has coordinates (h, k) , the major axis has length $2a$, the minor axis has length $2b$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$.
- The equation of a horizontal hyperbola in standard form is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ where the center has coordinates (h, k) , the vertices are located at $(h \pm a, k)$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 + b^2$.
- The eccentricity of an ellipse is less than 1, the eccentricity of a parabola is equal to 1, and the eccentricity of a hyperbola is greater than 1. The eccentricity of a circle is 0.
- The polar equation of a conic section with eccentricity e is $r = \frac{ep}{1 \pm e \cos \theta}$ or $r = \frac{ep}{1 \pm e \sin \theta}$, where p represents the focal parameter.
- To identify a conic generated by the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, first calculate the discriminant $D = 4AC - B^2$. If $D > 0$ then the conic is an ellipse, if $D = 0$ then the conic is a parabola, and if $D < 0$ then the conic is a hyperbola.

For the following exercises, determine the equation of the parabola using the information given.

Exercise:

Problem: Focus $(4, 0)$ and directrix $x = -4$

Solution:

$$y^2 = 16x$$

Exercise:

Problem: Focus $(0, -3)$ and directrix $y = 3$

Exercise:

Problem: Focus $(0, 0.5)$ and directrix $y = -0.5$

Solution:

$$x^2 = 2y$$

Exercise:

Problem: Focus $(2, 3)$ and directrix $x = -2$

Exercise:

Problem: Focus $(0, 2)$ and directrix $y = 4$

Solution:

$$x^2 = -4(y - 3)$$

Exercise:

Problem: Focus $(-1, 4)$ and directrix $x = 5$

Exercise:

Problem: Focus $(-3, 5)$ and directrix $y = 1$

Solution:

$$(x + 3)^2 = 8(y - 3)$$

Exercise:

Problem: Focus $(\frac{5}{2}, -4)$ and directrix $x = \frac{7}{2}$

For the following exercises, determine the equation of the ellipse using the information given.

Exercise:

Problem:

Endpoints of major axis at $(4, 0)$, $(-4, 0)$ and foci located at $(2, 0)$, $(-2, 0)$

Solution:

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

Exercise:

Problem:

Endpoints of major axis at $(0, 5)$, $(0, -5)$ and foci located at $(0, 3)$, $(0, -3)$

Exercise:

Problem:

Endpoints of major axis at $(0, 2)$, $(0, -2)$ and foci located at $(3, 0)$, $(-3, 0)$

Solution:

$$\frac{x^2}{13} + \frac{y^2}{4} = 1$$

Exercise:

Problem:

Endpoints of major axis at $(-3, 3)$, $(7, 3)$ and foci located at $(-2, 3)$, $(6, 3)$

Exercise:

Problem:

Endpoints of major axis at $(-3, 5)$, $(-3, -3)$ and foci located at $(-3, 3)$, $(-3, -1)$

Solution:

$$\frac{(y-1)^2}{16} + \frac{(x+3)^2}{12} = 1$$

Exercise:

Problem:

Endpoints of major axis at $(0, 0)$, $(0, 4)$ and foci located at $(5, 2)$, $(-5, 2)$

Exercise:

Problem: Foci located at $(2, 0)$, $(-2, 0)$ and eccentricity of $\frac{1}{2}$

Solution:

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

Exercise:

Problem: Foci located at $(0, -3)$, $(0, 3)$ and eccentricity of $\frac{3}{4}$

For the following exercises, determine the equation of the hyperbola using the information given.

Exercise:

Problem:

Vertices located at $(5, 0)$, $(-5, 0)$ and foci located at $(6, 0)$, $(-6, 0)$

Solution:

$$\frac{x^2}{25} - \frac{y^2}{11} = 1$$

Exercise:

Problem:

Vertices located at $(0, 2)$, $(0, -2)$ and foci located at $(0, 3)$, $(0, -3)$

Exercise:

Problem:

Endpoints of the conjugate axis located at $(0, 3)$, $(0, -3)$ and foci located $(4, 0)$, $(-4, 0)$

Solution:

$$\frac{x^2}{7} - \frac{y^2}{9} = 1$$

Exercise:

Problem: Vertices located at $(0, 1)$, $(6, 1)$ and focus located at $(8, 1)$

Exercise:

Problem:

Vertices located at $(-2, 0)$, $(-2, -4)$ and focus located at $(-2, -8)$

Solution:

$$\frac{(y+2)^2}{4} - \frac{(x+2)^2}{32} = 1$$

Exercise:

Problem:

Endpoints of the conjugate axis located at $(3, 2)$, $(3, 4)$ and focus located at $(3, 7)$

Exercise:

Problem: Foci located at $(6, -0)$, $(6, 0)$ and eccentricity of 3

Solution:

$$\frac{x^2}{4} - \frac{y^2}{32} = 1$$

Exercise:

Problem: $(0, 10)$, $(0, -10)$ and eccentricity of 2.5

For the following exercises, consider the following polar equations of conics. Determine the eccentricity and identify the conic.

Exercise:

Problem: $r = \frac{-1}{1+\cos \theta}$

Solution:

$$e = 1, \text{ parabola}$$

Exercise:

Problem: $r = \frac{8}{2-\sin \theta}$

Exercise:

Problem: $r = \frac{5}{2+\sin \theta}$

Solution:

$$e = \frac{1}{2}, \text{ ellipse}$$

Exercise:

Problem: $r = \frac{5}{-1+2 \sin \theta}$

Exercise:

Problem: $r = \frac{3}{2-6 \sin \theta}$

Solution:

$$e = 3, \text{ hyperbola}$$

Exercise:

Problem: $r = \frac{3}{-4+3 \sin \theta}$

For the following exercises, find a polar equation of the conic with focus at the origin and eccentricity and directrix as given.

Exercise:

Problem: Directrix: $x = 4$; $e = \frac{1}{5}$

Solution:

$$r = \frac{4}{5+\cos \theta}$$

Exercise:

Problem: Directrix: $x = -4$; $e = 5$

Exercise:

Problem: Directrix: $y = 2$; $e = 2$

Solution:

$$r = \frac{4}{1+2 \sin \theta}$$

Exercise:

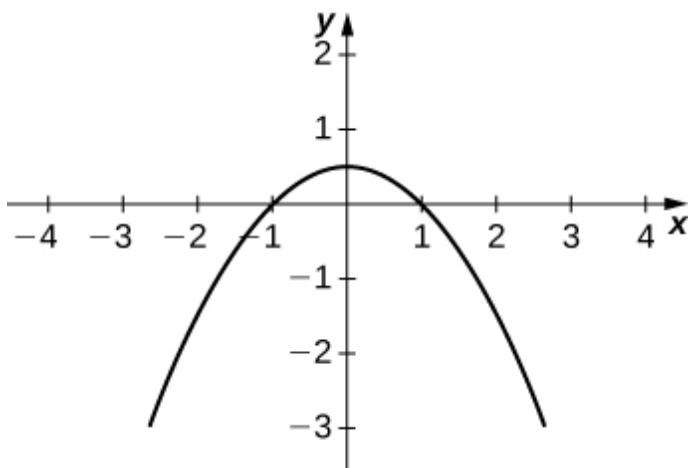
Problem: Directrix: $y = -2$; $e = \frac{1}{2}$

For the following exercises, sketch the graph of each conic.

Exercise:

Problem: $r = \frac{1}{1+\sin \theta}$

Solution:



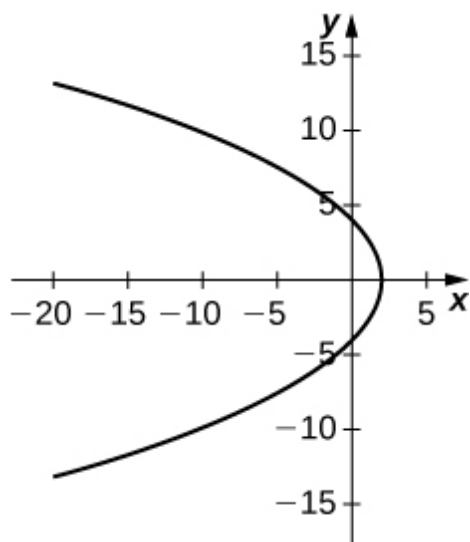
Exercise:

Problem: $r = \frac{1}{1-\cos \theta}$

Exercise:

Problem: $r = \frac{4}{1+\cos \theta}$

Solution:



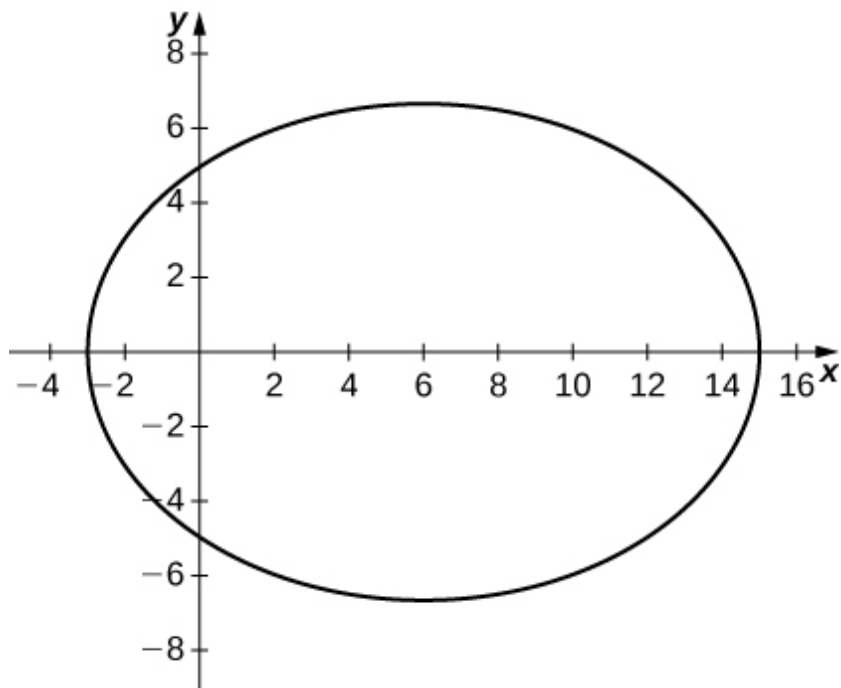
Exercise:

Problem: $r = \frac{10}{5+4 \sin \theta}$

Exercise:

Problem: $r = \frac{15}{3-2 \cos \theta}$

Solution:



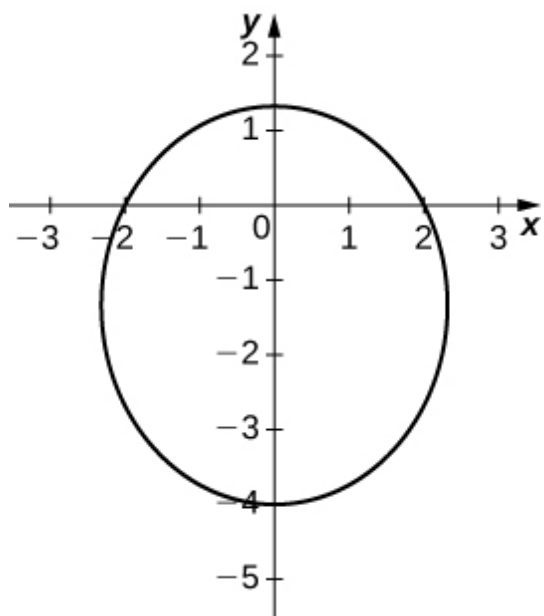
Exercise:

Problem: $r = \frac{32}{3+5 \sin \theta}$

Exercise:

Problem: $r(2 + \sin \theta) = 4$

Solution:



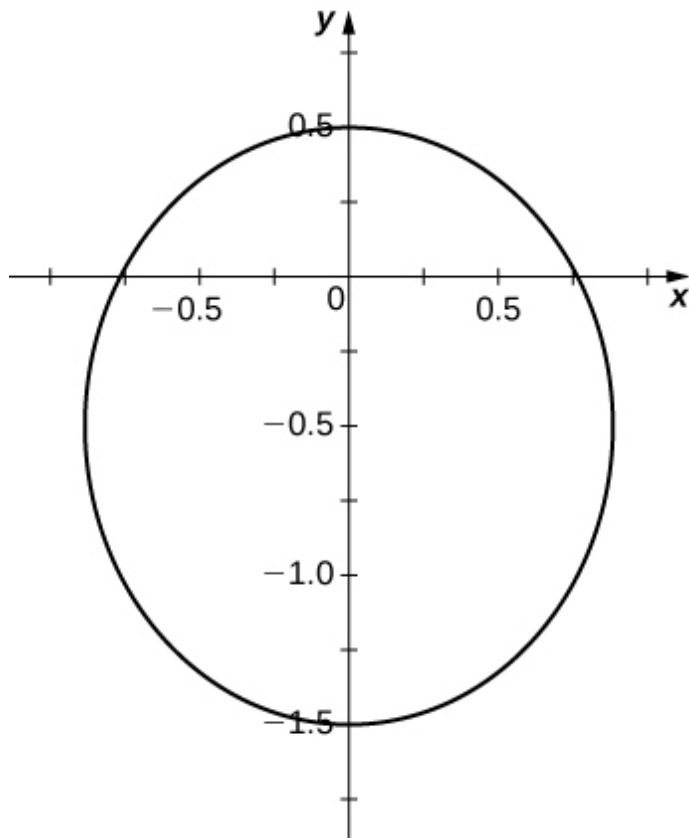
Exercise:

Problem: $r = \frac{3}{2+6 \sin \theta}$

Exercise:

Problem: $r = \frac{3}{-4+2 \sin \theta}$

Solution:



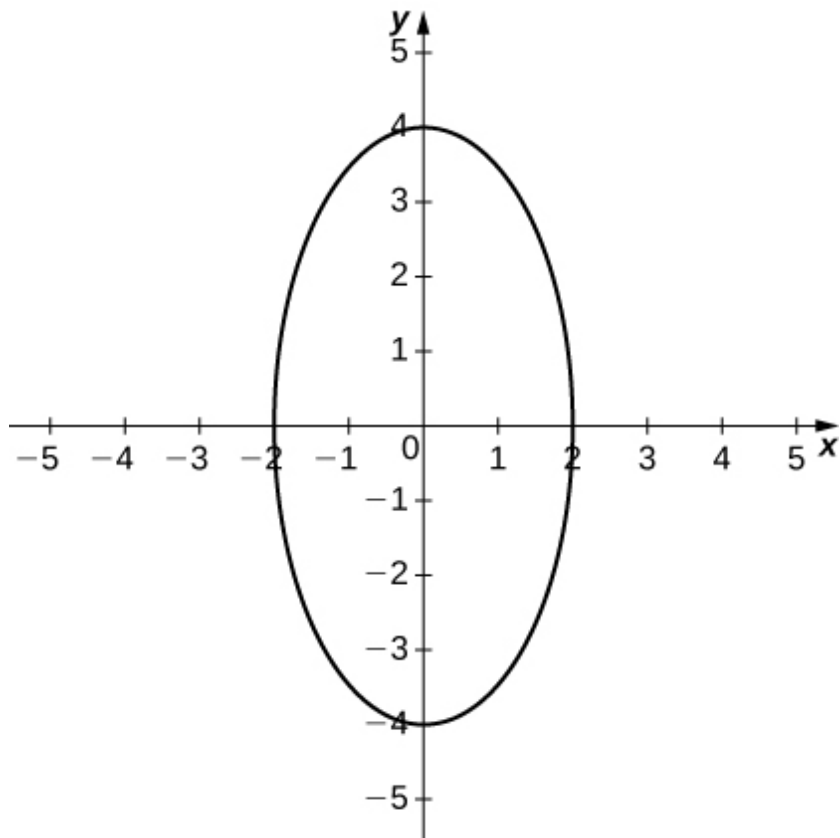
Exercise:

Problem: $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Exercise:

Problem: $\frac{x^2}{4} + \frac{y^2}{16} = 1$

Solution:



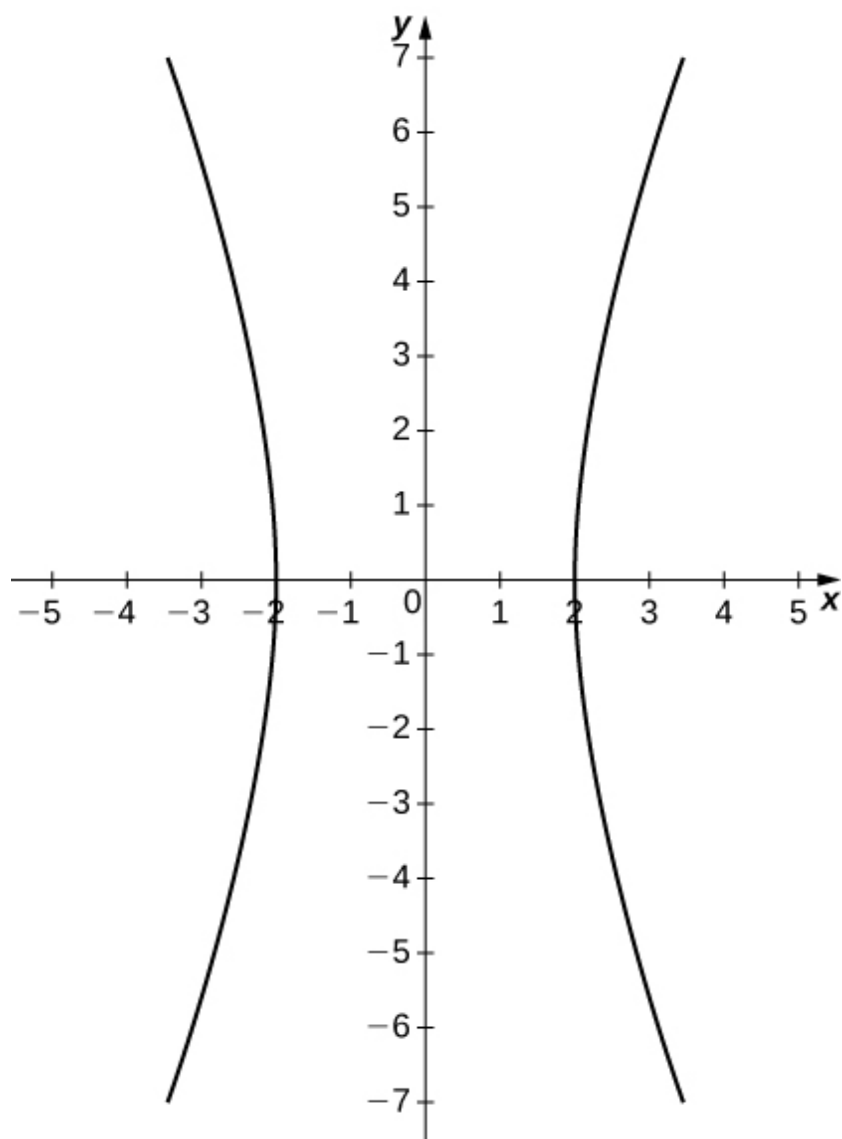
Exercise:

Problem: $4x^2 + 9y^2 = 36$

Exercise:

Problem: $25x^2 - 4y^2 = 100$

Solution:



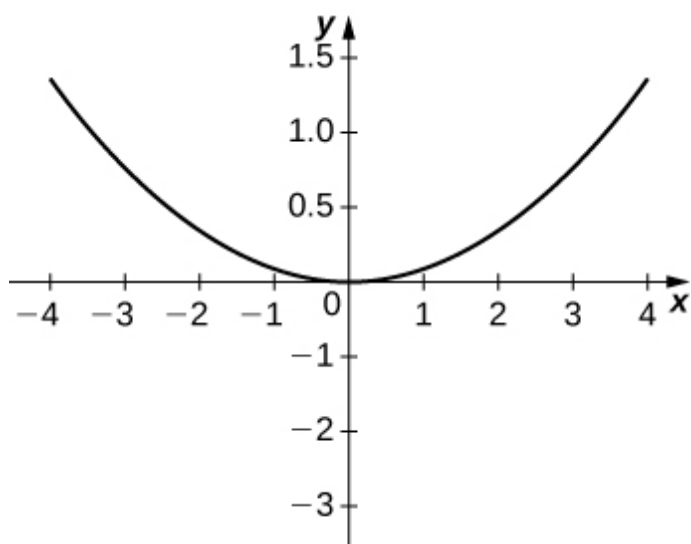
Exercise:

Problem: $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Exercise:

Problem: $x^2 = 12y$

Solution:



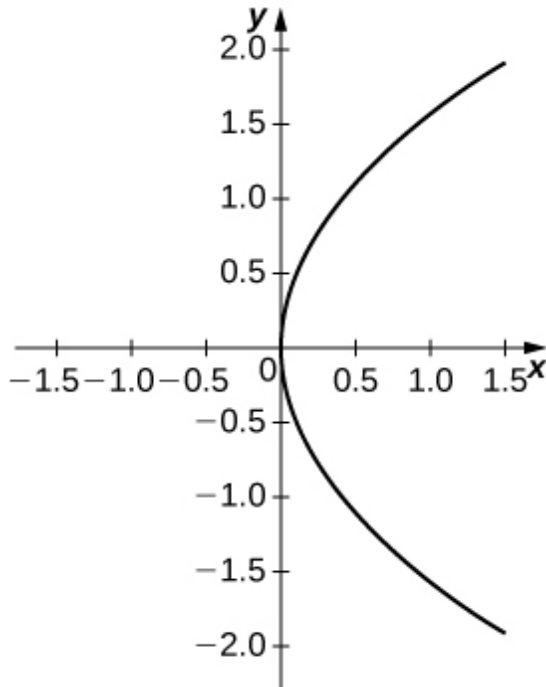
Exercise:

Problem: $y^2 = 20x$

Exercise:

Problem: $12x = 5y^2$

Solution:



For the following equations, determine which of the conic sections is described.

Exercise:

Problem: $xy = 4$

Exercise:

Problem: $x^2 + 4xy - 2y^2 - 6 = 0$

Solution:

Hyperbola

Exercise:

Problem: $x^2 + 2\sqrt{3}xy + 3y^2 - 6 = 0$

Exercise:

Problem: $x^2 - xy + y^2 - 2 = 0$

Solution:

Ellipse

Exercise:

Problem: $34x^2 - 24xy + 41y^2 - 25 = 0$

Exercise:

Problem: $52x^2 - 72xy + 73y^2 + 40x + 30y - 75 = 0$

Solution:

Ellipse

Exercise:

Problem:

The mirror in an automobile headlight has a parabolic cross section, with the lightbulb at the focus. On a schematic, the equation of the parabola is given as $x^2 = 4y$. At what coordinates should you place the lightbulb?

Exercise:

Problem:

A satellite dish is shaped like a paraboloid of revolution. The receiver is to be located at the focus. If the dish is 12 feet across at its opening and 4 feet deep at its center, where should the receiver be placed?

Solution:

At the point 2.25 feet above the vertex.

Exercise:

Problem:

Consider the satellite dish of the preceding problem. If the dish is 8 feet across at the opening and 2 feet deep, where should we place the receiver?

Exercise:**Problem:**

A searchlight is shaped like a paraboloid of revolution. A light source is located 1 foot from the base along the axis of symmetry. If the opening of the searchlight is 3 feet across, find the depth.

Solution:

0.5625 feet

Exercise:**Problem:**

Whispering galleries are rooms designed with elliptical ceilings. A person standing at one focus can whisper and be heard by a person standing at the other focus because all the sound waves that reach the ceiling are reflected to the other person. If a whispering gallery has a length of 120 feet and the foci are located 30 feet from the center, find the height of the ceiling at the center.

Exercise:**Problem:**

A person is standing 8 feet from the nearest wall in a whispering gallery. If that person is at one focus and the other focus is 80 feet away, what is the length and the height at the center of the gallery?

Solution:

Length is 96 feet and height is approximately 26.53 feet.

For the following exercises, determine the polar equation form of the orbit given the length of the major axis and eccentricity for the orbits of the comets or planets. Distance is given in astronomical units (AU).

Exercise:

Problem:

Halley's Comet: length of major axis = 35.88, eccentricity = 0.967

Exercise:

Problem:

Hale-Bopp Comet: length of major axis = 525.91, eccentricity = 0.995

Solution:

$$r = \frac{2.616}{1+0.995 \cos \theta}$$

Exercise:

Problem: Mars: length of major axis = 3.049, eccentricity = 0.0934

Exercise:

Problem: Jupiter: length of major axis = 10.408, eccentricity = 0.0484

Solution:

$$r = \frac{5.192}{1+0.0484 \cos \theta}$$

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

Exercise:

Problem:

The rectangular coordinates of the point $\left(4, \frac{5\pi}{6}\right)$ are $\left(2\sqrt{3}, -2\right)$.

Exercise:**Problem:**

The equations $x = \cosh(3t)$, $y = 2 \sinh(3t)$ represent a hyperbola.

Solution:

True.

Exercise:**Problem:**

The arc length of the spiral given by $r = \frac{\theta}{2}$ for $0 \leq \theta \leq 3\pi$ is $\frac{9}{4}\pi^3$.

Exercise:**Problem:**

Given $x = f(t)$ and $y = g(t)$, if $\frac{dx}{dy} = \frac{dy}{dx}$, then $f(t) = g(t) + C$, where C is a constant.

Solution:

False. Imagine $y = t + 1$, $x = -t + 1$.

For the following exercises, sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

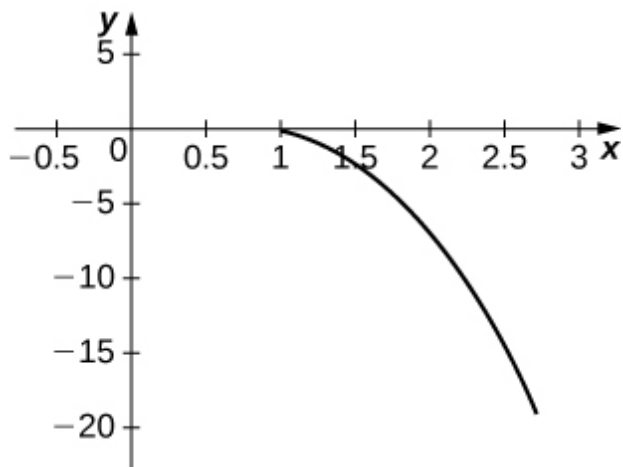
Exercise:

Problem: $x = 1 + t$, $y = t^2 - 1$, $-1 \leq t \leq 1$

Exercise:

Problem: $x = e^t$, $y = 1 - e^{3t}$, $0 \leq t \leq 1$

Solution:



$$y = 1 - x^3$$

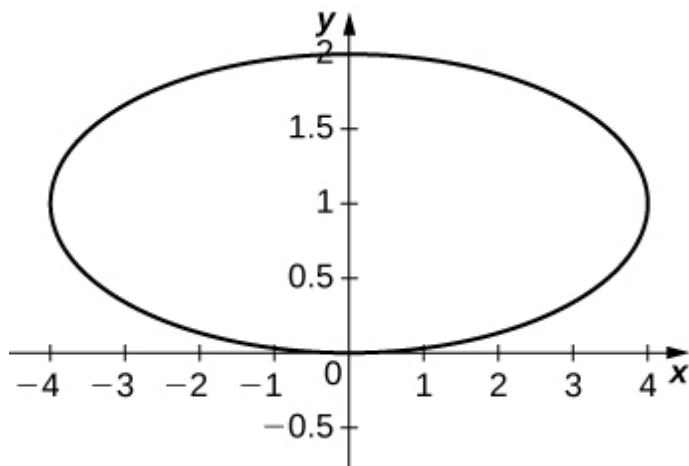
Exercise:

Problem: $x = \sin \theta, y = 1 - \csc \theta, 0 \leq \theta \leq 2\pi$

Exercise:

Problem: $x = 4 \cos \phi, y = 1 - \sin \phi, 0 \leq \phi \leq 2\pi$

Solution:



$$\frac{x^2}{16} + (y - 1)^2 = 1$$

For the following exercises, sketch the polar curve and determine what type of symmetry exists, if any.

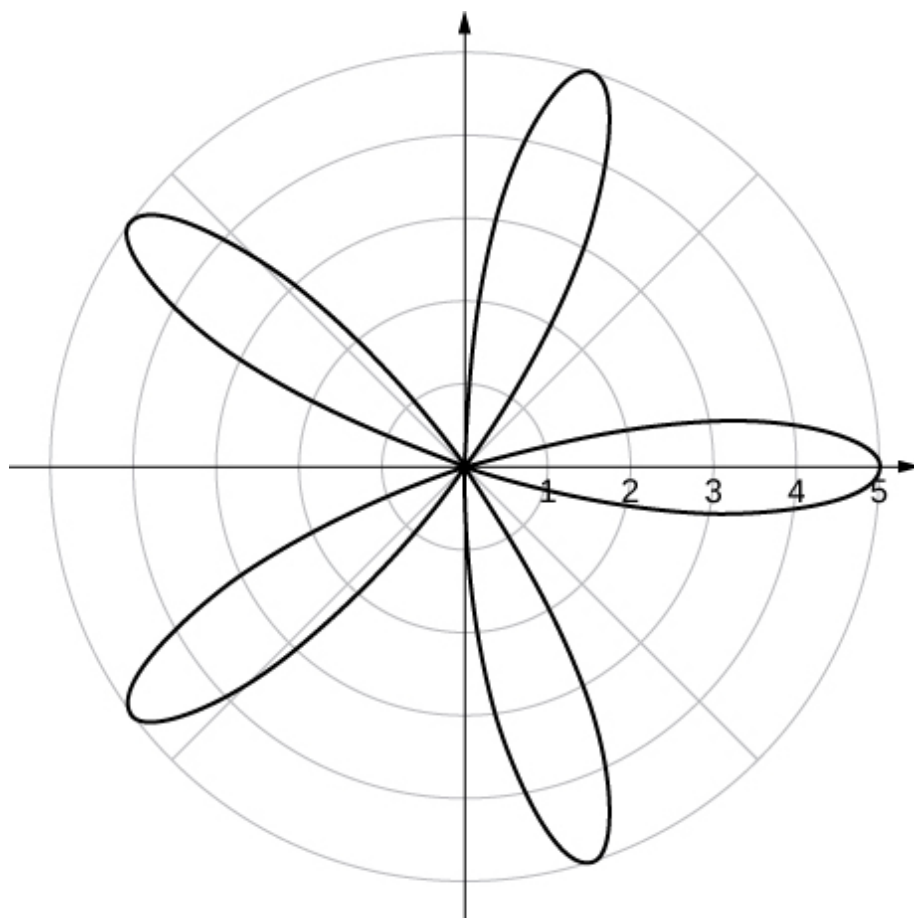
Exercise:

Problem: $r = 4 \sin \left(\frac{\theta}{3} \right)$

Exercise:

Problem: $r = 5 \cos (5\theta)$

Solution:



Symmetric about polar axis

For the following exercises, find the polar equation for the curve given as a Cartesian equation.

Exercise:

Problem: $x + y = 5$

Exercise:

Problem: $y^2 = 4 + x^2$

Solution:

$$r^2 = \frac{4}{\sin^2\theta - \cos^2\theta}$$

For the following exercises, find the equation of the tangent line to the given curve. Graph both the function and its tangent line.

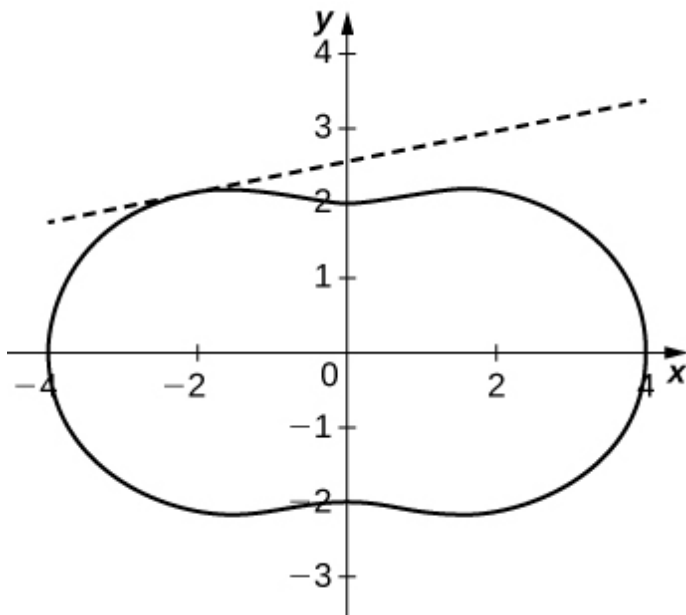
Exercise:

Problem: $x = \ln(t), y = t^2 - 1, t = 1$

Exercise:

Problem: $r = 3 + \cos(2\theta), \theta = \frac{3\pi}{4}$

Solution:



$$y = \frac{3\sqrt{2}}{2} + \frac{1}{5} \left(x + \frac{3\sqrt{2}}{2} \right)$$

Exercise:

Problem: Find $\frac{dy}{dx}$, $\frac{dx}{dy}$, and $\frac{d^2x}{dy^2}$ of $y = (2 + e^{-t})$, $x = 1 - \sin(t)$

For the following exercises, find the area of the region.

Exercise:

Problem: $x = t^2$, $y = \ln(t)$, $0 \leq t \leq e$

Solution:

$$\frac{e^2}{2}$$

Exercise:

Problem: $r = 1 - \sin \theta$ in the first quadrant

For the following exercises, find the arc length of the curve over the given interval.

Exercise:

Problem: $x = 3t + 4, y = 9t - 2, 0 \leq t \leq 3$

Solution:

$$9\sqrt{10}$$

Exercise:

Problem: $r = 6 \cos \theta, 0 \leq \theta \leq 2\pi$. Check your answer by geometry.

For the following exercises, find the Cartesian equation describing the given shapes.

Exercise:

Problem: A parabola with focus $(2, -5)$ and directrix $x = 6$

Solution:

$$(y + 5)^2 = -8x + 32$$

Exercise:

Problem:

An ellipse with a major axis length of 10 and foci at $(-7, 2)$ and $(1, 2)$

Exercise:

Problem:

A hyperbola with vertices at $(3, -2)$ and $(-5, -2)$ and foci at $(-2, -6)$ and $(-2, 4)$

Solution:

$$\frac{(y+1)^2}{16} - \frac{(x+2)^2}{9} = 1$$

For the following exercises, determine the eccentricity and identify the conic. Sketch the conic.

Exercise:

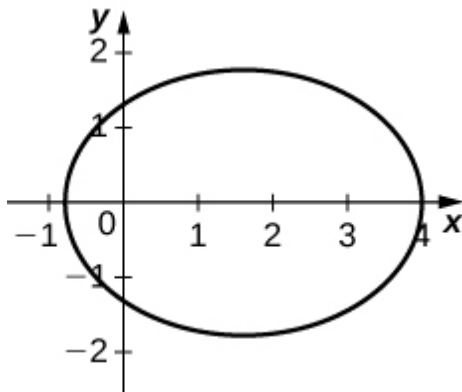
Problem: $r = \frac{6}{1+3\cos(\theta)}$

Exercise:

Problem: $r = \frac{4}{3-2\cos\theta}$

Solution:

$e = \frac{2}{3}$, ellipse



Exercise:

Problem: $r = \frac{7}{5-5\cos\theta}$

Exercise:

Problem:

Determine the Cartesian equation describing the orbit of Pluto, the most eccentric orbit around the Sun. The length of the major axis is 39.26 AU and minor axis is 38.07 AU. What is the eccentricity?

Solution:

$$\frac{y^2}{19.03^2} + \frac{x^2}{19.63^2} = 1, e = 0.2447$$

Exercise:**Problem:**

The C/1980 E1 comet was observed in 1980. Given an eccentricity of 1.057 and a perihelion (point of closest approach to the Sun) of 3.364 AU, find the Cartesian equations describing the comet's trajectory. Are we guaranteed to see this comet again? (*Hint:* Consider the Sun at point (0, 0).)

Glossary

conic section

a conic section is any curve formed by the intersection of a plane with a cone of two nappes

directrix

a directrix (plural: directrices) is a line used to construct and define a conic section; a parabola has one directrix; ellipses and hyperbolas have two

discriminant

the value $4AC - B^2$, which is used to identify a conic when the equation contains a term involving xy , is called a discriminant

focus

a focus (plural: foci) is a point used to construct and define a conic section; a parabola has one focus; an ellipse and a hyperbola have two

eccentricity

the eccentricity is defined as the distance from any point on the conic section to its focus divided by the perpendicular distance from that point to the nearest directrix

focal parameter

the focal parameter is the distance from a focus of a conic section to the nearest directrix

general form

an equation of a conic section written as a general second-degree equation

major axis

the major axis of a conic section passes through the vertex in the case of a parabola or through the two vertices in the case of an ellipse or hyperbola; it is also an axis of symmetry of the conic; also called the transverse axis

minor axis

the minor axis is perpendicular to the major axis and intersects the major axis at the center of the conic, or at the vertex in the case of the parabola; also called the conjugate axis

nappe

a nappe is one half of a double cone

standard form

an equation of a conic section showing its properties, such as location of the vertex or lengths of major and minor axes

vertex

a vertex is an extreme point on a conic section; a parabola has one vertex at its turning point. An ellipse has two vertices, one at each end of the major axis; a hyperbola has two vertices, one at the turning point of each branch

Introduction

class="introduction"

A garden
is to be
constructe
d adjoining
a rock
wall. What
dimensions
would
maximize
the size of
the
garden?
(credit:
Flickr)



A rectangular garden is to be constructed using an existing rock wall as one side of the garden and wire fencing for the other three sides. Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. (See [\[link\]](#).)

Making the best use of resources is one of the major uses of differentiation. In this chapter, you will investigate how derivatives can be used in optimization.

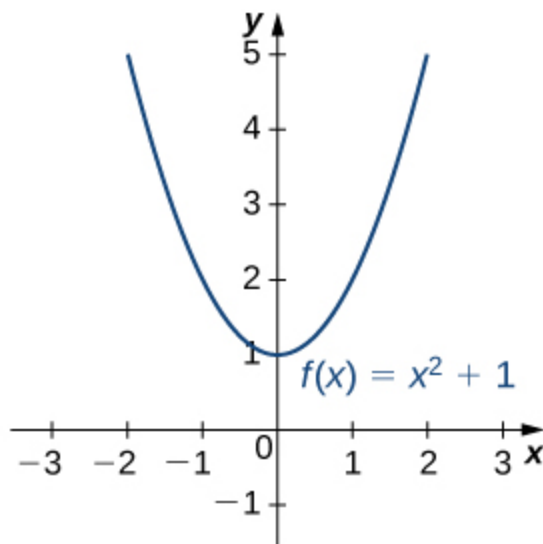
Maxima and Minima

- Define absolute extrema.
- Define local extrema.
- Explain how to find the critical points of a function over a closed interval.
- Describe how to use critical points to locate absolute extrema over a closed interval.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \rightarrow \pm\infty$, $f(x) \rightarrow \infty$. Therefore, the function does not have a largest value. However, since $x^2 + 1 \geq 1$ for all real numbers x and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$. We say that 1 is the absolute minimum of $f(x) = x^2 + 1$ and it occurs at $x = 0$. We say that $f(x) = x^2 + 1$ does not have an absolute maximum (see the following figure).



The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.

Note:

Definition

Let f be a function defined over an interval I and let $c \in I$. We say f has an **absolute maximum** on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an **absolute minimum** on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c , we say f has an **absolute extremum** on I at c .

Before proceeding, let's note two important issues regarding this definition. First, the term *absolute* here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c , the absolute extremum is $f(c)$. The real number c is a point in the domain at which the absolute extremum

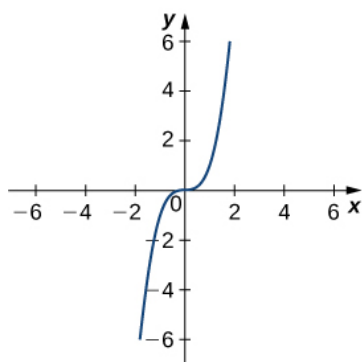
occurs. For example, consider the function $f(x) = 1/(x^2 + 1)$ over the interval $(-\infty, \infty)$. Since

Equation:

$$f(0) = 1 \geq \frac{1}{x^2 + 1} = f(x)$$

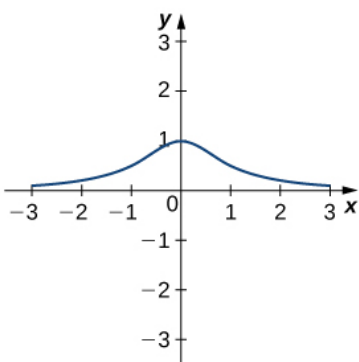
for all real numbers x , we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is $f(0) = 1$. It occurs at $x = 0$, as shown in [\[link\]](#)(b).

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. [\[link\]](#) shows several functions and some of the different possibilities regarding absolute extrema. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function f over a closed, bounded interval $[a, b]$ has both an absolute maximum and an absolute minimum.



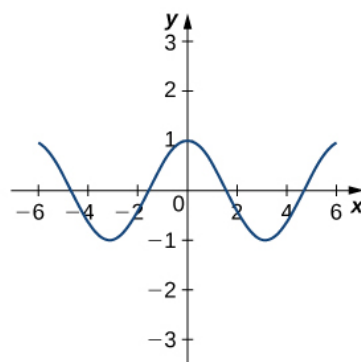
$f(x) = x^3$ on $(-\infty, \infty)$
No absolute maximum
No absolute minimum

(a)



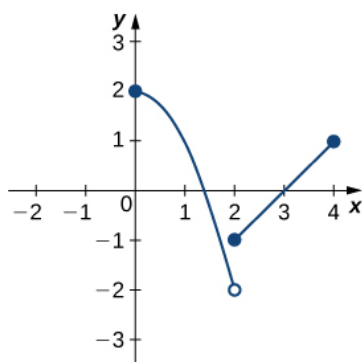
$f(x) = \frac{1}{x^2 + 1}$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0$
No absolute minimum

(b)



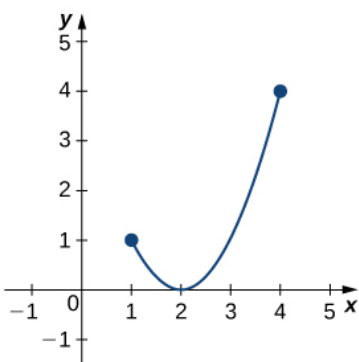
$f(x) = \cos(x)$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0, \pm 2\pi, \pm 4\pi \dots$
Absolute minimum of -1 at $x = \pm \pi, \pm 3\pi \dots$

(c)



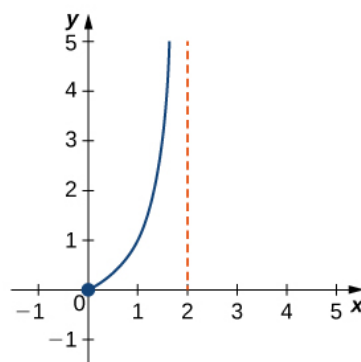
$f(x) = \begin{cases} 2 - x^2 & 0 \leq x < 2 \\ x - 3 & 2 \leq x \leq 4 \end{cases}$
Absolute maximum of 2 at $x = 0$
No absolute minimum

(d)



$f(x) = (x - 2)^2$ on $[1, 4]$
Absolute maximum of 4 at $x = 4$
Absolute minimum of 0 at $x = 2$

(e)



$f(x) = \frac{x}{2 - x}$ on $[0, 2)$
No absolute maximum
Absolute minimum of 0 at $x = 0$

(f)

Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty)$. Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

Note:

Extreme Value Theorem

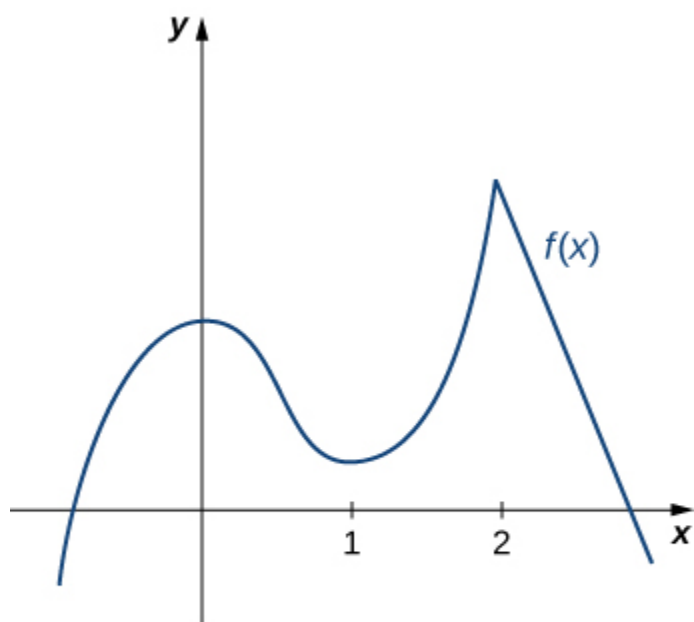
If f is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.

The proof of the extreme value theorem is beyond the scope of this text. Typically, it is proved in a course on real analysis. There are a couple of key points to note about the statement of this theorem. For the extreme value theorem to apply, the function must be continuous over a closed, bounded interval. If the interval I is open or the function has even one point of discontinuity, the function may not have an absolute maximum or absolute minimum over I . For example, consider the functions shown in [\[link\]](#)(d), (e), and (f). All three of these functions are defined over bounded intervals. However, the function in graph (e) is the only one that has both an absolute maximum and an absolute minimum over its domain. The extreme value theorem cannot be applied to the functions in graphs (d) and (f) because neither of these functions is continuous over a closed, bounded interval. Although the function in graph (d) is defined over the closed interval $[0, 4]$, the function is discontinuous at $x = 2$. The function has an absolute maximum over $[0, 4]$ but does not have an absolute minimum. The function in graph (f) is continuous over the half-open interval $[0, 2)$, but is not defined at $x = 2$, and therefore is not continuous over a closed, bounded interval. The function has an absolute minimum over $[0, 2)$, but does not have an absolute maximum over $[0, 2)$. These two graphs illustrate why a function over a bounded interval may fail to have an absolute maximum and/or absolute minimum.

Before looking at how to find absolute extrema, let's examine the related concept of local extrema. This idea is useful in determining where absolute extrema occur.

Local Extrema and Critical Points

Consider the function f shown in [\[link\]](#). The graph can be described as two mountains with a valley in the middle. The absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f , the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a local maximum at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.



$f(x)$ defined on $(-\infty, \infty)$

Local maxima at $x = 0$ and $x = 2$

Local minimum at $x = 1$

This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

Note:**Definition**

A function f has a **local maximum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$. A function f has a **local minimum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$. A function f has a **local extremum** at c if f has a local maximum at c or f has a local minimum at c .

Note that if f has an absolute extremum at c and f is defined over an interval containing c , then $f(c)$ is also considered a local extremum. If an absolute extremum for a function f occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum.

Given the graph of a function f , it is sometimes easy to see where a local maximum or local minimum occurs. However, it is not always easy to see, since the interesting features on the graph of a function may not be visible because they occur at a very small scale. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at [\[link\]](#) again. The local extrema occur at $x = 0$, $x = 1$, and $x = 2$. Notice that at $x = 0$ and $x = 1$, the derivative $f'(x) = 0$. At $x = 2$, the derivative $f'(x)$ does not exist, since the function f has a corner there. In fact, if f has a local extremum at a point $x = c$, the derivative $f'(c)$ must satisfy one of the following conditions: either $f'(c) = 0$ or $f'(c)$ is undefined. Such a value c is known as a critical point and it is important in finding extreme values for functions.

Note:**Definition**

Let c be an interior point in the domain of f . We say that c is a **critical point** of f if $f'(c) = 0$ or $f'(c)$ is undefined.

As mentioned earlier, if f has a local extremum at a point $x = c$, then c must be a critical point of f . This fact is known as **Fermat's theorem**.

Note:

Fermat's Theorem

If f has a local extremum at c and f is differentiable at c , then $f'(c) = 0$.

Proof

Suppose f has a local extremum at c and f is differentiable at c . We need to show that $f'(c) = 0$. To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, and therefore $f'(c) = 0$. Since f has a local extremum at c , f has a local maximum or local minimum at c . Suppose f has a local maximum at c . The case in which f has a local minimum at c can be handled similarly. There then exists an open interval I such that $f(c) \geq f(x)$ for all $x \in I$. Since f is differentiable at c , from the definition of the derivative, we know that

Equation:

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and equal $f'(c)$. Therefore,

Equation:

$$f'(c) = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c},$$

and

Equation:

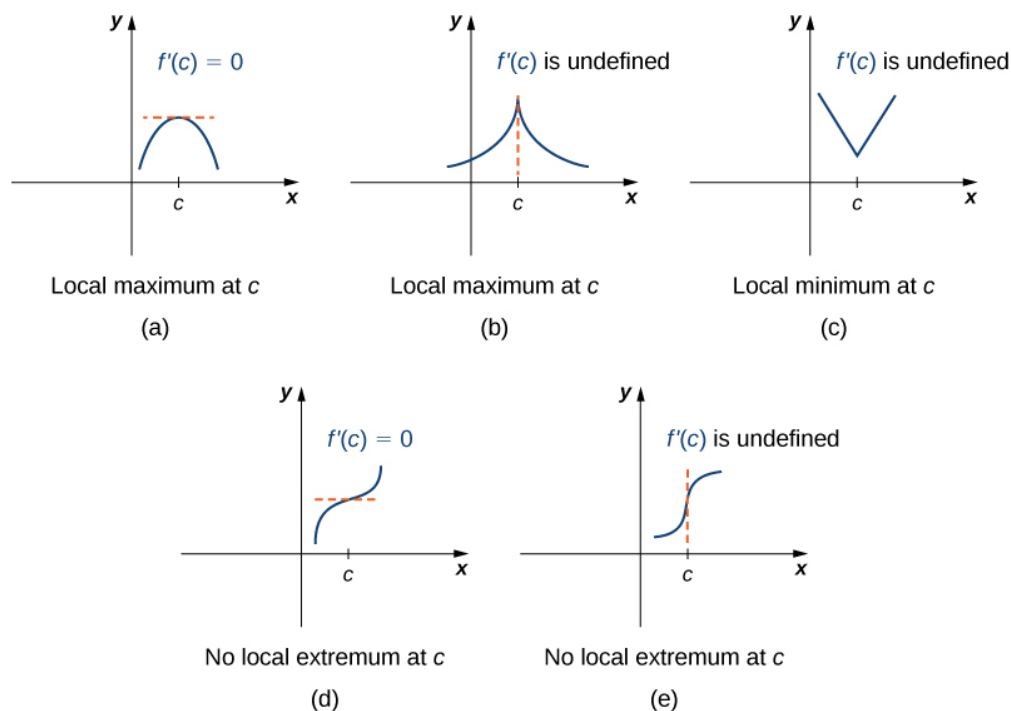
$$f'(c) = \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c}.$$

Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for x near c . Therefore, for x near c , but $x > c$, we have $\frac{f(x) - f(c)}{x - c} \leq 0$. From [\[link\]](#) we conclude that $f'(c) \leq 0$. Similarly, it can be shown that $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

□

From Fermat's theorem, we conclude that if f has a local extremum at c , then either $f'(c) = 0$ or $f'(c)$ is undefined. In other words, local extrema can only occur at critical points.

Note this theorem does not claim that a function f must have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema. For example, consider the function $f(x) = x^3$. We have $f'(x) = 3x^2 = 0$ when $x = 0$. Therefore, $x = 0$ is a critical point. However, $f(x) = x^3$ is increasing over $(-\infty, \infty)$, and thus f does not have a local extremum at $x = 0$. In [\[link\]](#), we see several different possibilities for critical points. In some of these cases, the functions have local extrema at critical points, whereas in other cases the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.



(a–e) A function f has a critical point at c if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example:

Exercise:

Problem:

Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

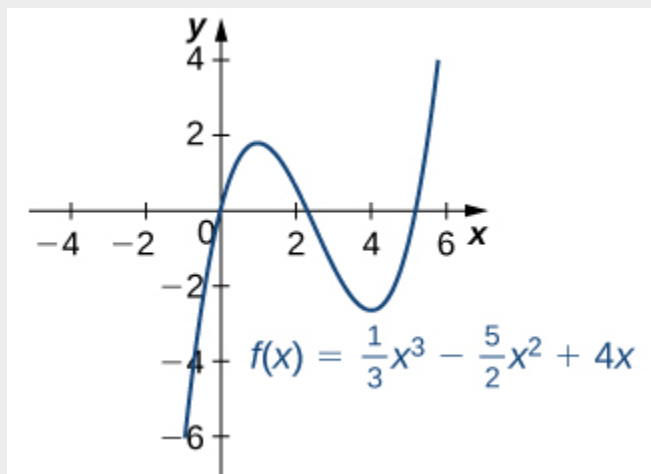
a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$

b. $f(x) = (x^2 - 1)^3$

c. $f(x) = \frac{4x}{1+x^2}$

Solution:

- a. The derivative $f'(x) = x^2 - 5x + 4$ is defined for all real numbers x . Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f'(x) = x^2 - 5x + 4 = (x - 4)(x - 1)$, the critical points are $x = 1$ and $x = 4$. From the graph of f in [\[link\]](#), we see that f has a local maximum at $x = 1$ and a local minimum at $x = 4$.

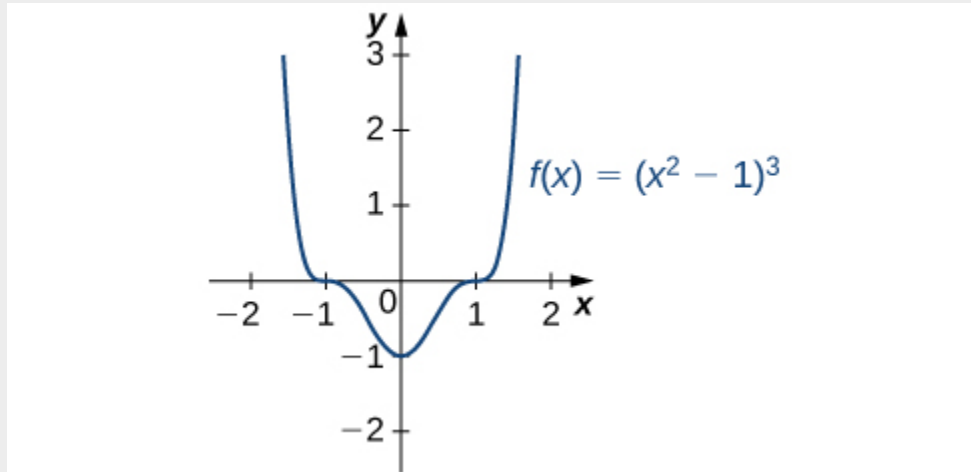


This function has a local maximum
and a local minimum.

- b. Using the chain rule, we see the derivative is
Equation:

$$f'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2.$$

Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical points are $x = 0, \pm 1$. From the graph of f in [\[link\]](#), we see that f has a local (and absolute) minimum at $x = 0$, but does not have a local extremum at $x = 1$ or $x = -1$.



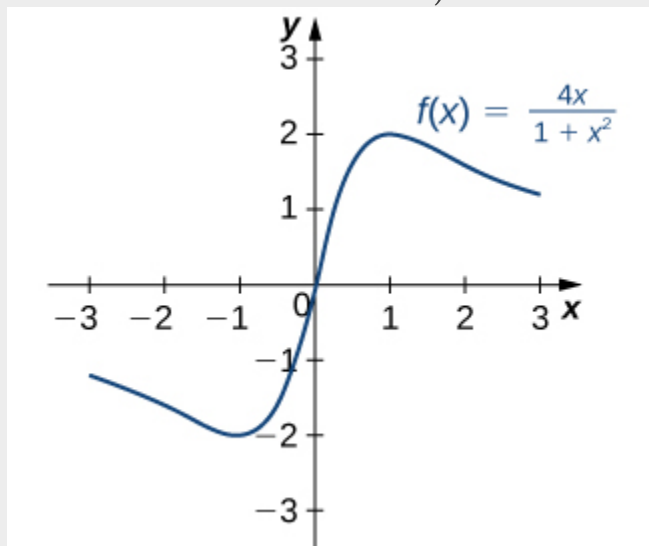
This function has three critical points: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.

c. By the chain rule, we see that the derivative is
Equation:

$$f'(x) = \frac{(1 + x^2)4 - 4x(2x)}{(1 + x^2)^2} = \frac{4 - 4x^2}{(1 + x^2)^2}.$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f'(x) = 0$. Solving $f'(x) = 0$, we see that $4 - 4x^2 = 0$, which implies $x = \pm 1$. Therefore, the critical points are $x = \pm 1$. From the graph of f in [\[link\]](#), we see that f

has an absolute maximum at $x = 1$ and an absolute minimum at $x = -1$. Hence, f has a local maximum at $x = 1$ and a local minimum at $x = -1$. (Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I , then f has a local extremum at c .)



This function has an absolute maximum and an absolute minimum.

Note:

Exercise:

Problem: Find all critical points for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$.

Solution:

$$x = -\frac{2}{3}, x = 1$$

Hint

Calculate $f'(x)$.

Locating Absolute Extrema

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in [\[link\]](#), one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by [\[link\]](#), the point c at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Note:

Location of Absolute Extrema

Let f be a continuous function over a closed, bounded interval I . The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I .

With this idea in mind, let's examine a procedure for locating absolute extrema.

Note:

Problem-Solving Strategy: Locating Absolute Extrema over a Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.

3. Compare all values found in (1) and (2). From [\[link\]](#), the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f . The smallest of these values is the absolute minimum of f .

Now let's look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

Example:

Exercise:

Problem:

Locating Absolute Extrema

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.

b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

Solution:

a. Step 1. Evaluate f at the endpoints $x = 1$ and $x = 3$.

Equation:

$$f(1) = 0 \text{ and } f(3) = -2$$

Step 2. Since $f'(x) = -2x + 3$, f' is defined for all real numbers x . Therefore, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0$. Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval

$[1, 3]$, $f\left(\frac{3}{2}\right)$ is a candidate for an absolute extremum of f over $[1, 3]$. We evaluate $f\left(\frac{3}{2}\right)$ and find

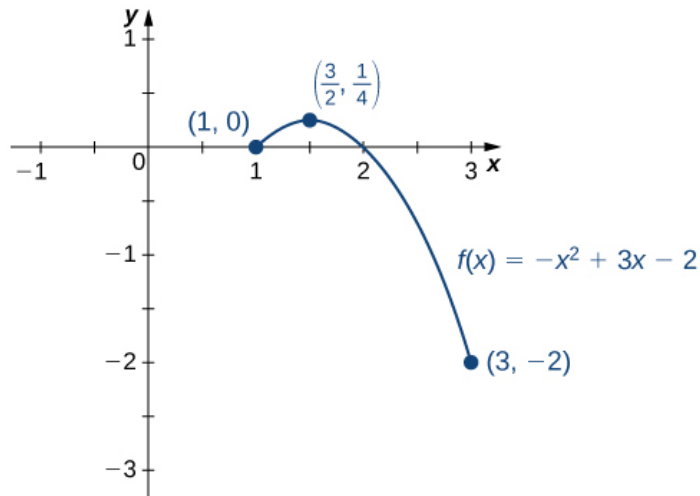
Equation:

$$f\left(\frac{3}{2}\right) = \frac{1}{4}.$$

Step 3. We set up the following table to compare the values found in steps 1 and 2.

x	$f(x)$	Conclusion
0	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum
3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval $[1, 3]$ is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$. The absolute minimum of f over the interval $[1, 3]$ is -2 , and it occurs at $x = 3$ as shown in the following graph.



This function has both an absolute maximum and an absolute minimum.

b. Step 1. Evaluate f at the endpoints $x = 0$ and $x = 2$.

Equation:

$$f(0) = 0 \text{ and } f(2) = 4 - 3\sqrt[3]{4} \approx -0.762$$

Step 2. The derivative of f is given by

Equation:

$$f'(x) = 2x - \frac{2}{x^{1/3}} = \frac{2x^{4/3} - 2}{x^{1/3}}$$

for $x \neq 0$. The derivative is zero when $2x^{4/3} - 2 = 0$, which implies $x = \pm 1$. The derivative is undefined at $x = 0$.

Therefore, the critical points of f are $x = 0, 1, -1$. The point $x = 0$ is an endpoint, so we already evaluated $f(0)$ in step 1. The point $x = -1$ is not in the interval of interest, so we need only evaluate $f(1)$. We find that

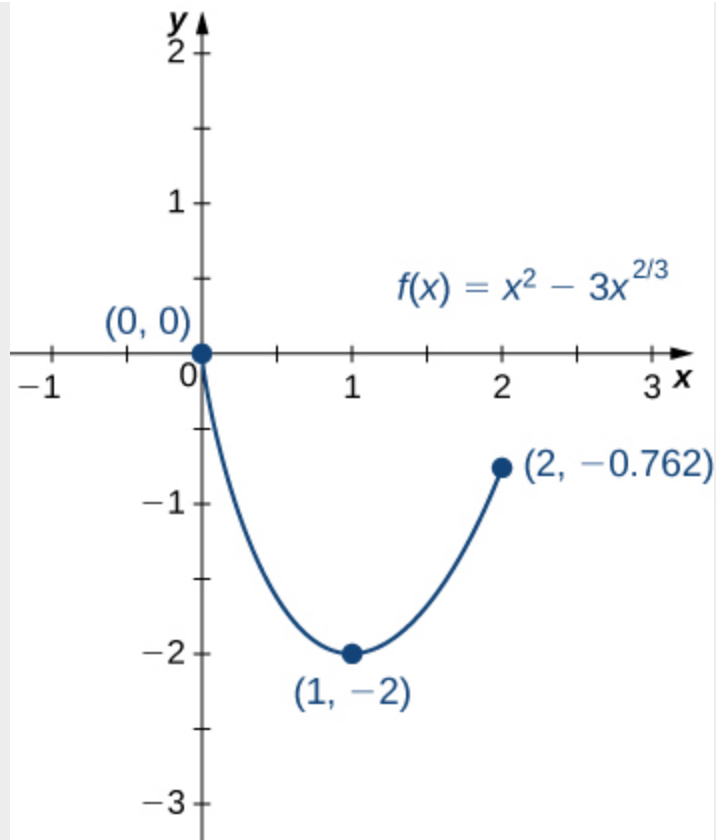
Equation:

$$f(1) = -2.$$

Step 3. We compare the values found in steps 1 and 2, in the following table.

x	$f(x)$	Conclusion
0	0	Absolute maximum
1	-2	Absolute minimum
2	-0.762	

We conclude that the absolute maximum of f over the interval $[0, 2]$ is zero, and it occurs at $x = 0$. The absolute minimum is -2, and it occurs at $x = 1$ as shown in the following graph.



This function has an absolute maximum at an endpoint of the interval.

Note:

Exercise:

Problem:

Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

Solution:

The absolute maximum is 3 and it occurs at $x = 4$. The absolute minimum is -1 and it occurs at $x = 2$.

Hint

Look for critical points. Evaluate f at all critical points and at the endpoints.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a point c , then c must be a critical point of f . However, c being a critical point is not a sufficient condition for f to have a local extremum at c . Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

Key Concepts

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical point. However, a function need not have a local extremum at a critical point.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical point or an endpoint.

Exercise:

Problem:

In precalculus, you learned a formula for the position of the maximum or minimum of a quadratic equation $y = ax^2 + bx + c$, which was $m = -\frac{b}{(2a)}$. Prove this formula using calculus.

Exercise:**Problem:**

If you are finding an absolute minimum over an interval $[a, b]$, why do you need to check the endpoints? Draw a graph that supports your hypothesis.

Solution:

Answers may vary

Exercise:**Problem:**

If you are examining a function over an interval (a, b) , for a and b finite, is it possible not to have an absolute maximum or absolute minimum?

Exercise:**Problem:**

When you are checking for critical points, explain why you also need to determine points where $f(x)$ is undefined. Draw a graph to support your explanation.

Solution:

Answers will vary

Exercise:

Problem:

Can you have a finite absolute maximum for $y = ax^2 + bx + c$ over $(-\infty, \infty)$? Explain why or why not using graphical arguments.

Exercise:**Problem:**

Can you have a finite absolute maximum for $y = ax^3 + bx^2 + cx + d$ over $(-\infty, \infty)$ assuming a is non-zero? Explain why or why not using graphical arguments.

Solution:

No; answers will vary

Exercise:**Problem:**

Let m be the number of local minima and M be the number of local maxima. Can you create a function where $M > m + 2$? Draw a graph to support your explanation.

Exercise:**Problem:**

Is it possible to have more than one absolute maximum? Use a graphical argument to prove your hypothesis.

Solution:

Since the absolute maximum is the function (output) value rather than the x value, the answer is no; answers will vary

Exercise:

Problem:

Is it possible to have no absolute minimum or maximum for a function? If so, construct such a function. If not, explain why this is not possible.

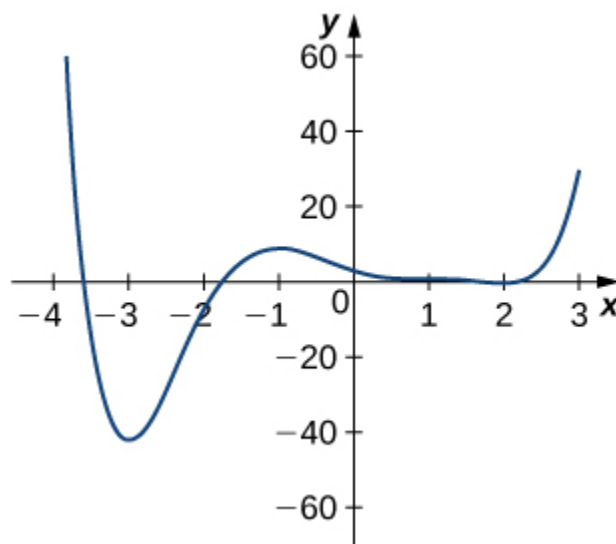
Exercise:**Problem:**

[T] Graph the function $y = e^{ax}$. For which values of a , on any infinite domain, will you have an absolute minimum and absolute maximum?

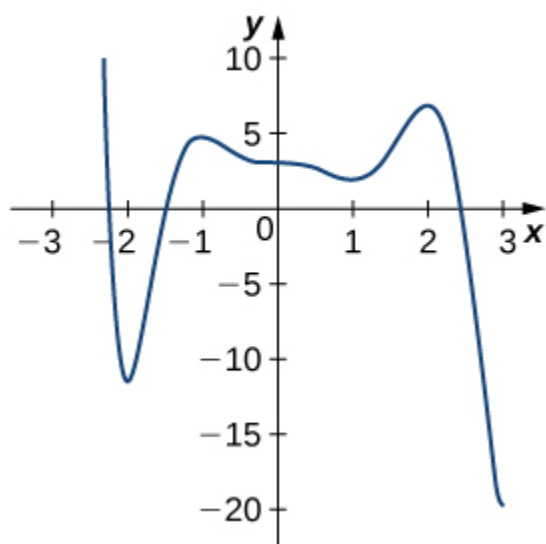
Solution:

When $a = 0$

For the following exercises, determine where the local and absolute maxima and minima occur on the graph given. Assume domains are closed intervals unless otherwise specified.

Exercise:**Problem:****Exercise:**

Problem:

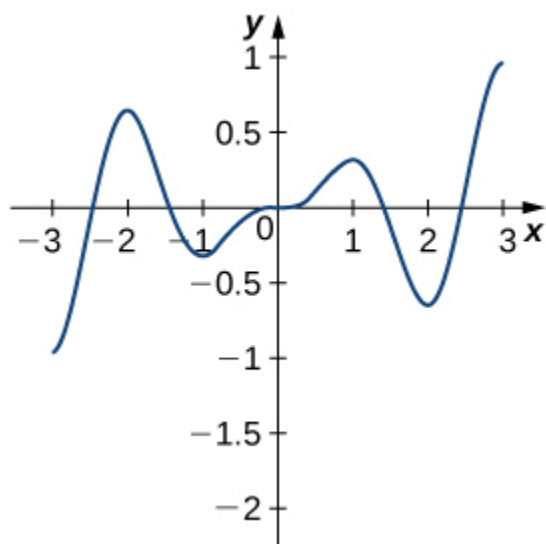


Solution:

Absolute minimum at 3; Absolute maximum at -2.2 ; local minima at $-2, 1$; local maxima at $-1, 2$

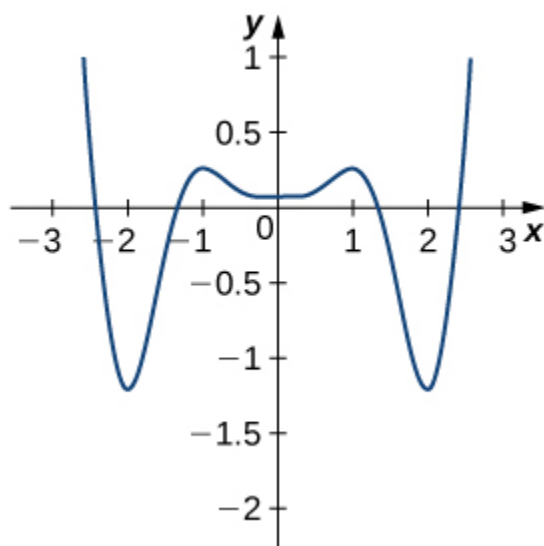
Exercise:

Problem:



Exercise:

Problem:



Solution:

Absolute minima at $-2, 2$; absolute maxima at $-2.5, 2.5$; local minimum at 0 ; local maxima at $-1, 1$

For the following problems, draw graphs of $f(x)$, which is continuous, over the interval $[-4, 4]$ with the following properties:

Exercise:

Problem:

Absolute maximum at $x = 2$ and absolute minima at $x = \pm 3$

Exercise:

Problem:

Absolute minimum at $x = 1$ and absolute maximum at $x = 2$

Solution:

Answers may vary.

Exercise:**Problem:**

Absolute maximum at $x = 4$, absolute minimum at $x = -1$, local maximum at $x = -2$, and a critical point that is not a maximum or minimum at $x = 2$

Exercise:**Problem:**

Absolute maxima at $x = 2$ and $x = -3$, local minimum at $x = 1$, and absolute minimum at $x = 4$

Solution:

Answers may vary.

For the following exercises, find the critical points in the domains of the following functions.

Exercise:

Problem: $y = 4x^3 - 3x$

Exercise:

Problem: $y = 4\sqrt{x} - x^2$

Solution:

$$x = 1$$

Exercise:

Problem: $y = \frac{1}{x-1}$

Exercise:

Problem: $y = \ln(x - 2)$

Solution:

None

Exercise:

Problem: $y = \tan(x)$

Exercise:

Problem: $y = \sqrt{4 - x^2}$

Solution:

$x = 0$

Exercise:

Problem: $y = x^{3/2} - 3x^{5/2}$

Exercise:

Problem: $y = \frac{x^2 - 1}{x^2 + 2x - 3}$

Solution:

None

Exercise:

Problem: $y = \sin^2(x)$

Exercise:

Problem: $y = x + \frac{1}{x}$

Solution:

$$x = -1, 1$$

For the following exercises, find the local and/or absolute maxima for the functions over the specified domain.

Exercise:

Problem: $f(x) = x^2 + 3$ over $[-1, 4]$

Exercise:

Problem: $y = x^2 + \frac{2}{x}$ over $[1, 4]$

Solution:

Absolute maximum: $x = 4, y = \frac{33}{2}$; absolute minimum: $x = 1, y = 3$

Exercise:

Problem: $y = (x - x^2)^2$ over $[-1, 1]$

Exercise:

Problem: $y = \frac{1}{(x-x^2)}$ over $[0, 1]$

Solution:

Absolute minimum: $x = \frac{1}{2}, y = 4$

Exercise:

Problem: $y = \sqrt{9 - x}$ over $[1, 9]$

Exercise:

Problem: $y = x + \sin(x)$ over $[0, 2\pi]$

Solution:

Absolute maximum: $x = 2\pi, y = 2\pi$; absolute minimum: $x = 0, y = 0$

Exercise:

Problem: $y = \frac{x}{1+x}$ over $[0, 100]$

Exercise:

Problem: $y = |x + 1| + |x - 1|$ over $[-3, 2]$

Solution:

Absolute maximum: $x = -3$; absolute minimum: $-1 \leq x \leq 1, y = 2$

Exercise:

Problem: $y = \sqrt{x} - \sqrt{x^3}$ over $[0, 4]$

Exercise:

Problem: $y = \sin x + \cos x$ over $[0, 2\pi]$

Solution:

Absolute maximum: $x = \frac{\pi}{4}, y = \sqrt{2}$; absolute minimum: $x = \frac{5\pi}{4}, y = -\sqrt{2}$

Exercise:

Problem: $y = 4\sin\theta - 3\cos\theta$ over $[0, 2\pi]$

For the following exercises, find the local and absolute minima and maxima for the functions over $(-\infty, \infty)$.

Exercise:

Problem: $y = x^2 + 4x + 5$

Solution:

Absolute minimum: $x = -2, y = 1$

Exercise:

Problem: $y = x^3 - 12x$

Exercise:

Problem: $y = 3x^4 + 8x^3 - 18x^2$

Solution:

Absolute minimum: $x = -3, y = -135$; local maximum: $x = 0, y = 0$; local minimum: $x = 1, y = -7$

Exercise:

Problem: $y = x^3(1 - x)^6$

Exercise:

Problem: $y = \frac{x^2+x+6}{x-1}$

Solution:

Local maximum: $x = 1 - 2\sqrt{2}, y = 3 - 4\sqrt{2}$; local minimum:
 $x = 1 + 2\sqrt{2}, y = 3 + 4\sqrt{2}$

Exercise:

Problem: $y = \frac{x^2-1}{x-1}$

For the following functions, use a calculator to graph the function and to estimate the absolute and local maxima and minima. Then, solve for them explicitly.

Exercise:

Problem: [T] $y = 3x\sqrt{1-x^2}$

Solution:

Absolute maximum: $x = \frac{\sqrt{2}}{2}$, $y = \frac{3}{2}$; absolute minimum: $x = -\frac{\sqrt{2}}{2}$, $y = -\frac{3}{2}$

Exercise:

Problem: [T] $y = x + \sin(x)$

Exercise:

Problem: [T] $y = 12x^5 + 45x^4 + 20x^3 - 90x^2 - 120x + 3$

Solution:

Local maximum: $x = -2$, $y = 59$; local minimum: $x = 1$, $y = -130$

Exercise:

Problem: [T] $y = \frac{x^3+6x^2-x-30}{x-2}$

Exercise:

Problem: [T] $y = \frac{\sqrt{4-x^2}}{\sqrt{4+x^2}}$

Solution:

Absolute maximum: $x = 0, y = 1$; absolute minimum: $x = -2, 2, y = 0$

Exercise:**Problem:**

A company that produces cell phones has a cost function of $C = x^2 - 1200x + 36,400$, where C is cost in dollars and x is number of cell phones produced (in thousands). How many units of cell phone (in thousands) minimizes this cost function?

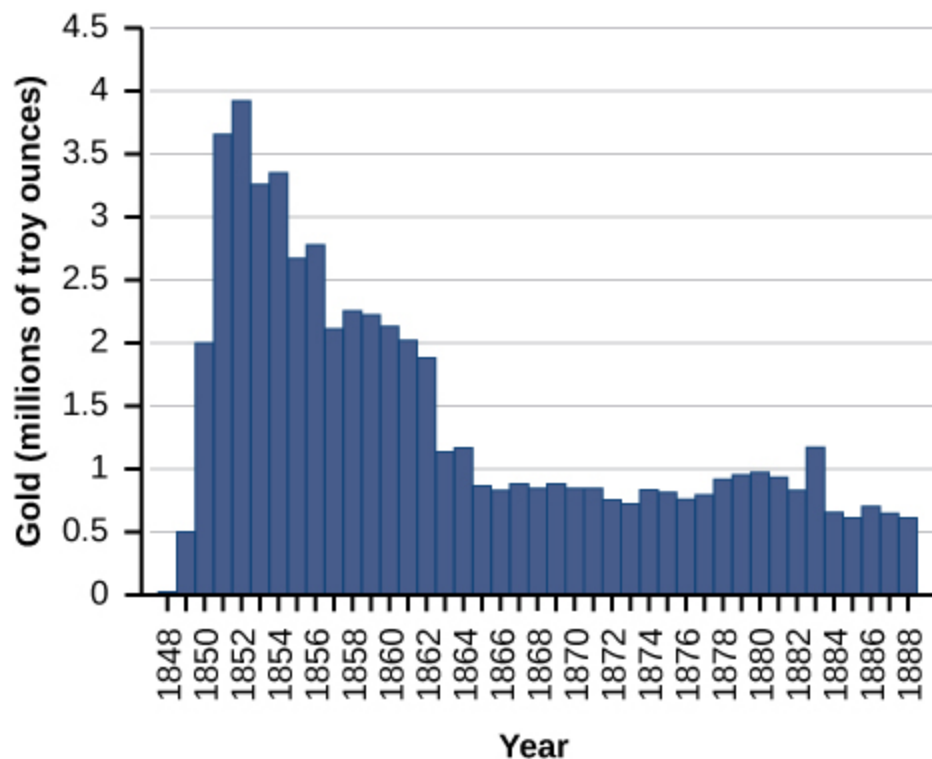
Exercise:**Problem:**

A ball is thrown into the air and its position is given by $h(t) = -4.9t^2 + 60t + 5$ m. Find the height at which the ball stops ascending. How long after it is thrown does this happen?

Solution:

$$h = \frac{9245}{49} \text{ m}, t = \frac{300}{49} \text{ s}$$

For the following exercises, consider the production of gold during the California gold rush (1848–1888). The production of gold can be modeled by $G(t) = \frac{(25t)}{(t^2+16)}$, where t is the number of years since the rush began ($0 \leq t \leq 40$) and G is ounces of gold produced (in millions). A summary of the data is shown in the following figure.



Exercise:

Problem:

Find when the maximum (local and global) gold production occurred, and the amount of gold produced during that maximum.

Exercise:

Problem:

Find when the minimum (local and global) gold production occurred. What was the amount of gold produced during this minimum?

Solution:

The global minimum was in 1848, when no gold was produced.

Find the critical points, maxima, and minima for the following piecewise functions.

Exercise:

Problem: $y = \begin{cases} x^2 - 4x & 0 \leq x \leq 1 \\ x^2 - 4 & 1 < x \leq 2 \end{cases}$

Exercise:

Problem: $y = \begin{cases} x^2 + 1 & x \leq 1 \\ x^2 - 4x + 5 & x > 1 \end{cases}$

Solution:

Absolute minima: $x = 0, x = 2, y = 1$; local maximum at $x = 1, y = 2$

For the following exercises, find the critical points of the following generic functions. Are they maxima, minima, or neither? State the necessary conditions.

Exercise:

Problem: $y = ax^2 + bx + c$, given that $a > 0$

Exercise:

Problem: $y = (x - 1)^a$, given that $a > 1$

Solution:

No maxima/minima if a is odd, minimum at $x = 1$ if a is even

Glossary

absolute extremum

if f has an absolute maximum or absolute minimum at c , we say f has an absolute extremum at c

absolute maximum

if $f(c) \geq f(x)$ for all x in the domain of f , we say f has an absolute maximum at c

absolute minimum

if $f(c) \leq f(x)$ for all x in the domain of f , we say f has an absolute minimum at c

critical point

if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of f

extreme value theorem

if f is a continuous function over a finite, closed interval, then f has an absolute maximum and an absolute minimum

Fermat's theorem

if f has a local extremum at c , then c is a critical point of f

local extremum

if f has a local maximum or local minimum at c , we say f has a local extremum at c

local maximum

if there exists an interval I such that $f(c) \geq f(x)$ for all $x \in I$, we say f has a local maximum at c

local minimum

if there exists an interval I such that $f(c) \leq f(x)$ for all $x \in I$, we say f has a local minimum at c

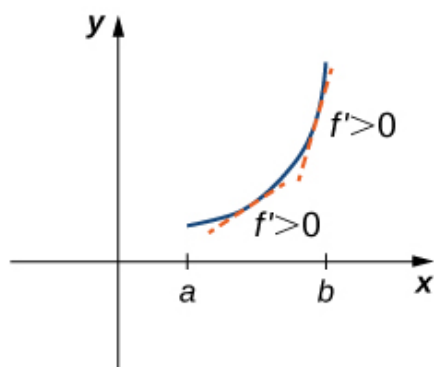
Derivatives and the Shape of a Graph

- Explain how the sign of the first derivative affects the shape of a function's graph.
- State the first derivative test for critical points.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the concavity test for a function over an open interval.
- Explain the relationship between a function and its first and second derivatives.
- State the second derivative test for local extrema.

Earlier in this chapter we stated that if a function f has a local extremum at a point c , then c must be a critical point of f . However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x) = x^3$ has a critical point at $x = 0$ since $f'(x) = 3x^2$ is zero at $x = 0$, but f does not have a local extremum at $x = 0$. Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

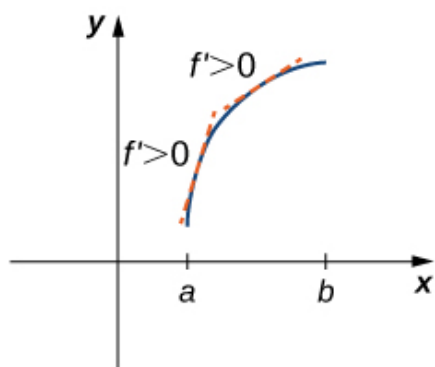
The First Derivative Test

Corollary 3 of the Mean Value Theorem showed that if the derivative of a function is positive over an interval I then the function is increasing over I . On the other hand, if the derivative of the function is negative over an interval I , then the function is decreasing over I as shown in the following figure.



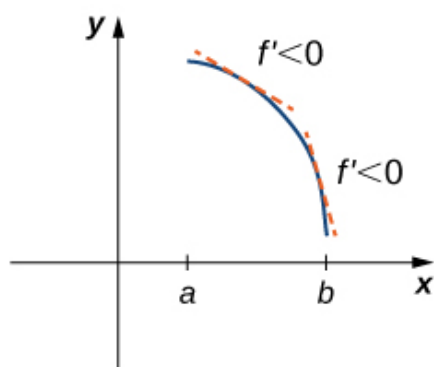
f is increasing

(a)



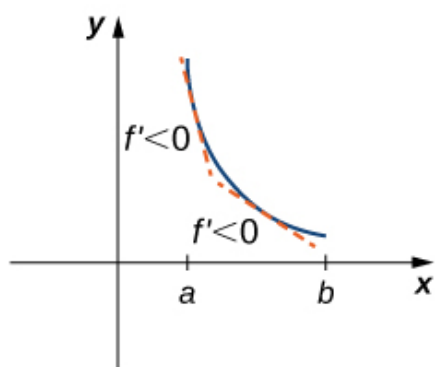
f is increasing

(b)



f is decreasing

(c)



f is decreasing

(d)

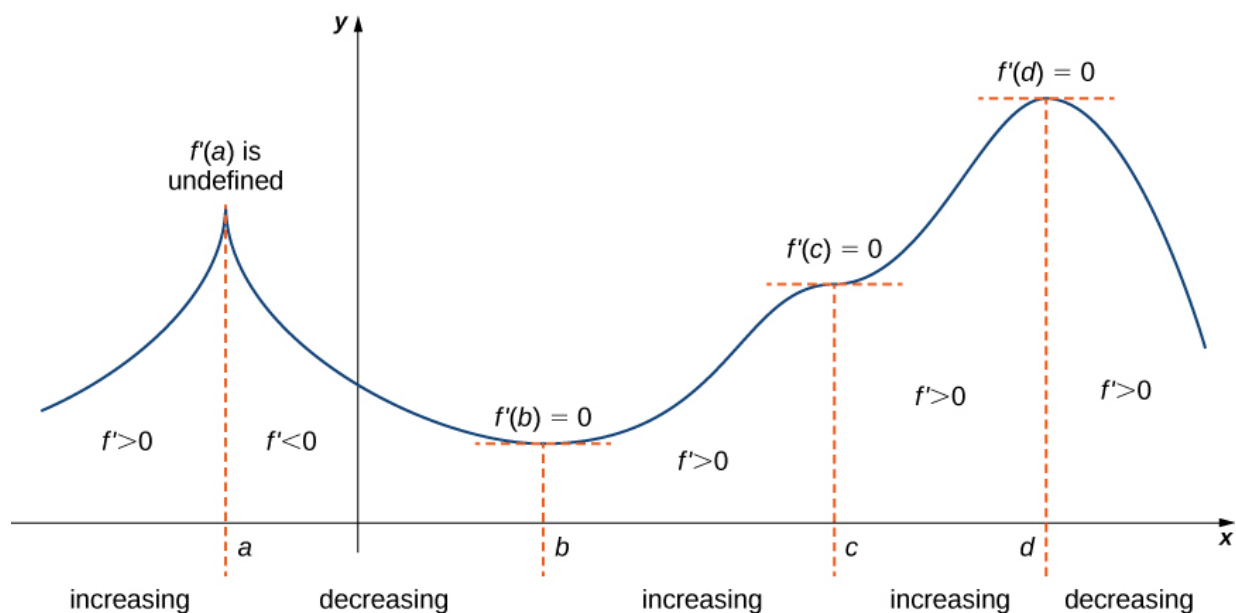
Both functions are increasing over the interval (a, b) .

At each point x , the derivative $f'(x) > 0$. Both functions are decreasing over the interval (a, b) . At each point x , the derivative $f'(x) < 0$.

A continuous function f has a local maximum at point c if and only if f switches from increasing to decreasing at point c . Similarly, f has a local minimum at c if and only if f switches from decreasing to increasing at c . If f is a continuous function over an interval I containing c and differentiable over I , except possibly at c , the only way f can switch from increasing to decreasing (or vice versa) at point c is if f' changes sign as x increases through c . If f is differentiable at c , the only way that f' can change sign as x increases through c is if $f'(c) = 0$. Therefore, for a function f that is continuous over an interval I containing c and

differentiable over I , except possibly at c , the only way f can switch from increasing to decreasing (or vice versa) is if $f'(c) = 0$ or $f'(c)$ is undefined. Consequently, to locate local extrema for a function f , we look for points c in the domain of f such that $f'(c) = 0$ or $f'(c)$ is undefined. Recall that such points are called critical points of f .

Note that f need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In [\[link\]](#), we show that if a continuous function f has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if f has a local extremum at a critical point, then the sign of f' switches as x increases through that point.



The function f has four critical points: a , b , c , and d . The function f has local maxima at a and d , and a local minimum at b . The function f does not have a local extremum at c . The sign of f' changes at all local extrema.

Using [\[link\]](#), we summarize the main results regarding local extrema.

- If a continuous function f has a local extremum, it must occur at a critical point c .

- The function has a local extremum at the critical point c if and only if the derivative f' switches sign as x increases through c .
- Therefore, to test whether a function has a local extremum at a critical point c , we must determine the sign of $f'(x)$ to the left and right of c .

This result is known as the **first derivative test**.

Note:

First Derivative Test

Suppose that f is a continuous function over an interval I containing a critical point c . If f is differentiable over I , except possibly at point c , then $f(c)$ satisfies one of the following descriptions:

- If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f .
- If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f .
- If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f .

We can summarize the first derivative test as a strategy for locating local extrema.

Note:

Problem-Solving Strategy: Using the First Derivative Test

Consider a function f that is continuous over an interval I .

1. Find all critical points of f and divide the interval I into smaller intervals using the critical points as endpoints.
2. Analyze the sign of f' in each of the subintervals. If f' is continuous over a given subinterval (which is typically the case), then the sign of f' in that subinterval does not change and, therefore, can be determined by choosing an arbitrary test point x in that subinterval and by evaluating the sign of f' at that test point. Use the sign analysis to determine whether f is increasing or decreasing over that interval.

3. Use [link](#) and the results of step 2 to determine whether f has a local maximum, a local minimum, or neither at each of the critical points.

Now let's look at how to use this strategy to locate all local extrema for particular functions.

Example:

Exercise:

Problem:

Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$. Use a graphing utility to confirm your results.

Solution:

Step 1. The derivative is $f'(x) = 3x^2 - 6x - 9$. To find the critical points, we need to find where $f'(x) = 0$. Factoring the polynomial, we conclude that the critical points must satisfy

Equation:

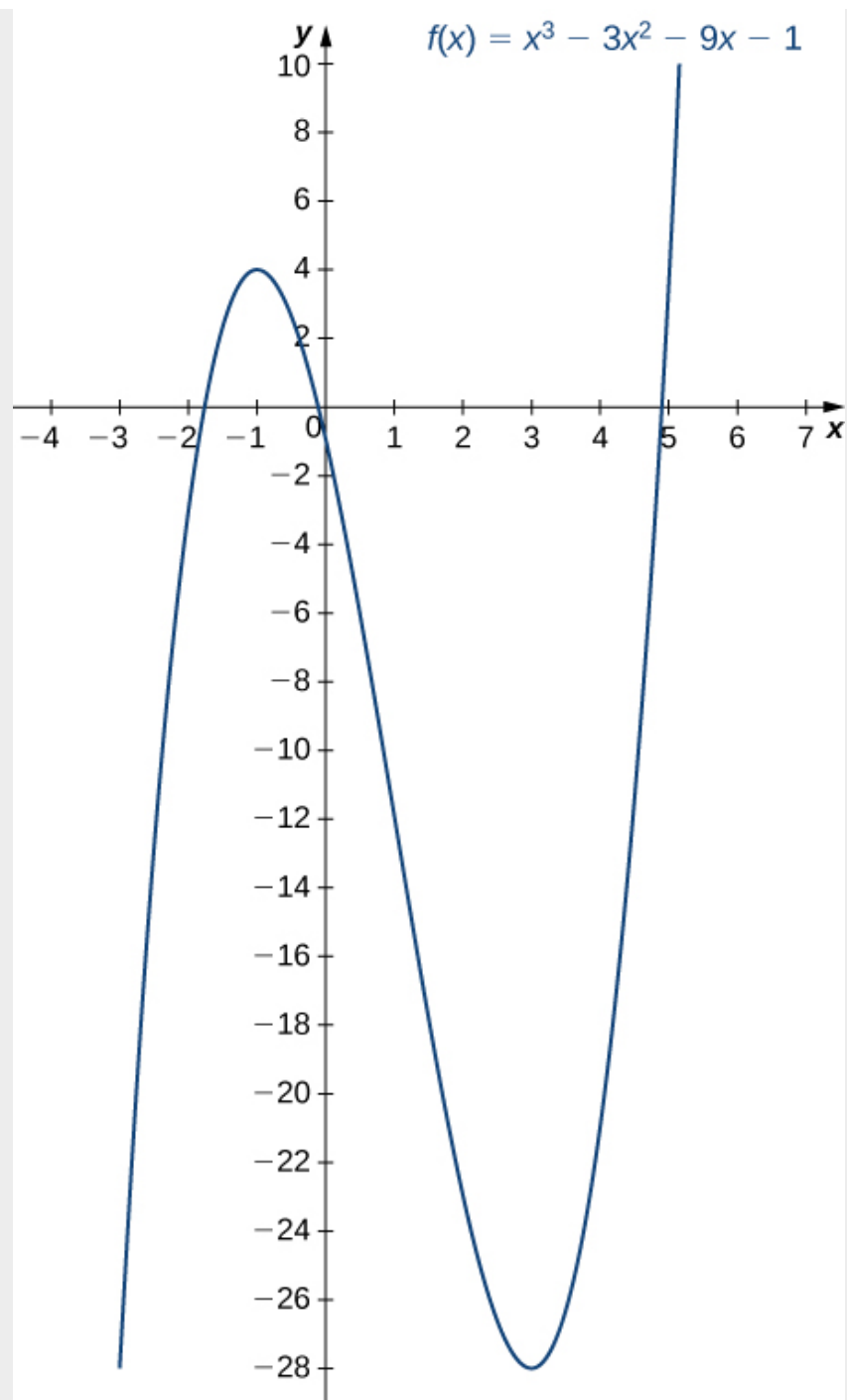
$$3(x^2 - 2x - 3) = 3(x - 3)(x + 1) = 0.$$

Therefore, the critical points are $x = 3, -1$. Now divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1)$, $(-1, 3)$ and $(3, \infty)$.

Step 2. Since f' is a continuous function, to determine the sign of $f'(x)$ over each subinterval, it suffices to choose a point over each of the intervals $(-\infty, -1)$, $(-1, 3)$ and $(3, \infty)$ and determine the sign of f' at each of these points. For example, let's choose $x = -2$, $x = 0$, and $x = 4$ as test points.

Interval	Test Point	Sign of $f'(x) = 3(x - 3)(x + 1)$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f is increasing.
$(-1, 3)$	$x = 0$	$(+)(-)(+) = -$	f is decreasing.
$(3, \infty)$	$x = 4$	$(+)(+)(+) = +$	f is increasing.

Step 3. Since f' switches sign from positive to negative as x increases through 1, f has a local maximum at $x = -1$. Since f' switches sign from negative to positive as x increases through 3, f has a local minimum at $x = 3$. These analytical results agree with the following graph.



The function f has a maximum at $x = -1$ and a minimum at $x = 3$

Note:**Exercise:****Problem:**

Use the first derivative test to locate all local extrema for $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$.

Solution:

f has a local minimum at -2 and a local maximum at 3 .

Hint

Find all critical points of f and determine the signs of $f'(x)$ over particular intervals determined by the critical points.

Example:**Exercise:****Problem:****Using the First Derivative Test**

Use the first derivative test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}$. Use a graphing utility to confirm your results.

Solution:

Step 1. The derivative is

Equation:

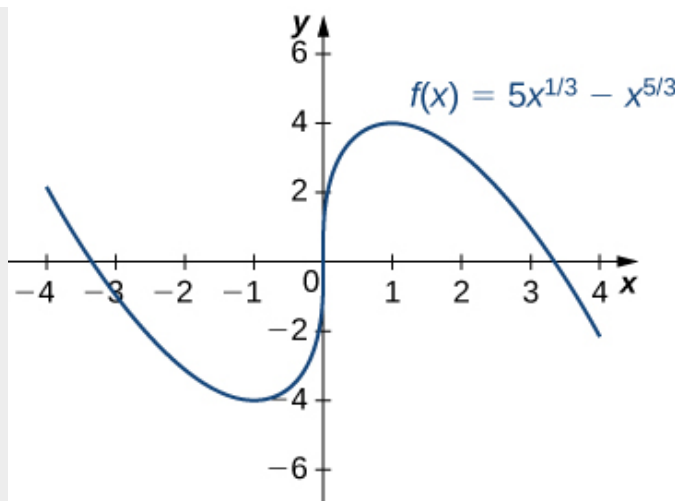
$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5}{3x^{2/3}} - \frac{5x^{2/3}}{3} = \frac{5 - 5x^{4/3}}{3x^{2/3}} = \frac{5(1 - x^{4/3})}{3x^{2/3}}.$$

The derivative $f'(x) = 0$ when $1 - x^{4/3} = 0$. Therefore, $f'(x) = 0$ at $x = \pm 1$. The derivative $f'(x)$ is undefined at $x = 0$. Therefore, we have three critical points: $x = 0$, $x = 1$, and $x = -1$. Consequently, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$.

Step 2: Since f' is continuous over each subinterval, it suffices to choose a test point x in each of the intervals from step 1 and determine the sign of f' at each of these points. The points $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are test points for these intervals.

Interval	Test Point	Sign of $f'(x) = \frac{5(1-x^{4/3})}{3x^{2/3}}$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	$\frac{(+)(-)}{+} = -$	f is decreasing.
$(-1, 0)$	$x = -\frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(0, 1)$	$x = \frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(1, \infty)$	$x = 2$	$\frac{(+)(-)}{+} = -$	f is decreasing.

Step 3: Since f is decreasing over the interval $(-\infty, -1)$ and increasing over the interval $(-1, 0)$, f has a local minimum at $x = -1$. Since f is increasing over the interval $(-1, 0)$ and the interval $(0, 1)$, f does not have a local extremum at $x = 0$. Since f is increasing over the interval $(0, 1)$ and decreasing over the interval $(1, \infty)$, f has a local maximum at $x = 1$. The analytical results agree with the following graph.



The function f has a local minimum at $x = -1$ and a local maximum at $x = 1$.

Note:

Exercise:

Problem:

Use the first derivative test to find all local extrema for $f(x) = \sqrt[3]{x-1}$.

Solution:

f has no local extrema because f' does not change sign at $x = 1$.

Hint

The only critical point of f is $x = 1$.

Concavity and Points of Inflection

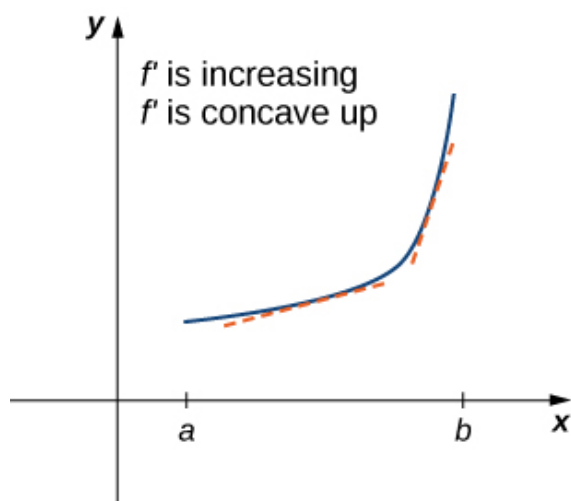
We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

[\[link\]](#)(a) shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is concave up. [\[link\]](#) (b) shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is concave down.

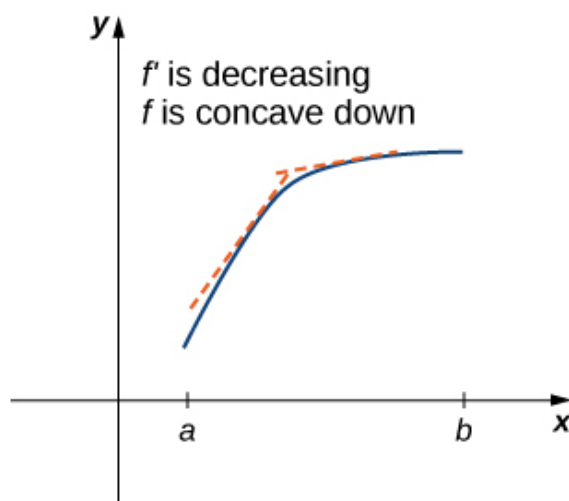
Note:

Definition

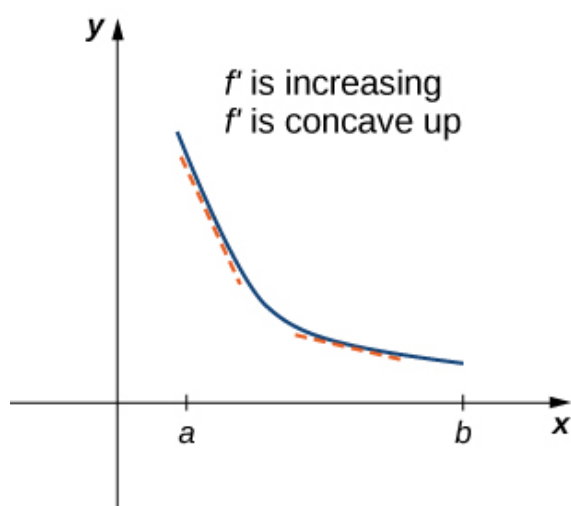
Let f be a function that is differentiable over an open interval I . If f' is increasing over I , we say f is **concave up** over I . If f' is decreasing over I , we say f is **concave down** over I .



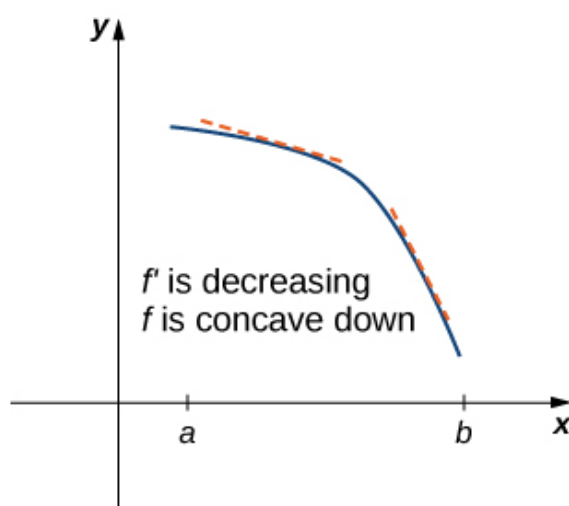
(a)



(b)



(c)



(d)

(a), (c) Since f' is increasing over the interval (a, b) , we say f is concave up over (a, b) . (b), (d) Since f' is decreasing over the interval (a, b) , we say f is concave down over (a, b) .

In general, without having the graph of a function f , how can we determine its concavity? By definition, a function f is concave up if f' is increasing. From Corollary 3, we know that if f' is a differentiable function, then f' is increasing if its derivative $f''(x) > 0$. Therefore, a function f that is twice differentiable is

concave up when $f''(x) > 0$. Similarly, a function f is concave down if f' is decreasing. We know that a differentiable function f' is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function f is concave down when $f''(x) < 0$. Applying this logic is known as the **concavity test**.

Note:

Test for Concavity

Let f be a function that is twice differentiable over an interval I .

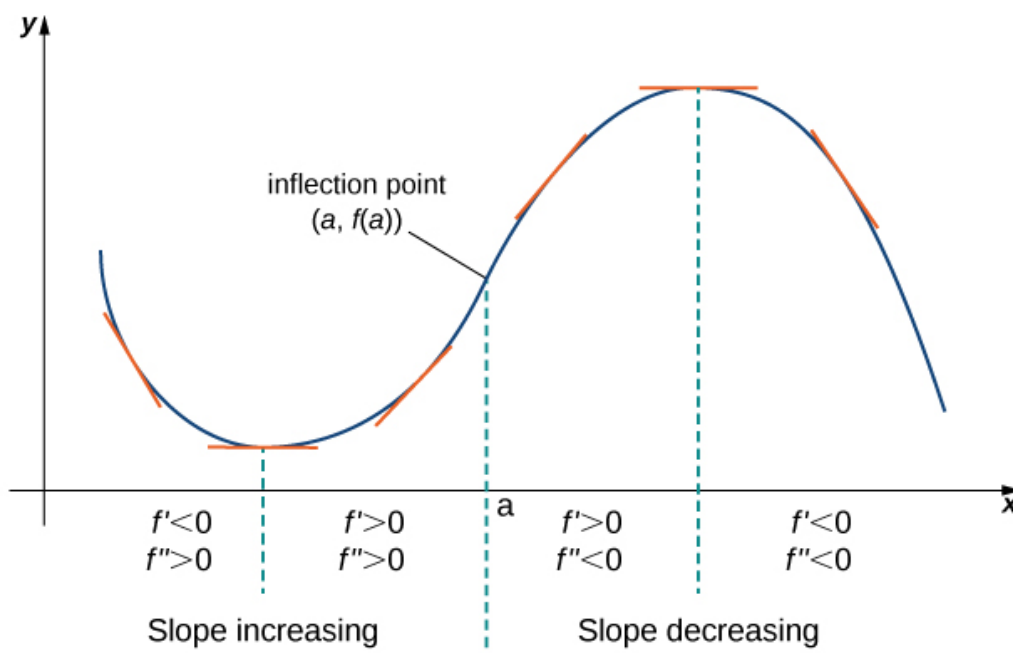
- i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I .
- ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I .

We conclude that we can determine the concavity of a function f by looking at the second derivative of f . In addition, we observe that a function f can switch concavity ([\[link\]](#)). However, a continuous function can switch concavity only at a point x if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point x , then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at a , we say the point $(a, f(a))$ is an inflection point of f .

Note:

Definition

If f is continuous at a and f changes concavity at a , the point $(a, f(a))$ is an **inflection point** of f .



Since $f''(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞) . The point $(a, f(a))$ is an inflection point of f .

Example:

Exercise:

Problem:

Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f . Use a graphing utility to confirm your results.

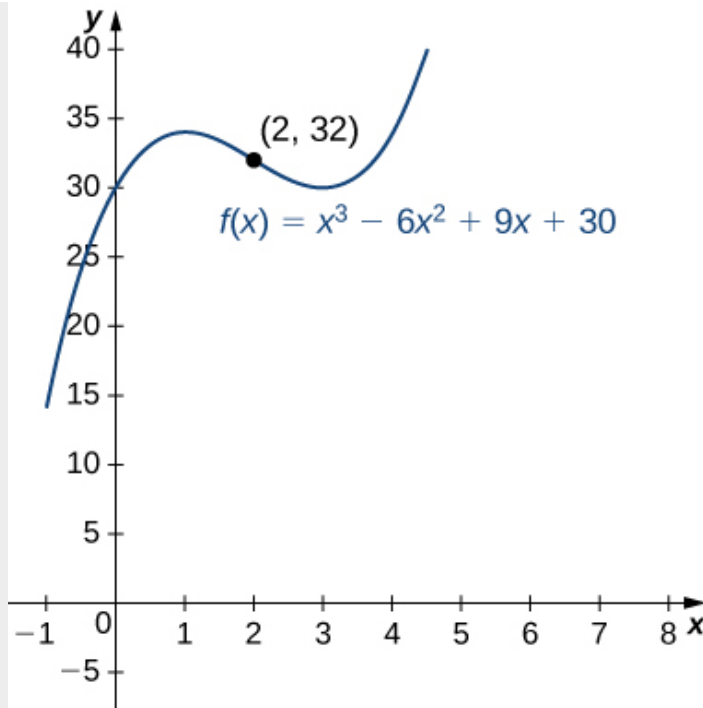
Solution:

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 - 12x + 9$, so the second derivative is $f''(x) = 6x - 12$. If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers

x , we need only find where $f''(x) = 0$. Solving the equation $6x - 12 = 0$, we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f . The points $x = 0$ and $x = 3$ are test points for these intervals.

Interval	Test Point	Sign of $f''(x) = 6x - 12$ at Test Point	Conclusion
$(-\infty, 2)$	$x = 0$	—	f is concave down
$(2, \infty)$	$x = 3$	+	f is concave up.

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since f changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point. [\[link\]](#) confirms the analytical results.



The given function has a point of inflection at $(2, 32)$ where the graph changes concavity.

Note:

Exercise:

Problem:

For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$, find all intervals where f is concave up and all intervals where f is concave down.

Solution:

f is concave up over the interval $(-\infty, \frac{1}{2})$ and concave down over the interval $(\frac{1}{2}, \infty)$

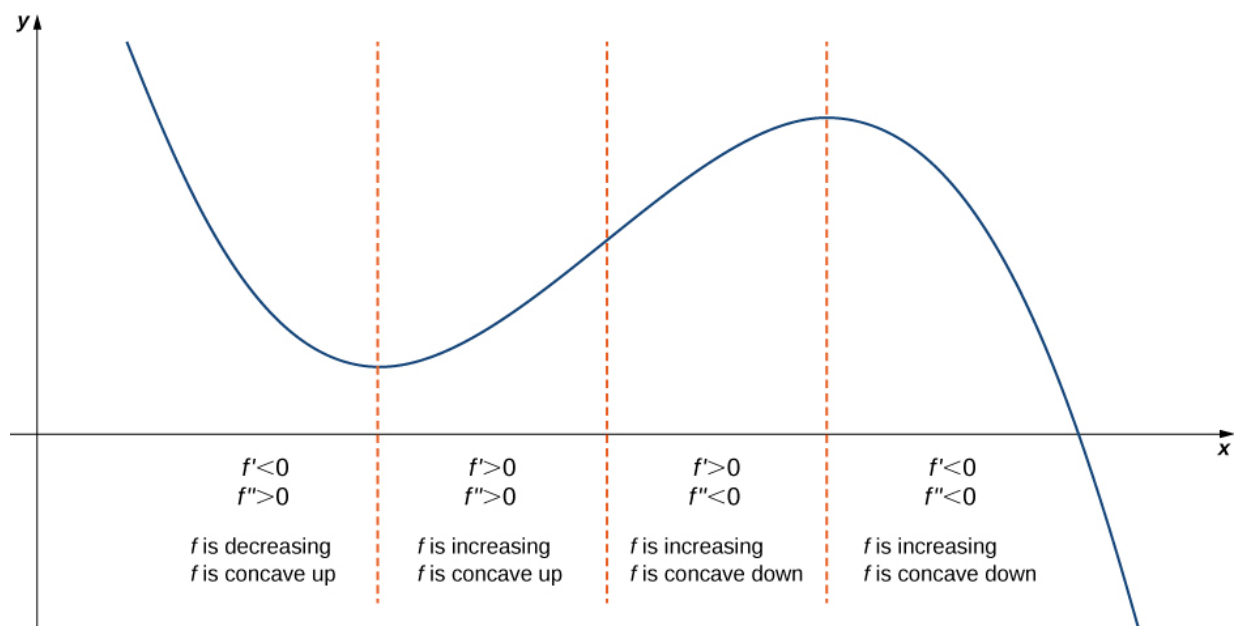
Hint

Find where $f''(x) = 0$.

We now summarize, in [\[link\]](#), the information that the first and second derivatives of a function f provide about the graph of f , and illustrate this information in [\[link\]](#).

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

What Derivatives Tell Us about Graphs



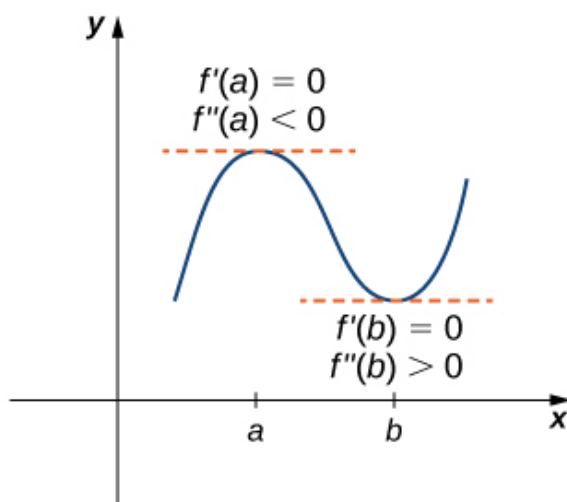
Consider a twice-differentiable function f over an open interval I . If $f'(x) > 0$ for all $x \in I$, the function is increasing over I . If $f'(x) < 0$ for all $x \in I$, the function is decreasing over I . If $f''(x) > 0$ for all $x \in I$, the function is concave up. If $f''(x) < 0$ for all $x \in I$, the function is concave down on I .

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extrema, it must occur at a critical point. However, a function need not have a local extrema at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing a . Suppose $f''(a) < 0$. Since f'' is continuous over I , $f''(x) < 0$ for all $x \in I$ ([link](#)). Then, by Corollary 3, f' is a decreasing function over I . Since $f'(a) = 0$, we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, f has a local maximum at $x = a$. On the other hand, suppose there exists a point b such that $f'(b) = 0$ but

$f''(b) > 0$. Since f'' is continuous over an open interval I containing b , then $f''(x) > 0$ for all $x \in I$ ([link](#)). Then, by Corollary 3, f' is an increasing function over I . Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, f has a local minimum at $x = b$.



Consider a twice-differentiable function f such that f'' is continuous. Since $f'(a) = 0$ and $f''(a) < 0$, there is an interval I containing a such that for all x in I , f is increasing if $x < a$ and f is decreasing if $x > a$. As a result, f has a local maximum at $x = a$. Since $f'(b) = 0$ and $f''(b) > 0$, there is an interval I containing b such that for all x in I , f is decreasing if $x < b$ and f is increasing if $x > b$. As a result, f has a local minimum at $x = b$.

Note:

Second Derivative Test

Suppose $f'(c) = 0$, f'' is continuous over an interval containing c .

- i. If $f''(c) > 0$, then f has a local minimum at c .

- ii. If $f''(c) < 0$, then f has a local maximum at c .
- iii. If $f''(c) = 0$, then the test is inconclusive.

Note that for case iii. when $f''(c) = 0$, then f may have a local maximum, local minimum, or neither at c . For example, the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all have critical points at $x = 0$. In each case, the second derivative is zero at $x = 0$. However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at x , and the function $f(x) = x^3$ does not have a local extremum at $x = 0$.

Let's now look at how to use the second derivative test to determine whether f has a local maximum or local minimum at a critical point c where $f'(c) = 0$.

Example:

Exercise:

Problem:

Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

Solution:

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0$. The derivative is $f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3) = 0$ when $x = 0, \pm\sqrt{3}$.

To determine whether f has a local extrema at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

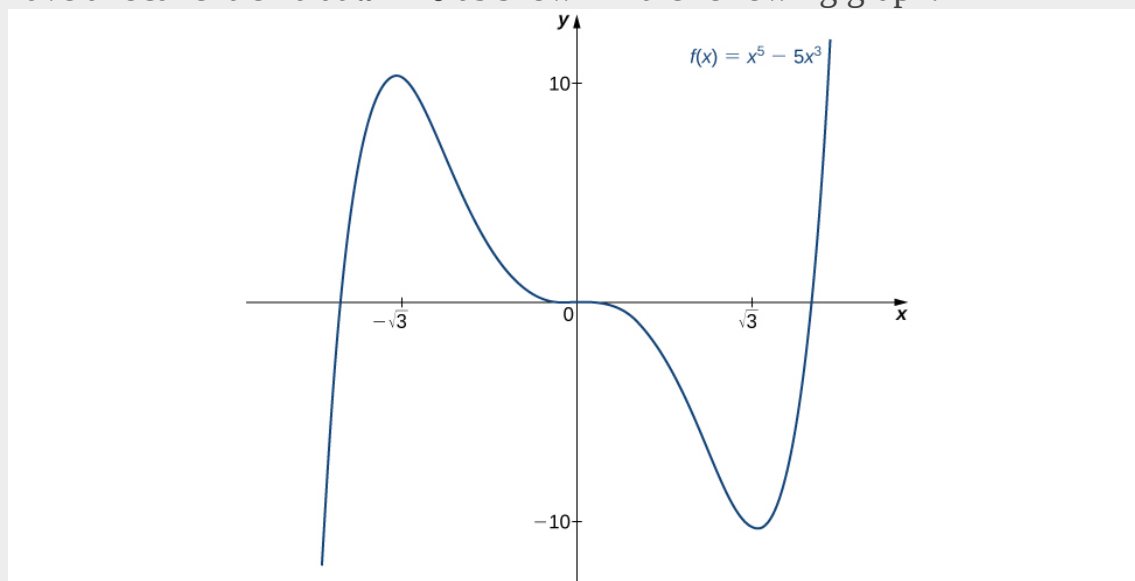
Equation:

$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

x	$f''(x)$	Conclusion
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}$. The second derivative test is inconclusive at $x = 0$. To determine whether f has a local extrema at $x = 0$, we apply the first derivative test. To evaluate the sign of $f'(x) = 5x^2(x^2 - 3)$ for $x \in (-\sqrt{3}, 0)$ and $x \in (0, \sqrt{3})$, let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x = 0$ as shown in the following graph.



The function f has a local maximum at $x = -\sqrt{3}$ and a local minimum at $x = \sqrt{3}$

Note:

Exercise:

Problem:

Consider the function $f(x) = x^3 - \left(\frac{3}{2}\right)x^2 - 18x$. The points $c = 3, -2$ satisfy $f'(c) = 0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

Solution:

f has a local maximum at -2 and a local minimum at 3 .

Hint

$$f''(x) = 6x - 3$$

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x \rightarrow \pm\infty$. At that point, we have enough tools to provide accurate graphs of a large variety of functions.

Key Concepts

- If c is a critical point of f and $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has a local maximum at c .
- If c is a critical point of f and $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has a local minimum at c .
- If $f''(x) > 0$ over an interval I , then f is concave up over I .
- If $f''(x) < 0$ over an interval I , then f is concave down over I .

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c .
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c .
- If $f'(c) = 0$ and $f''(c) = 0$, then evaluate $f'(x)$ at a test point x to the left of c and a test point x to the right of c , to determine whether f has a local extremum at c .

Exercise:

Problem:

If c is a critical point of $f(x)$, when is there no local maximum or minimum at c ? Explain.

Exercise:

Problem:

For the function $y = x^3$, is $x = 0$ both an inflection point and a local maximum/minimum?

Solution:

It is not a local maximum/minimum because f' does not change sign

Exercise:

Problem: For the function $y = x^3$, is $x = 0$ an inflection point?

Exercise:

Problem:

Is it possible for a point c to be both an inflection point and a local extrema of a twice differentiable function?

Solution:

No

Exercise:

Problem:

Why do you need continuity for the first derivative test? Come up with an example.

Exercise:

Problem:

Explain whether a concave-down function has to cross $y = 0$ for some value of x .

Solution:

False; for example, $y = \sqrt{x}$.

Exercise:

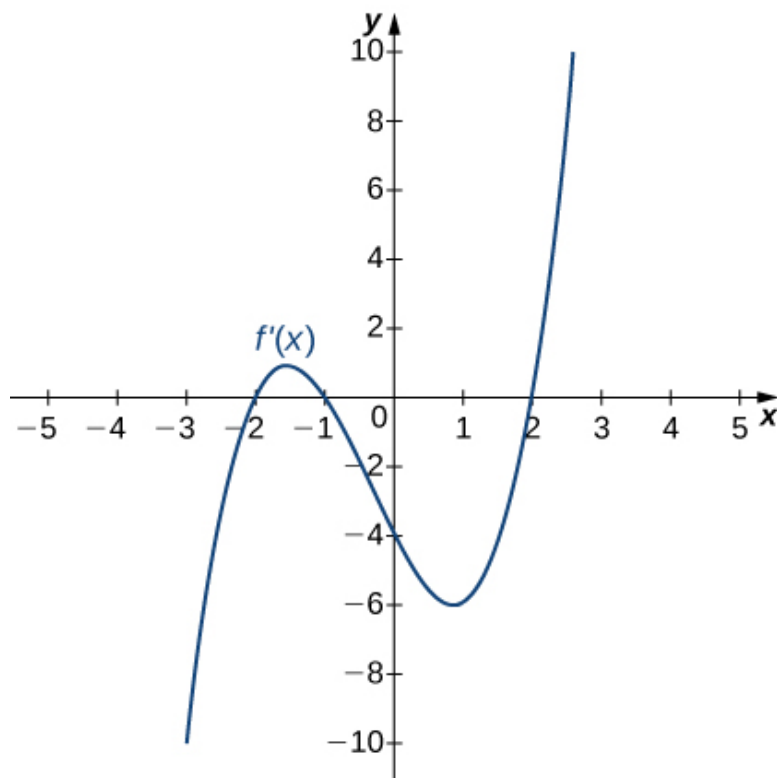
Problem:

Explain whether a polynomial of degree 2 can have an inflection point.

For the following exercises, analyze the graphs of f' , then list all intervals where f is increasing or decreasing.

Exercise:

Problem:

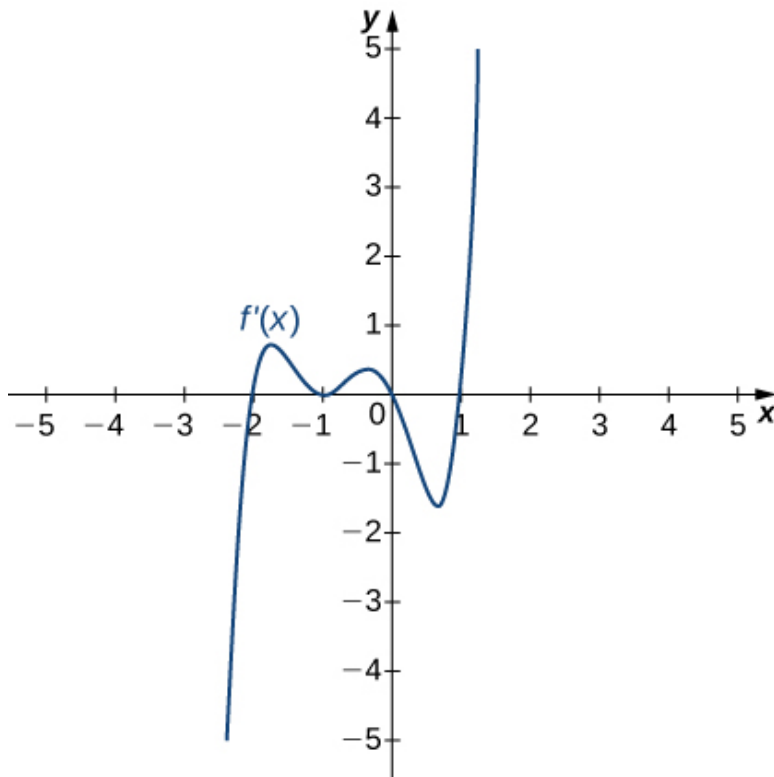


Solution:

Increasing for $-2 < x < -1$ and $x > 2$; decreasing for $x < -2$ and $-1 < x < 2$

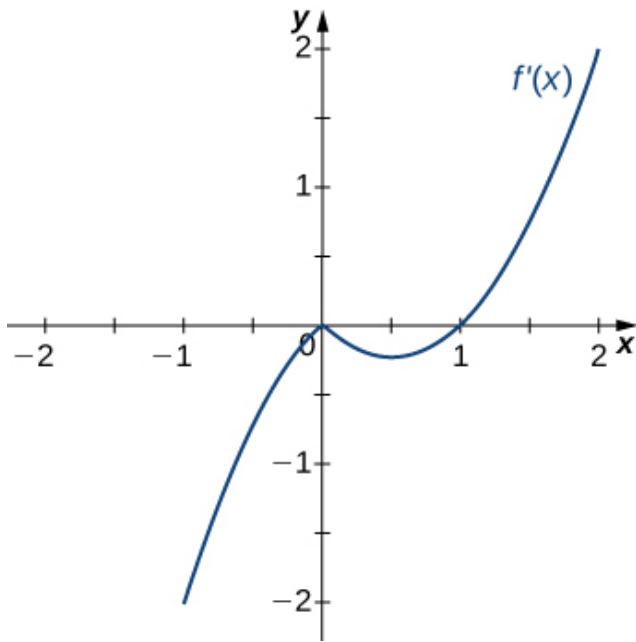
Exercise:

Problem:



Exercise:

Problem:

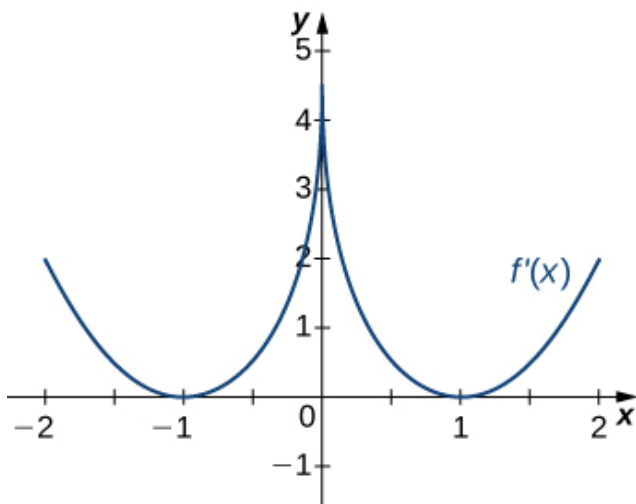


Solution:

Decreasing for $x < 1$, increasing for $x > 1$

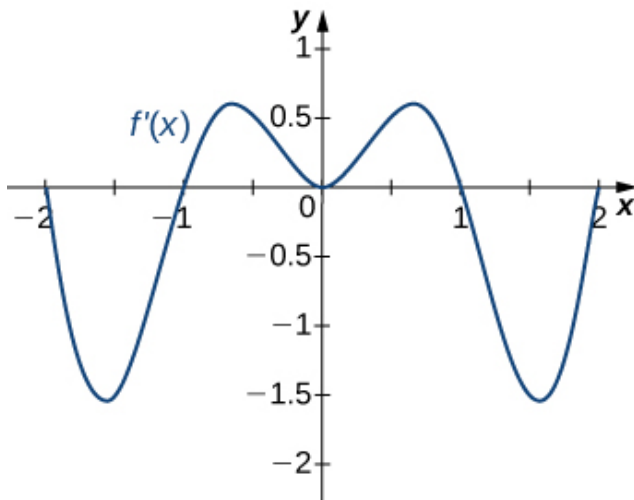
Exercise:

Problem:



Exercise:

Problem:



Solution:

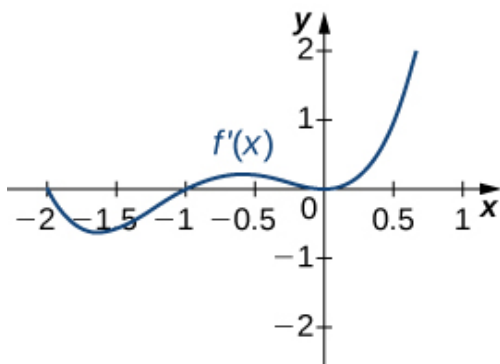
Decreasing for $-2 < x < -1$ and $1 < x < 2$; increasing for $-1 < x < 1$ and $x < -2$ and $x > 2$

For the following exercises, analyze the graphs of f' , then list all intervals where

- f is increasing and decreasing and
- the minima and maxima are located.

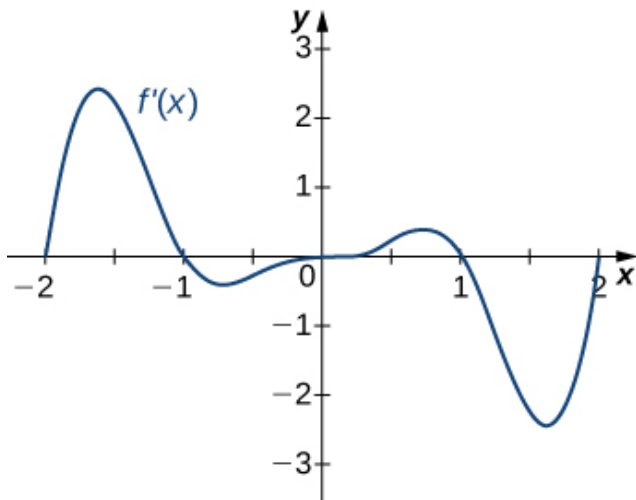
Exercise:

Problem:



Exercise:

Problem:

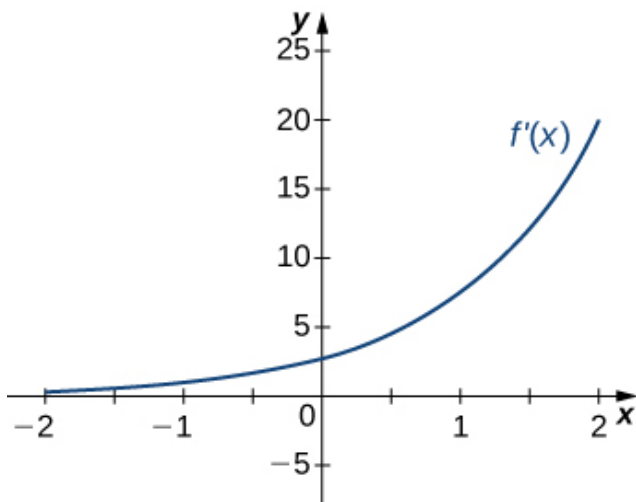


Solution:

a. Increasing over $-2 < x < -1$, $0 < x < 1$, $x > 2$, decreasing over $x < -2$, $-1 < x < 0$, $1 < x < 2$; b. maxima at $x = -1$ and $x = 1$, minima at $x = -2$ and $x = 0$ and $x = 2$

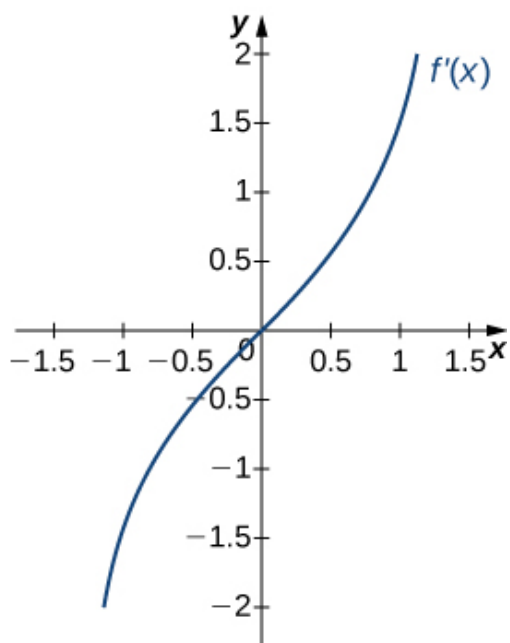
Exercise:

Problem:



Exercise:

Problem:

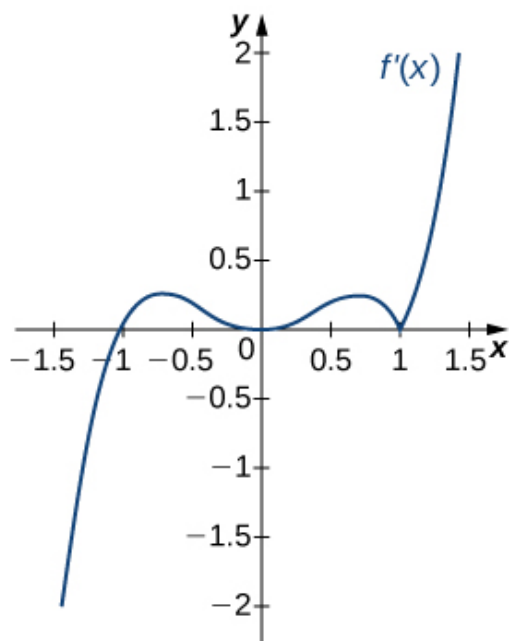


Solution:

a. Increasing over $x > 0$, decreasing over $x < 0$; b. Minimum at $x = 0$

Exercise:

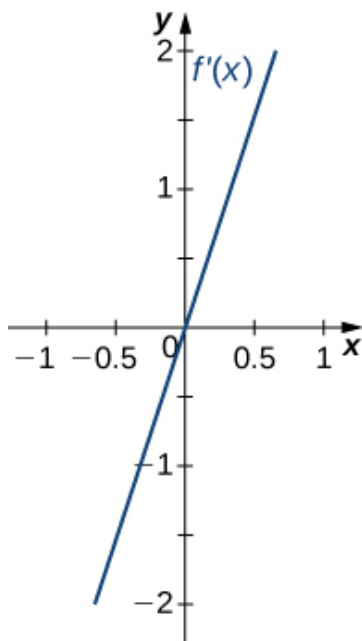
Problem:



For the following exercises, analyze the graphs of f' , then list all inflection points and intervals f that are concave up and concave down.

Exercise:

Problem:

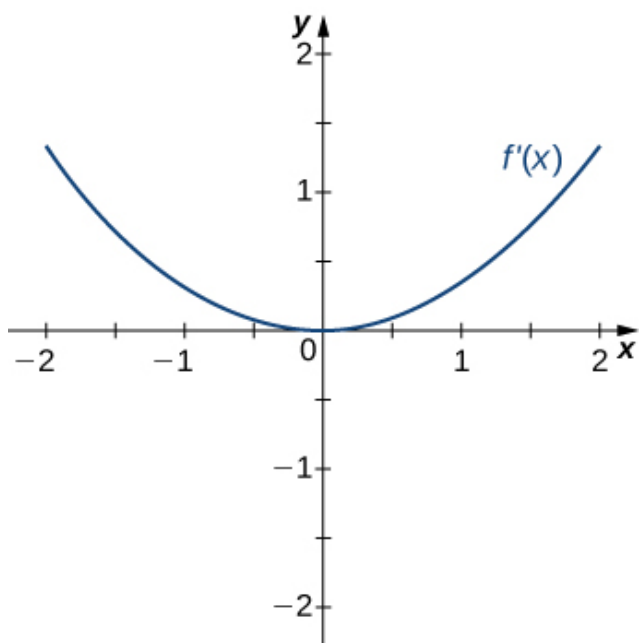


Solution:

Concave up on all x , no inflection points

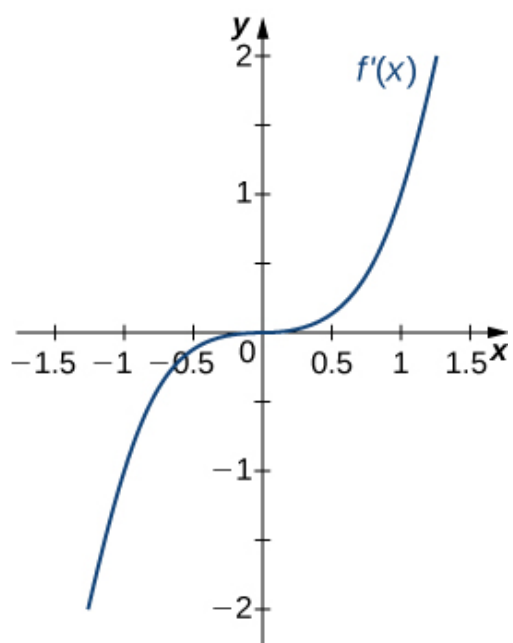
Exercise:

Problem:



Exercise:

Problem:

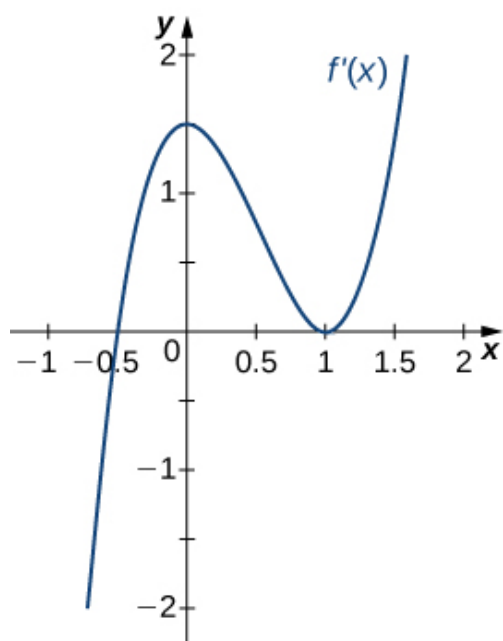


Solution:

Concave up on all x , no inflection points

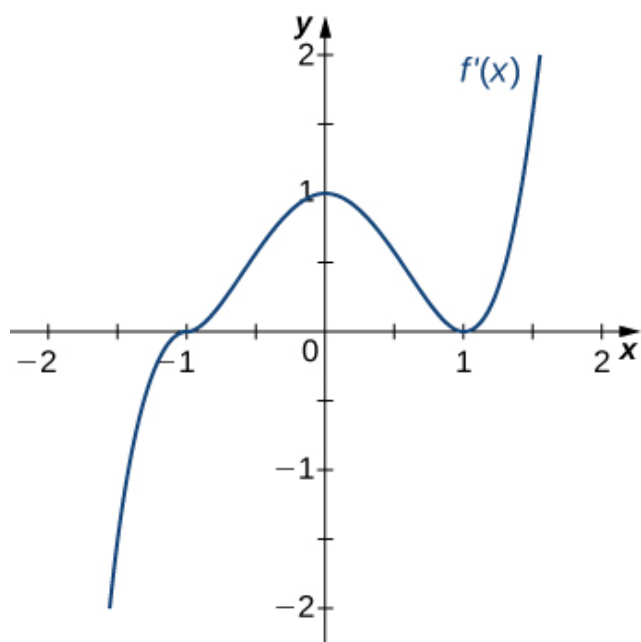
Exercise:

Problem:



Exercise:

Problem:



Solution:

Concave up for $x < 0$ and $x > 1$, concave down for $0 < x < 1$, inflection points at $x = 0$ and $x = 1$

For the following exercises, draw a graph that satisfies the given specifications for the domain $x = [-3, 3]$. The function does not have to be continuous or differentiable.

Exercise:**Problem:**

$$f(x) > 0, f'(x) > 0 \text{ over } x > 1, -3 < x < 0, f'(x) = 0 \text{ over } 0 < x < 1$$

Exercise:**Problem:**

$$f'(x) > 0 \text{ over } x > 2, -3 < x < -1, f'(x) < 0 \text{ over } -1 < x < 2, f''(x) < 0 \text{ for all } x$$

Solution:

Answers will vary

Exercise:**Problem:**

$$f''(x) < 0 \text{ over } -1 < x < 1, f''(x) > 0, -3 < x < -1, 1 < x < 3, \text{ local maximum at } x = 0, \text{ local minima at } x = \pm 2$$

Exercise:**Problem:**

There is a local maximum at $x = 2$, local minimum at $x = 1$, and the graph is neither concave up nor concave down.

Solution:

Answers will vary

Exercise:

Problem:

There are local maxima at $x = \pm 1$, the function is concave up for all x , and the function remains positive for all x .

For the following exercises, determine

- a. intervals where f is increasing or decreasing and
- b. local minima and maxima of f .

Exercise:

Problem: $f(x) = \sin x + \sin^3 x$ over $-\pi < x < \pi$

Solution:

- a. Increasing over $-\frac{\pi}{2} < x < \frac{\pi}{2}$, decreasing over $x < -\frac{\pi}{2}, x > \frac{\pi}{2}$
- b. Local maximum at $x = \frac{\pi}{2}$; local minimum at $x = -\frac{\pi}{2}$

Exercise:

Problem: $f(x) = x^2 + \cos x$

For the following exercises, determine a. intervals where f is concave up or concave down, and b. the inflection points of f .

Exercise:

Problem: $f(x) = x^3 - 4x^2 + x + 2$

Solution:

- a. Concave up for $x > \frac{4}{3}$, concave down for $x < \frac{4}{3}$
- b. Inflection point at $x = \frac{4}{3}$

For the following exercises, determine

- a. intervals where f is increasing or decreasing,
- b. local minima and maxima of f ,

- c. intervals where f is concave up and concave down, and
- d. the inflection points of f .

Exercise:

Problem: $f(x) = x^2 - 6x$

Exercise:

Problem: $f(x) = x^3 - 6x^2$

Solution:

- a. Increasing over $x < 0$ and $x > 4$, decreasing over $0 < x < 4$ b.
- Maximum at $x = 0$, minimum at $x = 4$ c. Concave up for $x > 2$, concave down for $x < 2$ d. Inflection point at $x = 2$

Exercise:

Problem: $f(x) = x^4 - 6x^3$

Exercise:

Problem: $f(x) = x^{11} - 6x^{10}$

Solution:

- a. Increasing over $x < 0$ and $x > \frac{60}{11}$, decreasing over $0 < x < \frac{60}{11}$ b.
- Minimum at $x = \frac{60}{11}$ c. Concave down for $x < \frac{54}{11}$, concave up for $x > \frac{54}{11}$
- d. Inflection point at $x = \frac{54}{11}$

Exercise:

Problem: $f(x) = x + x^2 - x^3$

Exercise:

Problem: $f(x) = x^2 + x + 1$

Solution:

- a. Increasing over $x > -\frac{1}{2}$, decreasing over $x < -\frac{1}{2}$ b. Minimum at $x = -\frac{1}{2}$ c. Concave up for all x d. No inflection points

Exercise:

Problem: $f(x) = x^3 + x^4$

For the following exercises, determine

- intervals where f is increasing or decreasing,
- local minima and maxima of f ,
- intervals where f is concave up and concave down, and
- the inflection points of f . Sketch the curve, then use a calculator to compare your answer. If you cannot determine the exact answer analytically, use a calculator.

Exercise:

Problem: [T] $f(x) = \sin(\pi x) - \cos(\pi x)$ over $x = [-1, 1]$

Solution:

- Increases over $-\frac{1}{4} < x < \frac{3}{4}$, decreases over $x > \frac{3}{4}$ and $x < -\frac{1}{4}$ b. Minimum at $x = -\frac{1}{4}$, maximum at $x = \frac{3}{4}$ c. Concave up for $-\frac{3}{4} < x < \frac{1}{4}$, concave down for $x < -\frac{3}{4}$ and $x > \frac{1}{4}$ d. Inflection points at $x = -\frac{3}{4}$, $x = \frac{1}{4}$

Exercise:

Problem: [T] $f(x) = x + \sin(2x)$ over $x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Exercise:

Problem: [T] $f(x) = \sin x + \tan x$ over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution:

- Increasing for all x b. No local minimum or maximum c. Concave up for $x > 0$, concave down for $x < 0$ d. Inflection point at $x = 0$

Exercise:

Problem: [T] $f(x) = (x - 2)^2(x - 4)^2$

Exercise:

Problem: [T] $f(x) = \frac{1}{1-x}, x \neq 1$

Solution:

a. Increasing for all x where defined b. No local minima or maxima c. Concave up for $x < 1$; concave down for $x > 1$ d. No inflection points in domain

Exercise:

Problem: [T] $f(x) = \frac{\sin x}{x}$ over $x = [-2\pi, 2\pi] \setminus \{0\}$

Exercise:

Problem: $f(x) = \sin(x)e^x$ over $x = [-\pi, \pi]$

Solution:

a. Increasing over $-\frac{\pi}{4} < x < \frac{3\pi}{4}$, decreasing over $x > \frac{3\pi}{4}, x < -\frac{\pi}{4}$ b. Minimum at $x = -\frac{\pi}{4}$, maximum at $x = \frac{3\pi}{4}$ c. Concave up for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, concave down for $x < -\frac{\pi}{2}, x > \frac{\pi}{2}$ d. Inflection points at $x = \pm \frac{\pi}{2}$

Exercise:

Problem: $f(x) = \ln x \sqrt{x}, x > 0$

Exercise:

Problem: $f(x) = \frac{1}{4}\sqrt{x} + \frac{1}{x}, x > 0$

Solution:

- a. Increasing over $x > 4$, decreasing over $0 < x < 4$ b. Minimum at $x = 4$
 c. Concave up for $0 < x < 8\sqrt[3]{2}$, concave down for $x > 8\sqrt[3]{2}$ d. Inflection point at $x = 8\sqrt[3]{2}$

Exercise:

Problem: $f(x) = \frac{e^x}{x}, x \neq 0$

For the following exercises, interpret the sentences in terms of f , f' , and f'' .

Exercise:

Problem: The population is growing more slowly. Here f is the population.

Solution:

$$f > 0, f' > 0, f'' < 0$$

Exercise:

Problem:

A bike accelerates faster, but a car goes faster. Here f = Bike's position minus Car's position.

Exercise:

Problem: The airplane lands smoothly. Here f is the plane's altitude.

Solution:

$$f > 0, f' < 0, f'' < 0$$

Exercise:

Problem: Stock prices are at their peak. Here f is the stock price.

Exercise:

Problem:

The economy is picking up speed. Here f is a measure of the economy, such as GDP.

Solution:

$$f > 0, f' > 0, f'' > 0$$

For the following exercises, consider a third-degree polynomial $f(x)$, which has the properties $f'(1) = 0$, $f'(3) = 0$. Determine whether the following statements are *true or false*. Justify your answer.

Exercise:

Problem: $f(x) = 0$ for some $1 \leq x \leq 3$

Exercise:

Problem: $f''(x) = 0$ for some $1 \leq x \leq 3$

Solution:

True, by the Mean Value Theorem

Exercise:

Problem: There is no absolute maximum at $x = 3$

Exercise:

Problem: If $f(x)$ has three roots, then it has 1 inflection point.

Solution:

True, examine derivative

Exercise:

Problem: If $f(x)$ has one inflection point, then it has three real roots.

Glossary

concave down

if f is differentiable over an interval I and f' is decreasing over I , then f is concave down over I

concave up

if f is differentiable over an interval I and f' is increasing over I , then f is concave up over I

concavity

the upward or downward curve of the graph of a function

concavity test

suppose f is twice differentiable over an interval I ; if $f'' > 0$ over I , then f is concave up over I ; if $f'' < 0$ over I , then f is concave down over I

first derivative test

let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c ; if f' changes sign from positive to negative as x increases through c , then f has a local maximum at c ; if f' changes sign from negative to positive as x increases through c , then f has a local minimum at c ; if f' does not change sign as x increases through c , then f does not have a local extremum at c

inflection point

if f is continuous at c and f changes concavity at c , the point $(c, f(c))$ is an inflection point of f

second derivative test

suppose $f'(c) = 0$ and f'' is continuous over an interval containing c ; if $f''(c) > 0$, then f has a local minimum at c ; if $f''(c) < 0$, then f has a local maximum at c ; if $f''(c) = 0$, then the test is inconclusive

Applied Optimization Problems

- Set up and solve optimization problems in several applied fields.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the **optimization problems** that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in [\[link\]](#), we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

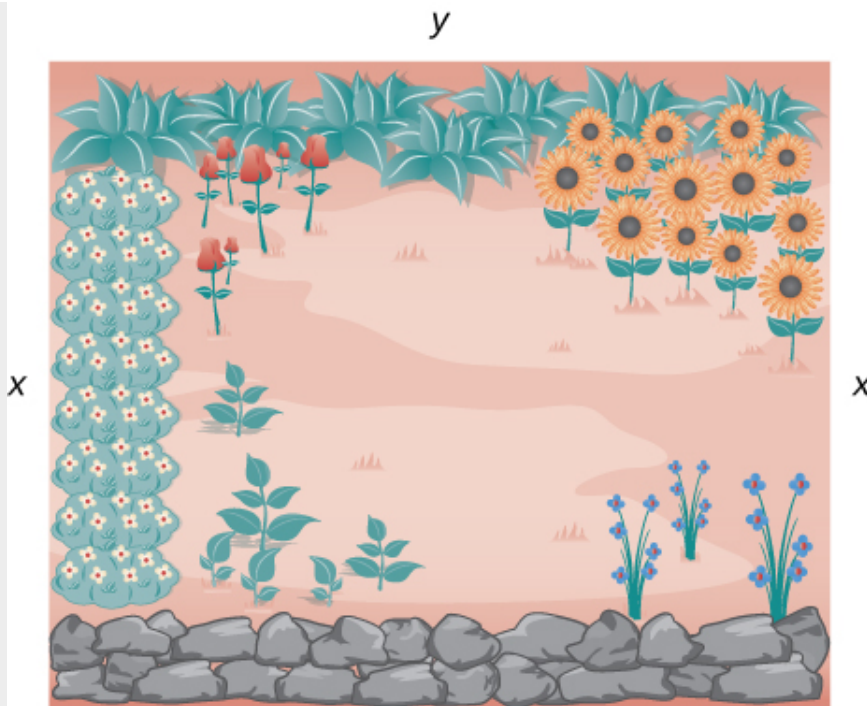
Example:

Exercise:

Problem:

Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides ([\[link\]](#)). Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?



We want to determine the measurements x and y that will create a garden with a maximum area using 100 ft of fencing.

Solution:

Let x denote the length of the side of the garden perpendicular to the rock wall and y denote the length of the side parallel to the rock wall. Then the area of the garden is

Equation:

$$A = x \cdot y.$$

We want to find the maximum possible area subject to the constraint that the total fencing is 100 ft. From [\[link\]](#), the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is

Equation:

$$2x + y = 100.$$

Solving this equation for y , we have $y = 100 - 2x$. Thus, we can write the area as

Equation:

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive.

Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for x over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value x in the open interval $(0, 50)$ that maximizes the area of the garden. Therefore, we consider the following problem:

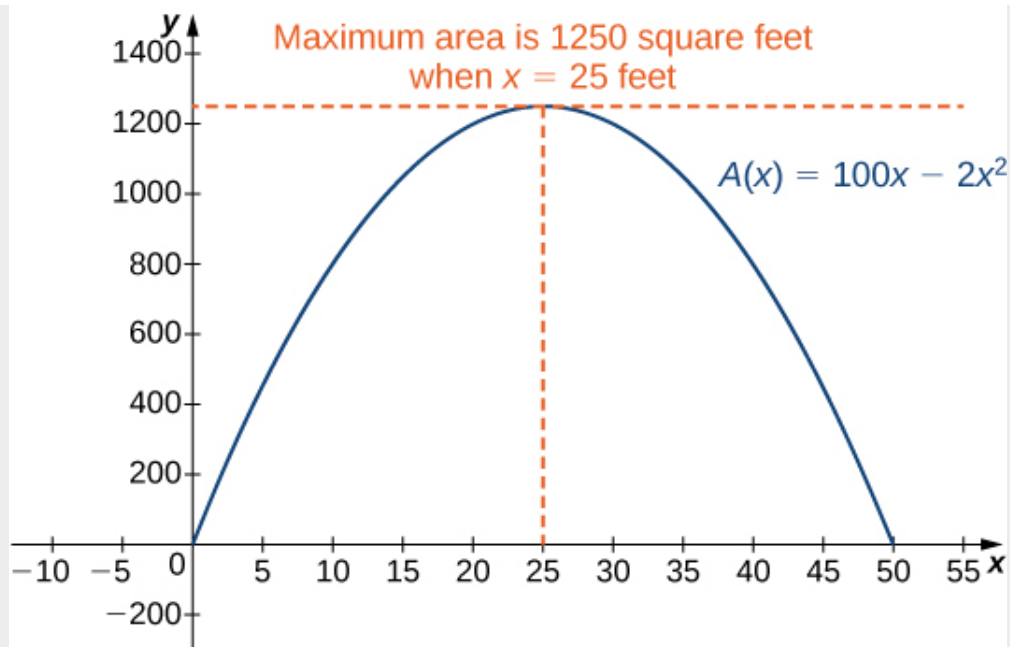
Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all x in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

Equation:

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$ ([link](#)). We conclude that the maximum area must occur when $x = 25$. Then we have $y = 100 - 2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25$ ft and $y = 50$ ft. The area of this garden is 1250 ft^2 .



To maximize the area of the garden, we need to find the maximum value of the function $A(x) = 100x - 2x^2$.

Note:

Exercise:

Problem:

Determine the maximum area if we want to make the same rectangular garden as in [\[link\]](#), but we have 200 ft of fencing.

Solution:

The maximum area is 5000 ft^2 .

Hint

We need to maximize the function $A(x) = 200x - 2x^2$ over the interval $[0, 100]$.

Now let's look at a general strategy for solving optimization problems similar to [\[link\]](#).

Note:

Problem-Solving Strategy: Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables in the formula from step 3. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step 4. This step typically involves looking for critical points and evaluating a function at endpoints.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

Example:

Exercise:

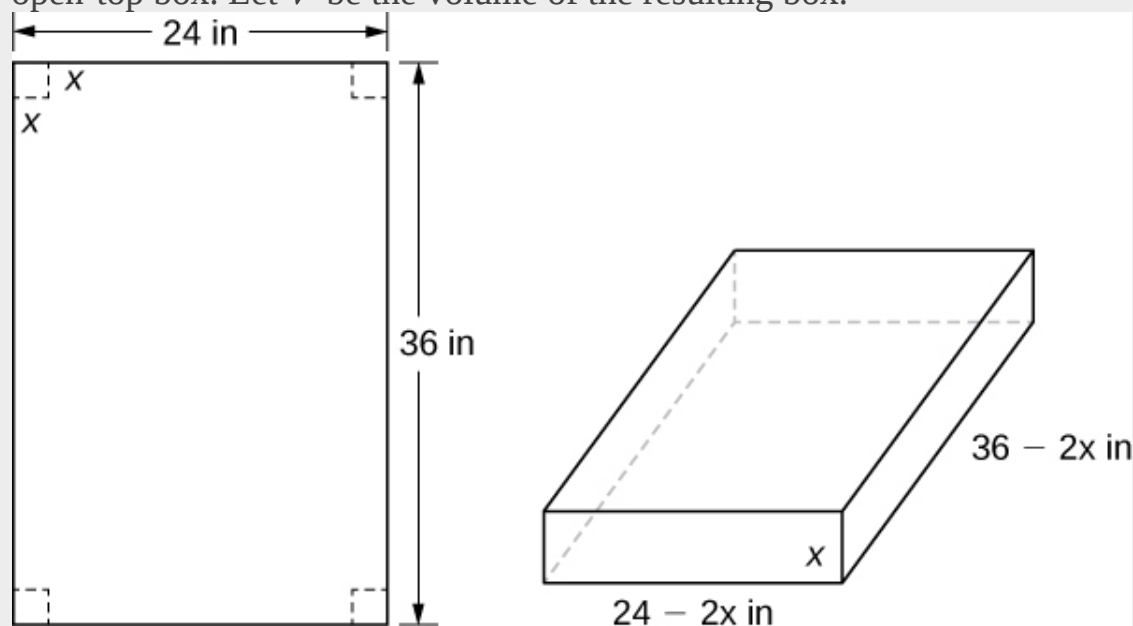
Problem:

Maximizing the Volume of a Box

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution:

Step 1: Let x be the side length of the square to be removed from each corner ([link](#)). Then, the remaining four flaps can be folded up to form an open-top box. Let V be the volume of the resulting box.



A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V .

Step 3: As mentioned in step 2, are trying to maximize the volume of a box. The volume of a box is $V = L \cdot W \cdot H$, where L , W , and H are the length, width, and height, respectively.

Step 4: From [link](#), we see that the height of the box is x inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

Equation:

$$V(x) = (36 - 2x)(24 - 2x)x = 4x^3 - 120x^2 + 864x.$$

Step 5: To determine the domain of consideration, let's examine [\[link\]](#). Certainly, we need $x > 0$. Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24 in.; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12)$. Since V is a continuous function over the closed interval $[0, 12]$, we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Step 6: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12]$, V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for $0 < x < 12$, the maximum must occur at a critical point. The derivative is **Equation:**

$$V'(x) = 12x^2 - 240x + 864.$$

To find the critical points, we need to solve the equation **Equation:**

$$12x^2 - 240x + 864 = 0.$$

Dividing both sides of this equation by 12, the problem simplifies to solving the equation

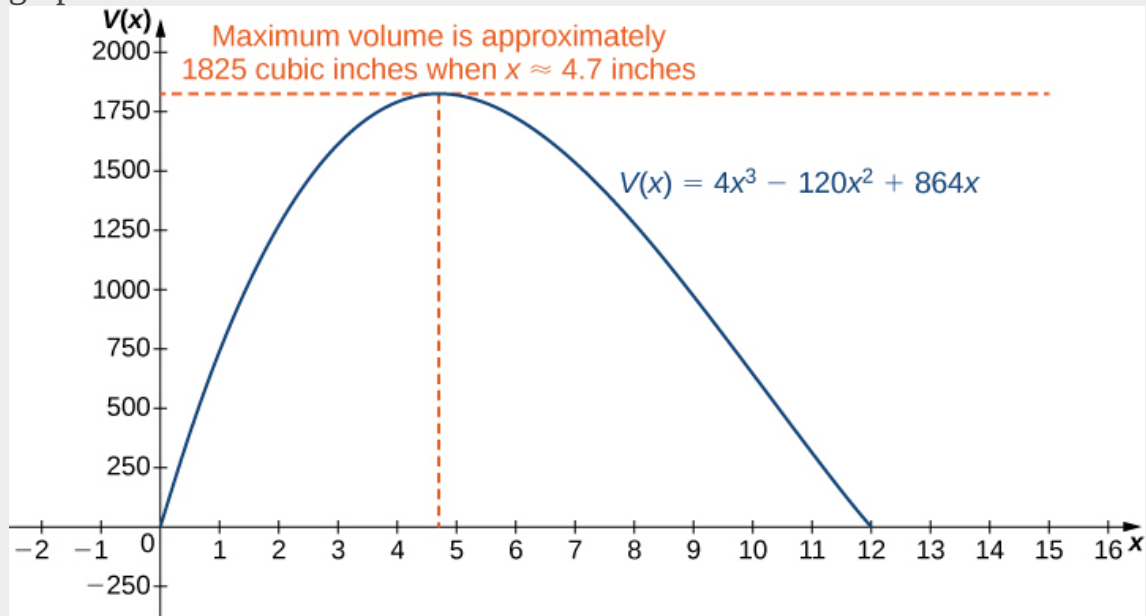
Equation:

$$x^2 - 20x + 72 = 0.$$

Using the quadratic formula, we find that the critical points are **Equation:**

$$x = \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} = \frac{20 \pm \sqrt{112}}{2} = \frac{20 \pm 4\sqrt{7}}{2} = 10 \pm 2\sqrt{7}.$$

Since $10 + 2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10 - 2\sqrt{7}$. Therefore, the volume is maximized if we let $x = 10 - 2\sqrt{7}$ in. The maximum volume is $V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825 \text{ in.}^3$ as shown in the following graph.



Maximizing the volume of the box leads to finding the maximum value of a cubic polynomial.

Note:

Watch a [video](#) about optimizing the volume of a box.

Note:

Exercise:

Problem:

Suppose the dimensions of the cardboard in [\[link\]](#) are 20 in. by 30 in. Let x be the side length of each square and write the volume of the open-top box as a function of x . Determine the domain of consideration for x .

Solution:

$V(x) = x(20 - 2x)(30 - 2x)$. The domain is $[0, 10]$.

Hint

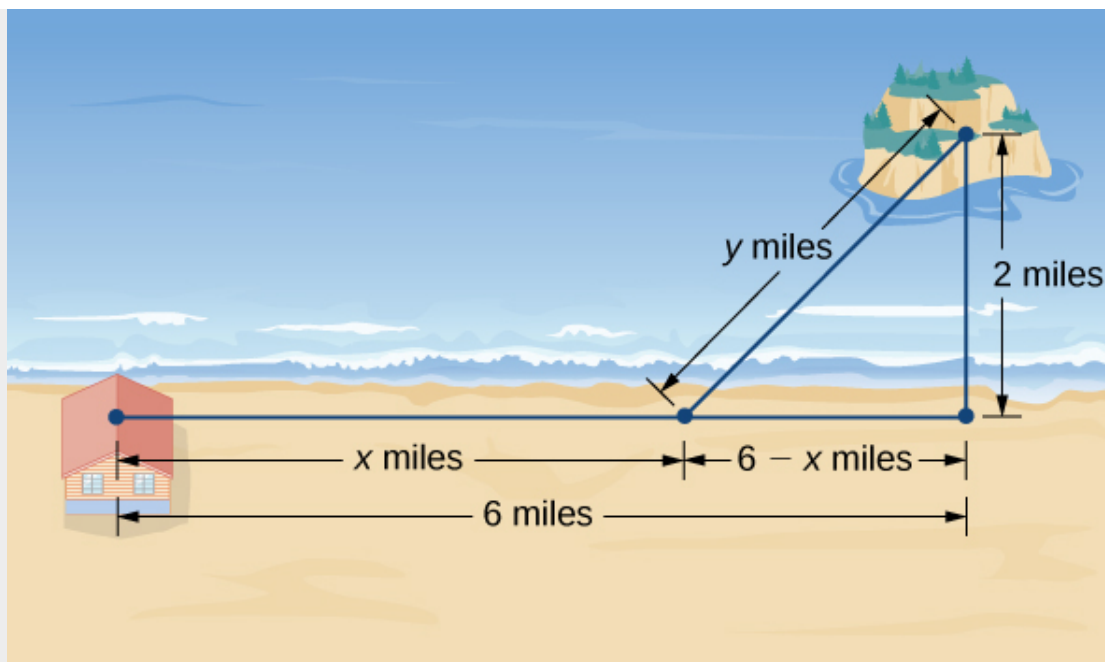
The volume of the box is $L \cdot W \cdot H$.

Example:**Exercise:****Problem:****Minimizing Travel Time**

An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution:

Step 1: Let x be the distance running and let y be the distance swimming ([\[link\]](#)). Let T be the time it takes to get from the cabin to the island.



How can we choose x and y to minimize the travel time from the cabin to the island?

Step 2: The problem is to minimize T .

Step 3: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance = Rate \times Time ($D = R \times T$), the time spent running is

Equation:

$$T_{\text{running}} = \frac{D_{\text{running}}}{R_{\text{running}}} = \frac{x}{8},$$

and the time spent swimming is

Equation:

$$T_{\text{swimming}} = \frac{D_{\text{swimming}}}{R_{\text{swimming}}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is

Equation:

$$T = \frac{x}{8} + \frac{y}{3}.$$

Step 4: From [\[link\]](#), the line segment of y miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x$ mi. Therefore, by the Pythagorean theorem, $2^2 + (6 - x)^2 = y^2$, and we obtain

$y = \sqrt{(6 - x)^2 + 4}$. Thus, the total time spent traveling is given by the function

Equation:

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6 - x)^2 + 4}}{3}.$$

Step 5: From [\[link\]](#), we see that $0 \leq x \leq 6$. Therefore, $[0, 6]$ is the domain of consideration.

Step 6: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval $[0, 6]$. The derivative is

Equation:

$$T'(x) = \frac{1}{8} - \frac{1}{2} \frac{\left[(6 - x)^2 + 4\right]^{-1/2}}{3} \cdot 2(6 - x) = \frac{1}{8} - \frac{(6 - x)}{3\sqrt{(6 - x)^2 + 4}}.$$

If $T'(x) = 0$, then

Equation:

$$\frac{1}{8} = \frac{6 - x}{3\sqrt{(6 - x)^2 + 4}}.$$

Therefore,

Equation:

$$3\sqrt{(6-x)^2 + 4} = 8(6-x).$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy

Equation:

$$9 \left[(6-x)^2 + 4 \right] = 64(6-x)^2,$$

which implies

Equation:

$$55(6-x)^2 = 36.$$

We conclude that if x is a critical point, then x satisfies

Equation:

$$(x-6)^2 = \frac{36}{55}.$$

Therefore, the possibilities for critical points are

Equation:

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + 6/\sqrt{55}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $x = 6 - 6/\sqrt{55}$ is in the domain. Since we squared both sides of [\[link\]](#) to arrive at the possible critical points, it remains to verify that $x = 6 - 6/\sqrt{55}$ satisfies [\[link\]](#). Since $x = 6 - 6/\sqrt{55}$ does satisfy that equation, we conclude that $x = 6 - 6/\sqrt{55}$ is a critical point, and it is the only one. To justify that the time is minimized for this value of x , we just need to check the values of

$T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical point $x = 6 - 6/\sqrt{55}$. We find that $T(0) \approx 2.108$ h and $T(6) \approx 1.417$ h, whereas $T\left(6 - 6/\sqrt{55}\right) \approx 1.368$ h. Therefore, we conclude that T has a local minimum at $x \approx 5.19$ mi.

Note:

Exercise:

Problem:

Suppose the island is 1 mi from shore, and the distance from the cabin to the point on the shore closest to the island is 15 mi. Suppose a visitor swims at the rate of 2.5 mph and runs at a rate of 6 mph. Let x denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

Solution:

$$T(x) = \frac{x}{6} + \frac{\sqrt{(15-x)^2 + 1}}{2.5}$$

Hint

The time $T = T_{\text{running}} + T_{\text{swimming}}$.

In business, companies are interested in maximizing revenue. In the following example, we consider a scenario in which a company has collected data on how many cars it is able to lease, depending on the price it charges its customers to rent a car. Let's use these data to determine the price the company should charge to maximize the amount of money it brings in.

Example:

Exercise:

Problem:
Maximizing Revenue

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \leq p \leq 200$, the number of cars n they rent per day can be modeled by the linear function $n(p) = 1000 - 5p$. If they charge \$50 per day or less, they will rent all their cars. If they charge \$200 per day or more, they will not rent any cars. Assuming the owners plan to charge customers between \$50 per day and \$200 per day to rent a car, how much should they charge to maximize their revenue?

Solution:

Step 1: Let p be the price charged per car per day and let n be the number of cars rented per day. Let R be the revenue per day.

Step 2: The problem is to maximize R .

Step 3: The revenue (per day) is equal to the number of cars rented per day times the price charged per car per day—that is, $R = n \times p$.

Step 4: Since the number of cars rented per day is modeled by the linear function $n(p) = 1000 - 5p$, the revenue R can be represented by the function

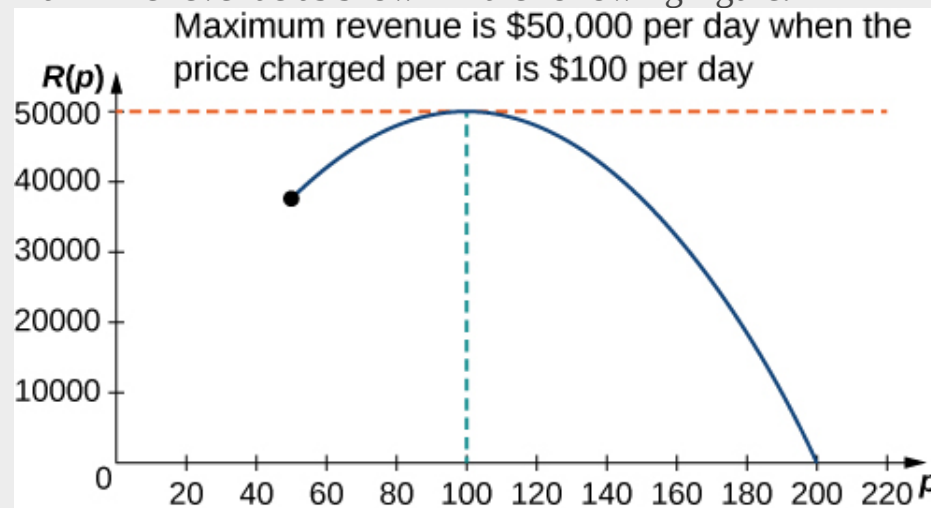
Equation:

$$R(p) = n \times p = (1000 - 5p)p = -5p^2 + 1000p.$$

Step 5: Since the owners plan to charge between \$50 per car per day and \$200 per car per day, the problem is to find the maximum revenue $R(p)$ for p in the closed interval $[50, 200]$.

Step 6: Since R is a continuous function over the closed, bounded interval $[50, 200]$, it has an absolute maximum (and an absolute minimum) in that interval. To find the maximum value, look for critical points. The derivative is $R'(p) = -10p + 1000$. Therefore, the critical point is $p = 100$. When $p = 100$, $R(100) = \$50,000$. When $p = 50$, $R(p) = \$37,500$. When $p = 200$, $R(p) = \$0$. Therefore, the absolute maximum occurs at

$p = \$100$. The car rental company should charge \$100 per day per car to maximize revenue as shown in the following figure.



To maximize revenue, a car rental company has to balance the price of a rental against the number of cars people will rent at that price.

Note:

Exercise:

Problem:

A car rental company charges its customers p dollars per day, where $60 \leq p \leq 150$. It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

Solution:

The company should charge \$75 per car per day.

Hint

$R(p) = n \times p$, where n is the number of cars rented and p is the price charged per car.

Example:

Exercise:

Problem:

Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse

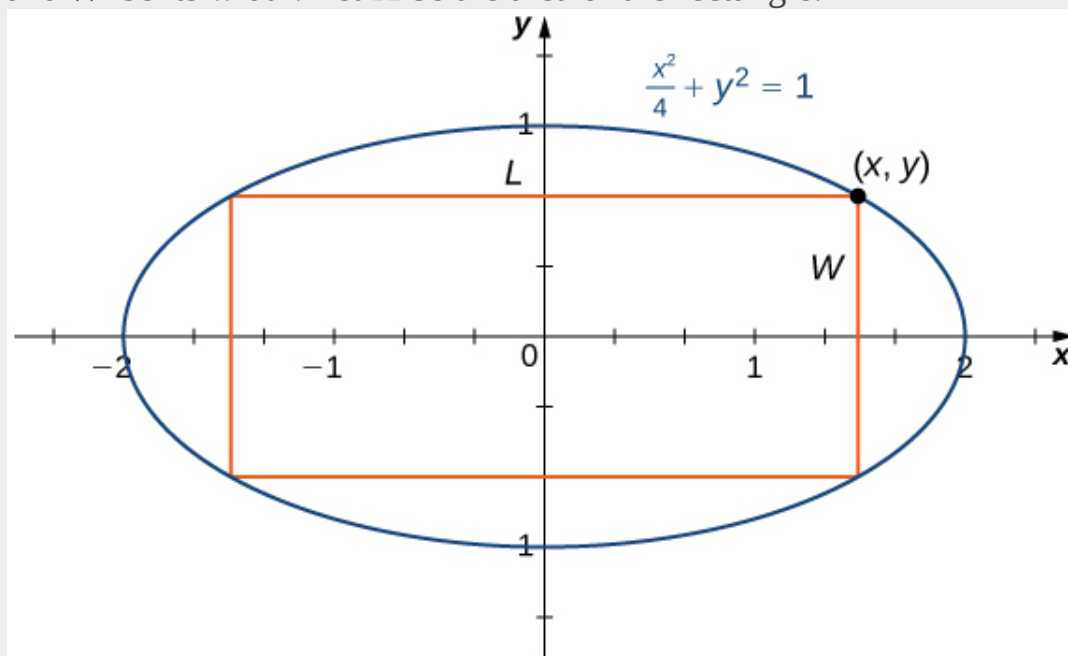
Equation:

$$\frac{x^2}{4} + y^2 = 1.$$

What should the dimensions of the rectangle be to maximize its area? What is the maximum area?

Solution:

Step 1: For a rectangle to be inscribed in the ellipse, the sides of the rectangle must be parallel to the axes. Let L be the length of the rectangle and W be its width. Let A be the area of the rectangle.



We want to maximize the area of a rectangle inscribed in an ellipse.

Step 2: The problem is to maximize A .

Step 3: The area of the rectangle is $A = LW$.

Step 4: Let (x, y) be the corner of the rectangle that lies in the first quadrant, as shown in [\[link\]](#). We can write length $L = 2x$ and width $W = 2y$. Since $\frac{x^2}{4+y^2}=1$ and $y > 0$, we have $y = \sqrt{\frac{1-x^2}{4}}$. Therefore, the area is

Equation:

$$A = LW = (2x)(2y) = 4x\sqrt{\frac{1-x^2}{4}} = 2x\sqrt{4-x^2}.$$

Step 5: From [\[link\]](#), we see that to inscribe a rectangle in the ellipse, the x -coordinate of the corner in the first quadrant must satisfy $0 < x < 2$. Therefore, the problem reduces to looking for the maximum value of $A(x)$ over the open interval $(0, 2)$. Since $A(x)$ will have an absolute maximum (and absolute minimum) over the closed interval $[0, 2]$, we consider $A(x) = 2x\sqrt{4-x^2}$ over the interval $[0, 2]$. If the absolute maximum occurs at an interior point, then we have found an absolute maximum in the open interval.

Step 6: As mentioned earlier, $A(x)$ is a continuous function over the closed, bounded interval $[0, 2]$. Therefore, it has an absolute maximum (and absolute minimum). At the endpoints $x = 0$ and $x = 2$, $A(x) = 0$. For $0 < x < 2$, $A(x) > 0$. Therefore, the maximum must occur at a critical point. Taking the derivative of $A(x)$, we obtain

Equation:

$$\begin{aligned} A'(x) &= 2\sqrt{4-x^2} + 2x \cdot \frac{1}{2\sqrt{4-x^2}}(-2x) \\ &= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}} \\ &= \frac{8-4x^2}{\sqrt{4-x^2}}. \end{aligned}$$

To find critical points, we need to find where $A'(x) = 0$. We can see that if x is a solution of

Equation:

$$\frac{8 - 4x^2}{\sqrt{4 - x^2}} = 0,$$

then x must satisfy

Equation:

$$8 - 4x^2 = 0.$$

Therefore, $x^2 = 2$. Thus, $x = \pm\sqrt{2}$ are the possible solutions of [\[link\]](#).

Since we are considering x over the interval $[0, 2]$, $x = \sqrt{2}$ is a possibility for a critical point, but $x = -\sqrt{2}$ is not. Therefore, we check whether $\sqrt{2}$ is a solution of [\[link\]](#). Since $x = \sqrt{2}$ is a solution of [\[link\]](#), we conclude that $\sqrt{2}$ is the only critical point of $A(x)$ in the interval $[0, 2]$. Therefore, $A(x)$ must have an absolute maximum at the critical point $x = \sqrt{2}$. To determine the dimensions of the rectangle, we need to find the length L and the width W . If $x = \sqrt{2}$ then

Equation:

$$y = \sqrt{1 - \frac{(\sqrt{2})^2}{4}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore, the dimensions of the rectangle are $L = 2x = 2\sqrt{2}$ and $W = 2y = \frac{2}{\sqrt{2}} = \sqrt{2}$. The area of this rectangle is

$$A = LW = (2\sqrt{2})(\sqrt{2}) = 4.$$

Note:

Exercise:

Problem:

Modify the area function A if the rectangle is to be inscribed in the unit circle $x^2 + y^2 = 1$. What is the domain of consideration?

Solution:

$A(x) = 4x\sqrt{1 - x^2}$. The domain of consideration is $[0, 1]$.

Hint

If (x, y) is the vertex of the square that lies in the first quadrant, then the area of the square is $A = (2x)(2y) = 4xy$.

Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded. For example, the function $f(x) = x^2 + 4$ over $(-\infty, \infty)$ has an absolute minimum of 4 at $x = 0$. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema. In the next example, we try to minimize a function over an unbounded domain. We will see that, although the domain of consideration is $(0, \infty)$, the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

Example:

Exercise:

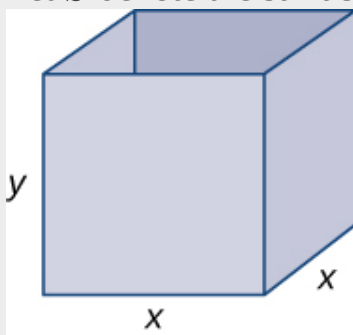
Problem:

Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of 216 in.^3 is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Solution:

Step 1: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.



We want to minimize the surface area of a square-based box with a given volume.

Step 2: We need to minimize the surface area. Therefore, we need to minimize S .

Step 3: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is x^2 . Therefore, the surface area of the box is

Equation:

$$S = 4xy + x^2.$$

Step 4: Since the volume of this box is x^2y and the volume is given as 216 in.³, the constraint equation is

Equation:

$$x^2y = 216.$$

Solving the constraint equation for y , we have $y = \frac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

Equation:

$$S(x) = 4x \left(\frac{216}{x^2} \right) + x^2.$$

Therefore, $S(x) = \frac{864}{x} + x^2$.

Step 5: Since we are requiring that $x^2y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, x is allowed to have any positive value. Note that as x becomes large, the height of the box y becomes correspondingly small so that $x^2y = 216$. Similarly, as x becomes small, the height of the box becomes correspondingly large. We conclude that the domain is the open, unbounded interval $(0, \infty)$. Note that, unlike the previous examples, we cannot reduce our problem to looking for an absolute maximum or absolute minimum over a closed, bounded interval. However, in the next step, we discover why this function must have an absolute minimum over the interval $(0, \infty)$.

Step 6: Note that as $x \rightarrow 0^+$, $S(x) \rightarrow \infty$. Also, as $x \rightarrow \infty$, $S(x) \rightarrow \infty$. Since S is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of S . The derivative is

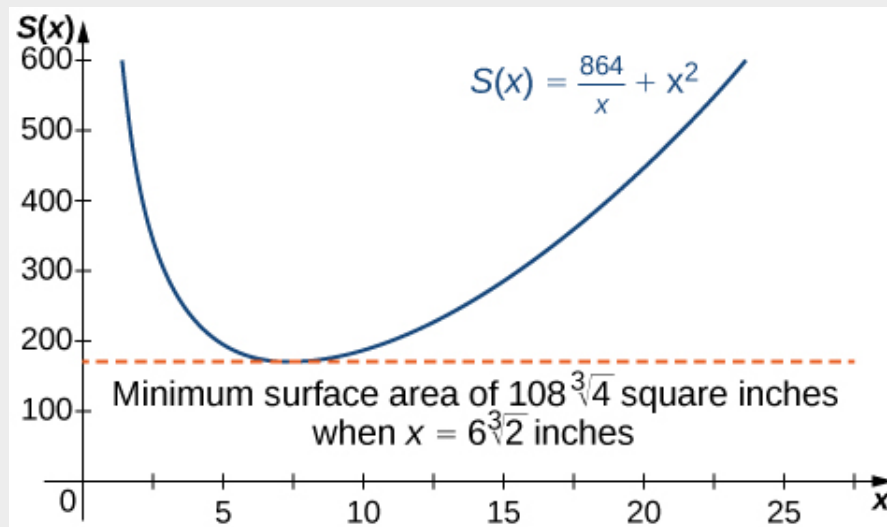
Equation:

$$S'(x) = -\frac{864}{x^2} + 2x.$$

Therefore, $S'(x) = 0$ when $2x = \frac{864}{x^2}$. Solving this equation for x , we obtain $x^3 = 432$, so $x = \sqrt[3]{432} = 6\sqrt[3]{2}$. Since this is the only critical point of S , the absolute minimum must occur at $x = 6\sqrt[3]{2}$ (see [\[link\]](#)). When $x = 6\sqrt[3]{2}$, $y = \frac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}$ in. Therefore, the dimensions of the box should be $x = 6\sqrt[3]{2}$ in. and $y = 3\sqrt[3]{2}$ in. With these dimensions, the surface area is

Equation:

$$S(6\sqrt[3]{2}) = \frac{864}{6\sqrt[3]{2}} + (6\sqrt[3]{2})^2 = 108\sqrt[3]{4} \text{ in.}^2$$



We can use a graph to determine the dimensions of a box of given the volume and the minimum surface area.

Note:

Exercise:

Problem:

Consider the same open-top box, which is to have volume 216 in.^3 . Suppose the cost of the material for the base is $20\text{¢}/\text{in.}^2$ and the cost of the material for the sides is $30\text{¢}/\text{in.}^2$ and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let x be the side length of the base and y be the height of the box.)

Solution:

$$c(x) = \frac{259.2}{x} + 0.2x^2 \text{ dollars}$$

Hint

If the cost of one of the sides is $30\text{¢}/\text{in.}^2$, the cost of that side is $0.30xy$.

Key Concepts

- To solve an optimization problem, begin by drawing a picture and introducing variables.
- Find an equation relating the variables.
- Find a function of one variable to describe the quantity that is to be minimized or maximized.
- Look for critical points to locate local extrema.

For the following exercises, answer by proof, counterexample, or explanation.

Exercise:**Problem:**

When you find the maximum for an optimization problem, why do you need to check the sign of the derivative around the critical points?

Solution:

The critical points can be the minima, maxima, or neither.

Exercise:

Problem:

Why do you need to check the endpoints for optimization problems?

Exercise:**Problem:**

True or False. For every continuous nonlinear function, you can find the value x that maximizes the function.

Solution:

False; $y = -x^2$ has a minimum only

Exercise:**Problem:**

True or False. For every continuous nonconstant function on a closed, finite domain, there exists at least one x that minimizes or maximizes the function.

For the following exercises, set up and evaluate each optimization problem.

Exercise:**Problem:**

To carry a suitcase on an airplane, the length + width + height of the box must be less than or equal to 62 in. Assuming the height is fixed, show that the maximum volume is $V = h\left(31 - \left(\frac{1}{2}\right)h\right)^2$. What height allows you to have the largest volume?

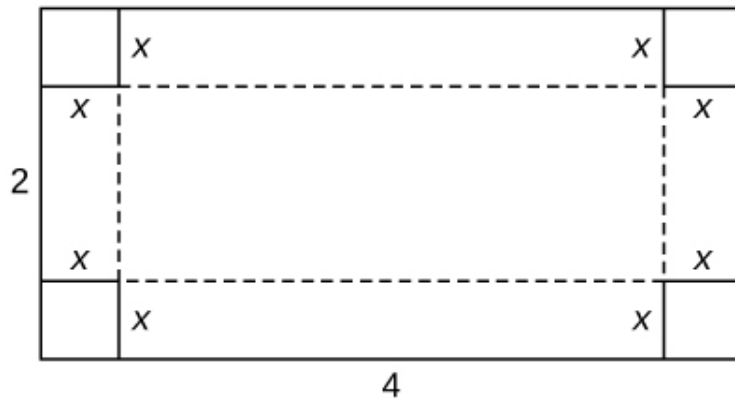
Solution:

$$h = \frac{62}{3} \text{ in.}$$

Exercise:

Problem:

You are constructing a cardboard box with the dimensions 2 m by 4 m. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?

**Exercise:****Problem:**

Find the positive integer that minimizes the sum of the number and its reciprocal.

Solution:

1

Exercise:**Problem:**

Find two positive integers such that their sum is 10, and minimize and maximize the sum of their squares.

For the following exercises, consider the construction of a pen to enclose an area.

Exercise:**Problem:**

You have 400 ft of fencing to construct a rectangular pen for cattle. What are the dimensions of the pen that maximize the area?

Solution:

100 ft by 100 ft

Exercise:**Problem:**

You have 800 ft of fencing to make a pen for hogs. If you have a river on one side of your property, what is the dimension of the rectangular pen that maximizes the area?

Exercise:**Problem:**

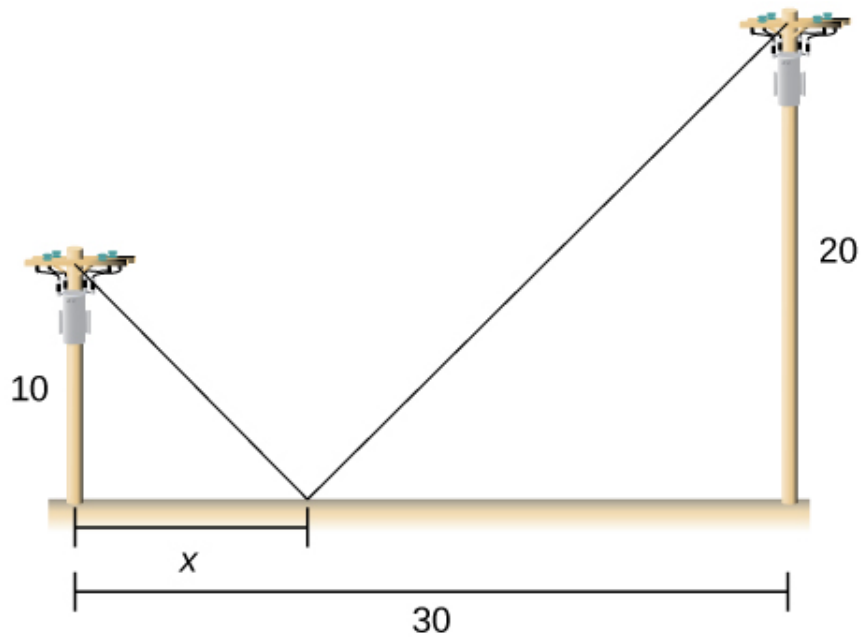
You need to construct a fence around an area of 1600 ft. What are the dimensions of the rectangular pen to minimize the amount of material needed?

Solution:

40 ft by 40 ft

Exercise:**Problem:**

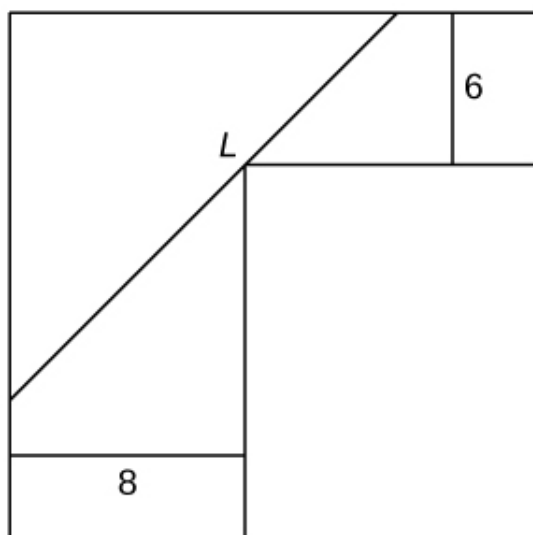
Two poles are connected by a wire that is also connected to the ground. The first pole is 20 ft tall and the second pole is 10 ft tall. There is a distance of 30 ft between the two poles. Where should the wire be anchored to the ground to minimize the amount of wire needed?



Exercise:

Problem:

[T] You are moving into a new apartment and notice there is a corner where the hallway narrows from 8 ft to 6 ft. What is the length of the longest item that can be carried horizontally around the corner?



Solution:

19.73 ft.

Exercise:

Problem:

A patient's pulse measures 70 bpm, 80 bpm, then 120 bpm. To determine an accurate measurement of pulse, the doctor wants to know what value minimizes the expression $(x - 70)^2 + (x - 80)^2 + (x - 120)^2$? What value minimizes it?

Exercise:

Problem:

In the previous problem, assume the patient was nervous during the third measurement, so we only weight that value half as much as the others. What is the value that minimizes $(x - 70)^2 + (x - 80)^2 + \frac{1}{2}(x - 120)^2$?

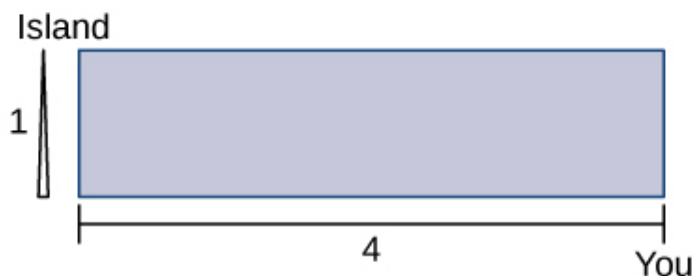
Solution:

84 bpm

Exercise:

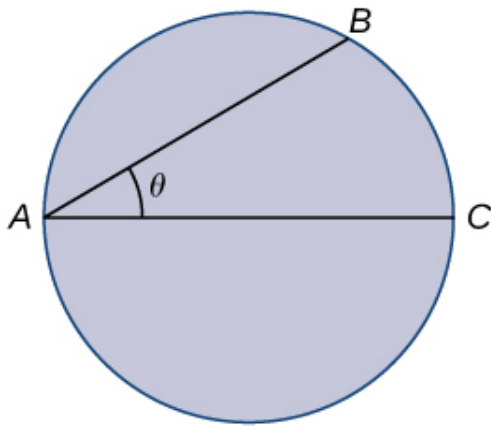
Problem:

You can run at a speed of 6 mph and swim at a speed of 3 mph and are located on the shore, 4 miles east of an island that is 1 mile north of the shoreline. How far should you run west to minimize the time needed to reach the island?



For the following problems, consider a lifeguard at a circular pool with diameter 40 m. He must reach someone who is drowning on the exact opposite side of the

pool, at position C . The lifeguard swims with a speed v and runs around the pool at speed $w = 3v$.



Exercise:

Problem:

Find a function that measures the total amount of time it takes to reach the drowning person as a function of the swim angle, θ .

Solution:

$$T(\theta) = \frac{40\theta}{3v} + \frac{40\cos\theta}{v}$$

Exercise:

Problem:

Find at what angle θ the lifeguard should swim to reach the drowning person in the least amount of time.

Exercise:

Problem:

A truck uses gas as $g(v) = av + \frac{b}{v}$, where v represents the speed of the truck and g represents the gallons of fuel per mile. At what speed is fuel consumption minimized?

Solution:

$$v = \sqrt{\frac{b}{a}}$$

For the following exercises, consider a limousine that gets

$m(v) = \frac{(120-2v)}{5}$ mi/gal at speed v , the chauffeur costs \$15/h, and gas is \$3.5/gal.

Exercise:

Problem: Find the cost per mile at speed v .

Exercise:

Problem: Find the cheapest driving speed.

Solution:

approximately 34.02 mph

For the following exercises, consider a pizzeria that sell pizzas for a revenue of $R(x) = ax$ and costs $C(x) = b + cx + dx^2$, where x represents the number of pizzas.

Exercise:

Problem:

Find the profit function for the number of pizzas. How many pizzas gives the largest profit per pizza?

Exercise:

Problem:

Assume that $R(x) = 10x$ and $C(x) = 2x + x^2$. How many pizzas sold maximizes the profit?

Solution:

4

Exercise:

Problem:

Assume that $R(x) = 15x$, and $C(x) = 60 + 3x + \frac{1}{2}x^2$. How many pizzas sold maximizes the profit?

For the following exercises, consider a wire 4 ft long cut into two pieces. One piece forms a circle with radius r and the other forms a square of side x .

Exercise:

Problem: Choose x to maximize the sum of their areas.

Solution:

0

Exercise:

Problem: Choose x to minimize the sum of their areas.

For the following exercises, consider two nonnegative numbers x and y such that $x + y = 10$. Maximize and minimize the quantities.

Exercise:

Problem: xy

Solution:

Maximal: $x = 5, y = 5$; minimal: $x = 0, y = 10$ and $y = 0, x = 10$

Exercise:

Problem: x^2y^2

Exercise:

Problem: $y - \frac{1}{x}$

Solution:

Maximal: $x = 1, y = 9$; minimal: none

Exercise:

Problem: $x^2 - y$

For the following exercises, draw the given optimization problem and solve.

Exercise:

Problem:

Find the volume of the largest right circular cylinder that fits in a sphere of radius 1.

Solution:

$$\frac{4\pi}{3\sqrt{3}}$$

Exercise:

Problem:

Find the volume of the largest right cone that fits in a sphere of radius 1.

Exercise:

Problem:

Find the area of the largest rectangle that fits into the triangle with sides $x = 0, y = 0$ and $\frac{x}{4} + \frac{y}{6} = 1$.

Solution:

6

Exercise:

Problem:

Find the largest volume of a cylinder that fits into a cone that has base radius R and height h .

Exercise:

Problem:

Find the dimensions of the closed cylinder volume $V = 16\pi$ that has the least amount of surface area.

Solution:

$$r = 2, h = 4$$

Exercise:**Problem:**

Find the dimensions of a right cone with surface area $S = 4\pi$ that has the largest volume.

For the following exercises, consider the points on the given graphs. Use a calculator to graph the functions.

Exercise:

Problem: [T] Where is the line $y = 5 - 2x$ closest to the origin?

Solution:

$$(2, 1)$$

Exercise:

Problem: [T] Where is the line $y = 5 - 2x$ closest to point $(1, 1)$?

Exercise:

Problem: [T] Where is the parabola $y = x^2$ closest to point $(2, 0)$?

Solution:

$$(0.8351, 0.6974)$$

Exercise:

Problem: [T] Where is the parabola $y = x^2$ closest to point $(0, 3)$?

For the following exercises, set up, but do not evaluate, each optimization problem.

Exercise:

Problem:

A window is composed of a semicircle placed on top of a rectangle. If you have 20 ft of window-framing materials for the outer frame, what is the maximum size of the window you can create? Use r to represent the radius of the semicircle.



Solution:

$$A = 20r - 2r^2 - \frac{1}{2}\pi r^2$$

Exercise:

Problem:

You have a garden row of 20 watermelon plants that produce an average of 30 watermelons apiece. For any additional watermelon plants planted, the output per watermelon plant drops by one watermelon. How many extra watermelon plants should you plant?

Exercise:**Problem:**

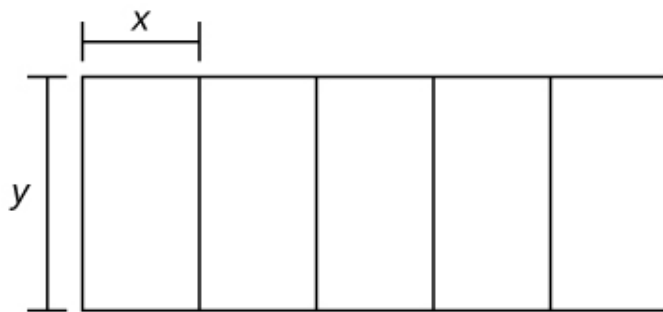
You are constructing a box for your cat to sleep in. The plush material for the square bottom of the box costs $\$5/\text{ft}^2$ and the material for the sides costs $\$2/\text{ft}^2$. You need a box with volume 4 ft^3 . Find the dimensions of the box that minimize cost. Use x to represent the length of the side of the box.

Solution:

$$C(x) = 5x^2 + \frac{32}{x}$$

Exercise:**Problem:**

You are building five identical pens adjacent to each other with a total area of 1000 m^2 , as shown in the following figure. What dimensions should you use to minimize the amount of fencing?

**Exercise:**

Problem:

You are the manager of an apartment complex with 50 units. When you set rent at \$800/month, all apartments are rented. As you increase rent by \$25/month, one fewer apartment is rented. Maintenance costs run \$50/month for each occupied unit. What is the rent that maximizes the total amount of profit?

Solution:

$$P(x) = (50 - x)(800 + 25x - 50)$$

Glossary

optimization problems

problems that are solved by finding the maximum or minimum value of a function

Introduction

class="introduction"

Sir Isaac
Newton
developed
many of
the
concepts
calculus
is based
upon.
(credit:
modified
from
image by
P Pix,
Flickr)



The sculpture of Sir Isaac Newton measuring the universe by Scottish sculptor Sir Eduardo Paolozzi is located in the plaza of the British Library in London, England. A technique known as Newton's Method will be studied in this chapter. (See [\[link\]](#).)

Newton's Method uses the tangent line to approximate function values.

Linear Approximations and Differentials

- Describe the linear approximation to a function at a point.
- Write the linearization of a given function.
- Draw a graph that illustrates the use of differentials to approximate the change in a quantity.
- Calculate the relative error and percentage error in using a differential approximation.

We have just seen how derivatives allow us to compare related quantities that are changing over time. In this section, we examine another application of derivatives: the ability to approximate functions locally by linear functions. Linear functions are the easiest functions with which to work, so they provide a useful tool for approximating function values. In addition, the ideas presented in this section are generalized later in the text when we study how to approximate functions by higher-degree polynomials [Introduction to Power Series and Functions](#).

Linear Approximation of a Function at a Point

Consider a function f that is differentiable at a point $x = a$. Recall that the tangent line to the graph of f at a is given by the equation

Equation:

$$y = f(a) + f'(a)(x - a).$$

For example, consider the function $f(x) = \frac{1}{x}$ at $a = 2$. Since f is differentiable at $x = 2$ and $f'(x) = -\frac{1}{x^2}$, we see that $f'(2) = -\frac{1}{4}$. Therefore, the tangent line to the graph of f at $a = 2$ is given by the equation

Equation:

$$y = \frac{1}{2} - \frac{1}{4}(x - 2).$$

[\[link\]](#)(a) shows a graph of $f(x) = \frac{1}{x}$ along with the tangent line to f at $x = 2$. Note that for x near 2, the graph of the tangent line is close to the graph of f . As a result, we can use the equation of the tangent line to approximate $f(x)$ for x near 2. For example, if $x = 2.1$, the y value of the corresponding point on the tangent line is

Equation:

$$y = \frac{1}{2} - \frac{1}{4}(2.1 - 2) = 0.475.$$

The actual value of $f(2.1)$ is given by

Equation:

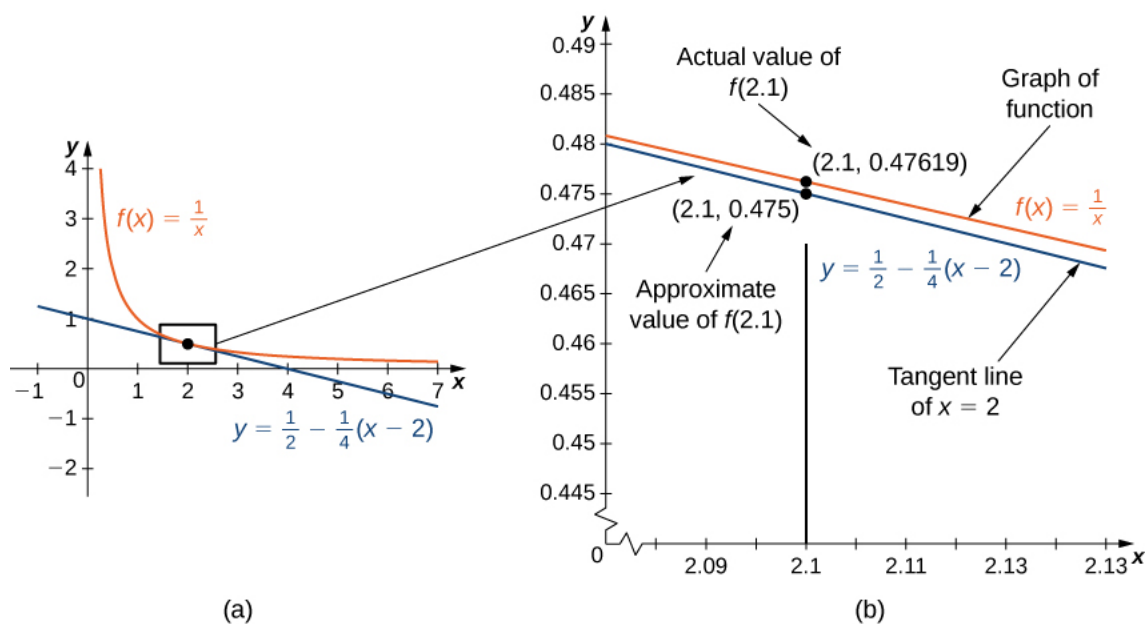
$$f(2.1) = \frac{1}{2.1} \approx 0.47619.$$

Therefore, the tangent line gives us a fairly good approximation of $f(2.1)$ ([\[link\]](#)(b)). However, note that for values of x far from 2, the equation of the tangent line does not give us a good approximation. For example, if $x = 10$, the y -value of the corresponding point on the tangent line is

Equation:

$$y = \frac{1}{2} - \frac{1}{4}(10 - 2) = \frac{1}{2} - 2 = -1.5,$$

whereas the value of the function at $x = 10$ is $f(10) = 0.1$.



(a) The tangent line to $f(x) = 1/x$ at $x = 2$ provides a good approximation to f for x near 2. (b) At $x = 2.1$, the value of y on the tangent line to $f(x) = 1/x$ is 0.475. The actual value of $f(2.1)$ is $1/2.1$, which is approximately 0.47619.

In general, for a differentiable function f , the equation of the tangent line to f at $x = a$ can be used to approximate $f(x)$ for x near a . Therefore, we can write

Equation:

$$f(x) \approx f(a) + f'(a)(x - a) \text{ for } x \text{ near } a.$$

We call the linear function

Equation:

$$L(x) = f(a) + f'(a)(x - a)$$

the **linear approximation**, or **tangent line approximation**, of f at $x = a$. This function L is also known as the **linearization** of f at $x = a$.

To show how useful the linear approximation can be, we look at how to find the linear approximation for $f(x) = \sqrt{x}$ at $x = 9$.

Example:

Exercise:

Problem:

Linear Approximation of \sqrt{x}

Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ and use the approximation to estimate $\sqrt{9.1}$.

Solution:

Since we are looking for the linear approximation at $x = 9$, using [\[link\]](#) we know the linear approximation is given by

Equation:

$$L(x) = f(9) + f'(9)(x - 9).$$

We need to find $f(9)$ and $f'(9)$.

Equation:

$$\begin{aligned} f(x) = \sqrt{x} &\Rightarrow f(9) = \sqrt{9} = 3 \\ f'(x) = \frac{1}{2\sqrt{x}} &\Rightarrow f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6} \end{aligned}$$

Therefore, the linear approximation is given by [\[link\]](#).

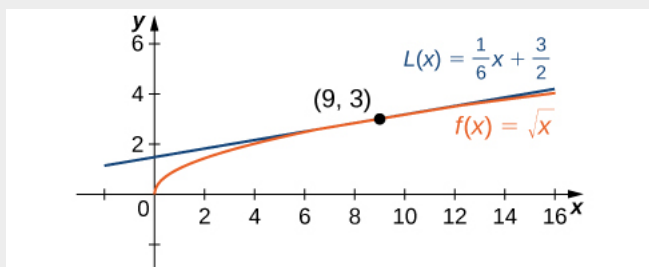
Equation:

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Using the linear approximation, we can estimate $\sqrt{9.1}$ by writing

Equation:

$$\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.0167.$$



The local linear approximation to $f(x) = \sqrt{x}$ at $x = 9$ provides an approximation to f for x near 9.

Analysis

Using a calculator, the value of $\sqrt{9.1}$ to four decimal places is 3.0166. The value given by the linear approximation, 3.0167, is very close to the value obtained with a calculator, so it appears that using this linear approximation is a good way to estimate \sqrt{x} , at least for x near 9. At the same time, it may seem odd to use a linear approximation when we can just push a few buttons on a calculator to evaluate $\sqrt{9.1}$. However, how does the calculator evaluate $\sqrt{9.1}$? The calculator uses an approximation! In fact, calculators and computers use approximations all the time to evaluate mathematical expressions; they just use higher-degree approximations.

Note:

Exercise:

Problem:

Find the local linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 8$. Use it to approximate $\sqrt[3]{8.1}$ to five decimal places.

Solution:

$$L(x) = 2 + \frac{1}{12}(x - 8); 2.00833$$

Hint

$$L(x) = f(a) + f'(a)(x - a)$$

Example:**Exercise:****Problem:****Linear Approximation of $\sin x$**

Find the linear approximation of $f(x) = \sin x$ at $x = \frac{\pi}{3}$ and use it to approximate $\sin(62^\circ)$.

Solution:

First we note that since $\frac{\pi}{3}$ rad is equivalent to 60° , using the linear approximation at $x = \pi/3$ seems reasonable. The linear approximation is given by

Equation:

$$L(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right).$$

We see that

Equation:

$$\begin{aligned} f(x) = \sin x &\Rightarrow f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \\ f'(x) = \cos x &\Rightarrow f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \end{aligned}$$

Therefore, the linear approximation of f at $x = \pi/3$ is given by [\[link\]](#).

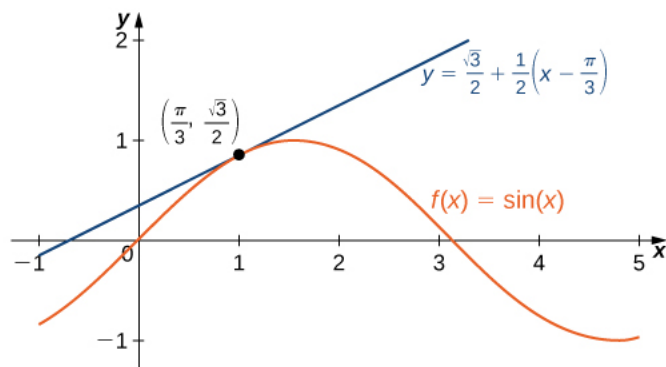
Equation:

$$L(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

To estimate $\sin(62^\circ)$ using L , we must first convert 62° to radians. We have $62^\circ = \frac{62\pi}{180}$ radians, so the estimate for $\sin(62^\circ)$ is given by

Equation:

$$\sin(62^\circ) = f\left(\frac{62\pi}{180}\right) \approx L\left(\frac{62\pi}{180}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{62\pi}{180} - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{2\pi}{180}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{180} \approx 0.88$$



The linear approximation to $f(x) = \sin x$ at $x = \pi/3$ provides an approximation to $\sin x$ for x near $\pi/3$.

Note:

Exercise:

Problem: Find the linear approximation for $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

Solution:

$$L(x) = -x + \frac{\pi}{2}$$

Hint

$$L(x) = f(a) + f'(a)(x - a)$$

Linear approximations may be used in estimating roots and powers. In the next example, we find the linear approximation for $f(x) = (1 + x)^n$ at $x = 0$, which can be used to estimate roots and powers for real numbers near 1. The same idea can be extended to a function of the form $f(x) = (m + x)^n$ to estimate roots and powers near a different number m .

Example:

Exercise:

Problem:
Approximating Roots and Powers

Find the linear approximation of $f(x) = (1 + x)^n$ at $x = 0$. Use this approximation to estimate $(1.01)^3$.

Solution:

The linear approximation at $x = 0$ is given by

Equation:

$$L(x) = f(0) + f'(0)(x - 0).$$

Because

Equation:

$$\begin{aligned}f(x) &= (1+x)^n \Rightarrow f(0) = 1 \\f'(x) &= n(1+x)^{n-1} \Rightarrow f'(0) = n,\end{aligned}$$

the linear approximation is given by [link](#)(a).

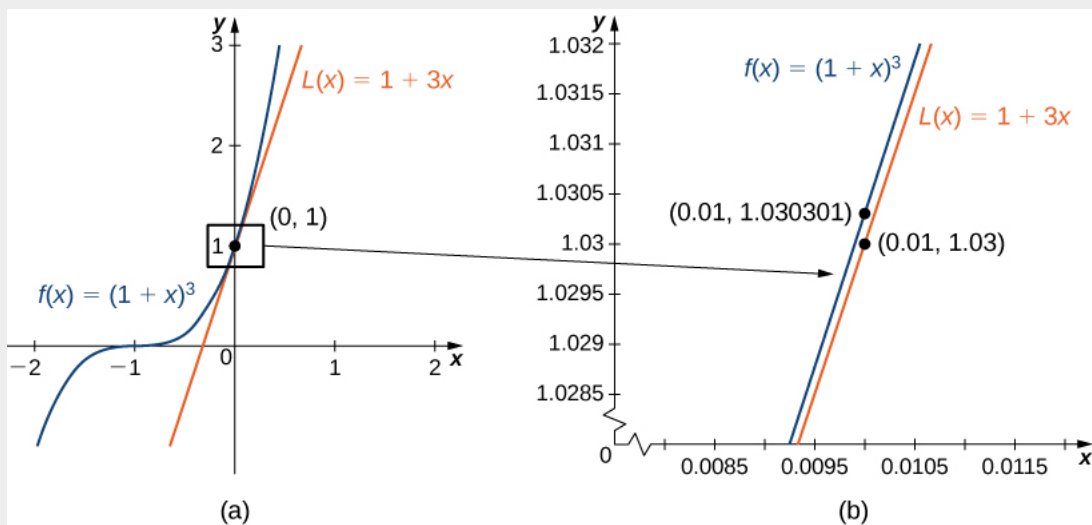
Equation:

$$L(x) = 1 + n(x - 0) = 1 + nx$$

We can approximate $(1.01)^3$ by evaluating $L(0.01)$ when $n = 3$. We conclude that

Equation:

$$(1.01)^3 = f(1.01) \approx L(1.01) = 1 + 3(0.01) = 1.03.$$



(a) The linear approximation of $f(x)$ at $x = 0$ is $L(x)$. (b) The actual value of 1.01^3 is 1.030301. The linear approximation of $f(x)$ at $x = 0$ estimates 1.01^3 to be 1.03.

Note:

Exercise:

Problem:

Find the linear approximation of $f(x) = (1+x)^4$ at $x = 0$ without using the result from the preceding example.

Solution:

$$L(x) = 1 + 4x$$

Hint

$$f'(x) = 4(1+x)^3$$

Differentials

We have seen that linear approximations can be used to estimate function values. They can also be used to estimate the amount a function value changes as a result of a small change in the input. To discuss this more formally, we define a related concept: **differentials**. Differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values.

When we first looked at derivatives, we used the Leibniz notation dy/dx to represent the derivative of y with respect to x . Although we used the expressions dy and dx in this notation, they did not have meaning on their own. Here we see a meaning to the expressions dy and dx . Suppose $y = f(x)$ is a differentiable function. Let dx be an independent variable that can be assigned any nonzero real number, and define the dependent variable dy by

Equation:

$$dy = f'(x)dx.$$

It is important to notice that dy is a function of both x and dx . The expressions dy and dx are called *differentials*. We can divide both sides of [\[link\]](#) by dx , which yields

Equation:

$$\frac{dy}{dx} = f'(x).$$

This is the familiar expression we have used to denote a derivative. [\[link\]](#) is known as the **differential form** of [\[link\]](#).

Example:

Exercise:

Problem:

Computing differentials

For each of the following functions, find dy and evaluate when $x = 3$ and $dx = 0.1$.

- a. $y = x^2 + 2x$
- b. $y = \cos x$

Solution:

The key step is calculating the derivative. When we have that, we can obtain dy directly.

- a. Since $f(x) = x^2 + 2x$, we know $f'(x) = 2x + 2$, and therefore

Equation:

$$dy = (2x + 2)dx.$$

When $x = 3$ and $dx = 0.1$,

Equation:

$$dy = (2 \cdot 3 + 2)(0.1) = 0.8.$$

b. Since $f(x) = \cos x$, $f'(x) = -\sin(x)$. This gives us

Equation:

$$dy = -\sin x \, dx.$$

When $x = 3$ and $dx = 0.1$,

Equation:

$$dy = -\sin(3)(0.1) = -0.1\sin(3).$$

Note:

Exercise:

Problem: For $y = e^{x^2}$, find dy .

Solution:

$$dy = 2xe^{x^2}dx$$

Hint

$$dy = f'(x)dx$$

We now connect differentials to linear approximations. Differentials can be used to estimate the change in the value of a function resulting from a small change in input values. Consider a function f that is differentiable at point a . Suppose the input x changes by a small amount. We are interested in how much the output y changes. If x changes from a to $a + dx$, then the change in x is dx (also denoted Δx), and the change in y is given by

Equation:

$$\Delta y = f(a + dx) - f(a).$$

Instead of calculating the exact change in y , however, it is often easier to approximate the change in y by using a linear approximation. For x near a , $f(x)$ can be approximated by the linear approximation

Equation:

$$L(x) = f(a) + f'(a)(x - a).$$

Therefore, if dx is small,

Equation:

$$f(a + dx) \approx L(a + dx) = f(a) + f'(a)(a + dx - a).$$

That is,

Equation:

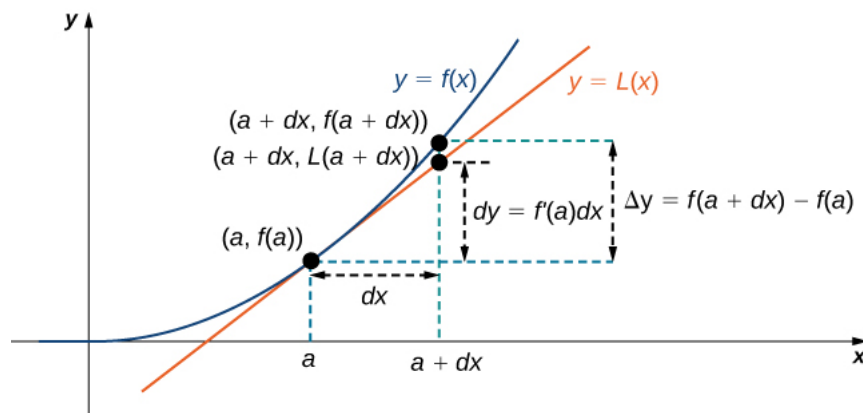
$$f(a + dx) - f(a) \approx L(a + dx) - f(a) = f'(a)dx.$$

In other words, the actual change in the function f if x increases from a to $a + dx$ is approximately the difference between $L(a + dx)$ and $f(a)$, where $L(x)$ is the linear approximation of f at a . By definition of $L(x)$, this difference is equal to $f'(a)dx$. In summary,

Equation:

$$\Delta y = f(a + dx) - f(a) \approx L(a + dx) - f(a) = f'(a)dx = dy.$$

Therefore, we can use the differential $dy = f'(a)dx$ to approximate the change in y if x increases from $x = a$ to $x = a + dx$. We can see this in the following graph.



The differential $dy = f'(a)dx$ is used to approximate the actual change in y if x increases from a to $a + dx$.

We now take a look at how to use differentials to approximate the change in the value of the function that results from a small change in the value of the input. Note the calculation with differentials is much simpler than calculating actual values of functions and the result is very close to what we would obtain with the more exact calculation.

Example:

Exercise:

Problem:

Approximating Change with Differentials

Let $y = x^2 + 2x$. Compute Δy and dy at $x = 3$ if $dx = 0.1$.

Solution:

The actual change in y if x changes from $x = 3$ to $x = 3.1$ is given by

Equation:

$$\Delta y = f(3.1) - f(3) = [(3.1)^2 + 2(3.1)] - [3^2 + 2(3)] = 0.81.$$

The approximate change in y is given by $dy = f'(3)dx$. Since $f'(x) = 2x + 2$, we have

Equation:

$$dy = f'(3)dx = (2(3) + 2)(0.1) = 0.8.$$

Note:

Exercise:

Problem: For $y = x^2 + 2x$, find Δy and dy at $x = 3$ if $dx = 0.2$.

Solution:

$$dy = 1.6, \Delta y = 1.64$$

Hint

$$dy = f'(3)dx, \Delta y = f(3.2) - f(3)$$

Calculating the Amount of Error

Any type of measurement is prone to a certain amount of error. In many applications, certain quantities are calculated based on measurements. For example, the area of a circle is calculated by measuring the radius of the circle. An error in the measurement of the radius leads to an error in the computed value of the area. Here we examine this type of error and study how differentials can be used to estimate the error.

Consider a function f with an input that is a measured quantity. Suppose the exact value of the measured quantity is a , but the measured value is $a + dx$. We say the measurement error is dx (or Δx). As a result, an error occurs in the calculated quantity $f(x)$. This type of error is known as a **propagated error** and is given by

Equation:

$$\Delta y = f(a + dx) - f(a).$$

Since all measurements are prone to some degree of error, we do not know the exact value of a measured quantity, so we cannot calculate the propagated error exactly. However, given an estimate of the accuracy of a measurement, we can use differentials to approximate the propagated error Δy . Specifically, if f is a differentiable function at a , the propagated error is

Equation:

$$\Delta y \approx dy = f'(a)dx.$$

Unfortunately, we do not know the exact value a . However, we can use the measured value $a + dx$, and estimate

Equation:

$$\Delta y \approx dy \approx f'(a + dx)dx.$$

In the next example, we look at how differentials can be used to estimate the error in calculating the volume of a box if we assume the measurement of the side length is made with a certain amount of accuracy.

Example:

Exercise:

Problem:

Volume of a Cube

Suppose the side length of a cube is measured to be 5 cm with an accuracy of 0.1 cm.

- Use differentials to estimate the error in the computed volume of the cube.
- Compute the volume of the cube if the side length is (i) 4.9 cm and (ii) 5.1 cm to compare the estimated error with the actual potential error.

Solution:

- a. The measurement of the side length is accurate to within ± 0.1 cm. Therefore,

Equation:

$$-0.1 \leq dx \leq 0.1.$$

The volume of a cube is given by $V = x^3$, which leads to

Equation:

$$dV = 3x^2 dx.$$

Using the measured side length of 5 cm, we can estimate that

Equation:

$$-3(5)^2(0.1) \leq dV \leq 3(5)^2(0.1).$$

Therefore,

Equation:

$$-7.5 \leq dV \leq 7.5.$$

- b. If the side length is actually 4.9 cm, then the volume of the cube is

Equation:

$$V(4.9) = (4.9)^3 = 117.649 \text{ cm}^3.$$

If the side length is actually 5.1 cm, then the volume of the cube is

Equation:

$$V(5.1) = (5.1)^3 = 132.651 \text{ cm}^3.$$

Therefore, the actual volume of the cube is between 117.649 and 132.651. Since the side length is measured to be 5 cm, the computed volume is $V(5) = 5^3 = 125$. Therefore, the error in the computed volume is

Equation:

$$117.649 - 125 \leq \Delta V \leq 132.651 - 125.$$

That is,

Equation:

$$-7.351 \leq \Delta V \leq 7.651.$$

We see the estimated error dV is relatively close to the actual potential error in the computed volume.

Note:

Exercise:

Problem:

Estimate the error in the computed volume of a cube if the side length is measured to be 6 cm with an accuracy of 0.2 cm.

Solution:

The volume measurement is accurate to within 21.6 cm^3 .

Hint

$$dV = 3x^2 dx$$

The measurement error dx ($=\Delta x$) and the propagated error Δy are absolute errors. We are typically interested in the size of an error relative to the size of the quantity being measured or calculated. Given an absolute error Δq for a particular quantity, we define the **relative error** as $\frac{\Delta q}{q}$, where q is the actual value of the quantity. The **percentage error** is the relative error expressed as a percentage. For example, if we measure the height of a ladder to be 63 in. when the actual height is 62 in., the absolute error is 1 in. but the relative error is $\frac{1}{62} = 0.016$, or 1.6%. By comparison, if we measure the width of a piece of cardboard to be 8.25 in. when the actual width is 8 in., our absolute error is $\frac{1}{4}$ in., whereas the relative error is $\frac{0.25}{8} = \frac{1}{32}$, or 3.1%. Therefore, the percentage error in the measurement of the cardboard is larger, even though 0.25 in. is less than 1 in.

Example:

Exercise:

Problem:

Relative and Percentage Error

An astronaut using a camera measures the radius of Earth as 4000 mi with an error of ± 80 mi. Let's use differentials to estimate the relative and percentage error of using this radius measurement to calculate the volume of Earth, assuming the planet is a perfect sphere.

Solution:

If the measurement of the radius is accurate to within ± 80 , we have

Equation:

$$-80 \leq dr \leq 80.$$

Since the volume of a sphere is given by $V = \left(\frac{4}{3}\right)\pi r^3$, we have

Equation:

$$dV = 4\pi r^2 dr.$$

Using the measured radius of 4000 mi, we can estimate

Equation:

$$-4\pi(4000)^2(80) \leq dV \leq 4\pi(4000)^2(80).$$

To estimate the relative error, consider $\frac{dV}{V}$. Since we do not know the exact value of the volume V , use the measured radius $r = 4000$ mi to estimate V . We obtain $V \approx \left(\frac{4}{3}\right)\pi(4000)^3$. Therefore the relative error satisfies

Equation:

$$\frac{-4\pi(4000)^2(80)}{4\pi(4000)^3/3} \leq \frac{dV}{V} \leq \frac{4\pi(4000)^2(80)}{4\pi(4000)^3/3},$$

which simplifies to

Equation:

$$-0.06 \leq \frac{dV}{V} \leq 0.06.$$

The relative error is 0.06 and the percentage error is 6%.

Note:

Exercise:

Problem:

Determine the percentage error if the radius of Earth is measured to be 3950 mi with an error of ± 100 mi.

Solution:

7.6%

Hint

Use the fact that $dV = 4\pi r^2 dr$ to find dV/V .

Key Concepts

- A differentiable function $y = f(x)$ can be approximated at a by the linear function

Equation:

$$L(x) = f(a) + f'(a)(x - a).$$

- For a function $y = f(x)$, if x changes from a to $a + dx$, then

Equation:

$$dy = f'(x)dx$$

is an approximation for the change in y . The actual change in y is

Equation:

$$\Delta y = f(a + dx) - f(a).$$

- A measurement error dx can lead to an error in a calculated quantity $f(x)$. The error in the calculated quantity is known as the *propagated error*. The propagated error can be estimated by
Equation:

$$dy \approx f'(x)dx.$$

- To estimate the relative error of a particular quantity q , we estimate $\frac{\Delta q}{q}$.

Key Equations

- **Linear approximation**
 $L(x) = f(a) + f'(a)(x - a)$
- **A differential**
 $dy = f'(x)dx.$

Exercise:

Problem: What is the linear approximation for any generic linear function $y = mx + b$?

Exercise:

Problem:

Determine the necessary conditions such that the linear approximation function is constant. Use a graph to prove your result.

Solution:

$$f'(a) = 0$$

Exercise:

Problem:

Explain why the linear approximation becomes less accurate as you increase the distance between x and a . Use a graph to prove your argument.

Exercise:

Problem: When is the linear approximation exact?

Solution:

The linear approximation exact when $y = f(x)$ is linear or constant.

For the following exercises, find the linear approximation $L(x)$ to $y = f(x)$ near $x = a$ for the function.

Exercise:

Problem: [T] $f(x) = x + x^4, a = 0$

Exercise:

Problem: [T] $f(x) = \frac{1}{x}, a = 2$

Solution:

$$L(x) = \frac{1}{2} - \frac{1}{4}(x - 2)$$

Exercise:

Problem: [T] $f(x) = \tan x, a = \frac{\pi}{4}$

Exercise:

Problem: [T] $f(x) = \sin x, a = \frac{\pi}{2}$

Solution:

$$L(x) = 1$$

Exercise:

Problem: [T] $f(x) = x \sin x, a = 2\pi$

Exercise:

Problem: [T] $f(x) = \sin^2 x, a = 0$

Solution:

$$L(x) = 0$$

For the following exercises, compute the values given within 0.01 by deciding on the appropriate $f(x)$ and a , and evaluating $L(x) = f(a) + f'(a)(x - a)$. Check your answer using a calculator.

Exercise:

Problem: [T] $(2.001)^6$

Exercise:

Problem: [T] $\sin(0.02)$

Solution:

$$0.02$$

Exercise:

Problem: [T] $\cos(0.03)$

Exercise:

Problem: [T] $(15.99)^{1/4}$

Solution:

$$1.9996875$$

Exercise:

Problem: [T] $\frac{1}{0.98}$

Exercise:

Problem: [T] $\sin(3.14)$

Solution:

0.001593

For the following exercises, determine the appropriate $f(x)$ and a , and evaluate $L(x) = f(a) + f'(a)(x - a)$. Calculate the numerical error in the linear approximations that follow.

Exercise:

Problem: $(1.01)^3$

Exercise:

Problem: $\cos(0.01)$

Solution:

1; error, ~ 0.00005

Exercise:

Problem: $(\sin(0.01))^2$

Exercise:

Problem: $(1.01)^{-3}$

Solution:

0.97; error, ~ 0.0006

Exercise:

Problem: $\left(1 + \frac{1}{10}\right)^{10}$

Exercise:

Problem: $\sqrt{8.99}$

Solution:

$3 - \frac{1}{600}$; error, $\sim 4.632 \times 10^{-7}$

For the following exercises, find the differential of the function.

Exercise:

Problem: $y = 3x^4 + x^2 - 2x + 1$

Exercise:

Problem: $y = x \cos x$

Solution:

$$dy = (\cos x - x \sin x)dx$$

Exercise:

Problem: $y = \sqrt{1+x}$

Exercise:

Problem: $y = \frac{x^2+2}{x-1}$

Solution:

$$dy = \left(\frac{x^2-2x-2}{(x-1)^2} \right) dx$$

For the following exercises, find the differential and evaluate for the given x and dx .

Exercise:

Problem: $y = 3x^2 - x + 6$, $x = 2$, $dx = 0.1$

Exercise:

Problem: $y = \frac{1}{x+1}$, $x = 1$, $dx = 0.25$

Solution:

$$dy = -\frac{1}{(x+1)^2} dx, -\frac{1}{16}$$

Exercise:

Problem: $y = \tan x$, $x = 0$, $dx = \frac{\pi}{10}$

Exercise:

Problem: $y = \frac{3x^2+2}{\sqrt{x+1}}$, $x = 0$, $dx = 0.1$

Solution:

$$dy = \frac{9x^2+12x-2}{2(x+1)^{3/2}} dx, -0.1$$

Exercise:

Problem: $y = \frac{\sin(2x)}{x}$, $x = \pi$, $dx = 0.25$

Exercise:

Problem: $y = x^3 + 2x + \frac{1}{x}$, $x = 1$, $dx = 0.05$

Solution:

$$dy = \left(3x^2 + 2 - \frac{1}{x^2} \right) dx, 0.2$$

For the following exercises, find the change in volume dV or in surface area dA .

Exercise:

Problem: dV if the sides of a cube change from 10 to 10.1.

Exercise:

Problem: dA if the sides of a cube change from x to $x + dx$.

Solution:

$$12x \, dx$$

Exercise:

Problem: dA if the radius of a sphere changes from r by dr .

Exercise:

Problem: dV if the radius of a sphere changes from r by dr .

Solution:

$$4\pi r^2 dr$$

Exercise:

Problem: dV if a circular cylinder with $r = 2$ changes height from 3 cm to 3.05 cm.

Exercise:

Problem: dV if a circular cylinder of height 3 changes from $r = 2$ to $r = 1.9$ cm.

Solution:

$$-1.2\pi \text{ cm}^3$$

For the following exercises, use differentials to estimate the maximum and relative error when computing the surface area or volume.

Exercise:

Problem:

A spherical golf ball is measured to have a radius of 5 mm, with a possible measurement error of 0.1 mm. What is the possible change in volume?

Exercise:

Problem:

A pool has a rectangular base of 10 ft by 20 ft and a depth of 6 ft. What is the change in volume if you only fill it up to 5.5 ft?

Solution:

$$-100 \text{ ft}^3$$

Exercise:

Problem:

An ice cream cone has height 4 in. and radius 1 in. If the cone is 0.1 in. thick, what is the difference between the volume of the cone, including the shell, and the volume of the ice cream you can fit inside the shell?

For the following exercises, confirm the approximations by using the linear approximation at $x = 0$.

Exercise:

Problem: $\sqrt{1-x} \approx 1 - \frac{1}{2}x$

Exercise:

Problem: $\frac{1}{\sqrt{1-x^2}} \approx 1$

Exercise:

Problem: $\sqrt{c^2 + x^2} \approx c$

Glossary**differential**

the differential dx is an independent variable that can be assigned any nonzero real number; the differential dy is defined to be $dy = f'(x)dx$

differential form

given a differentiable function $y = f(x)$, the equation $dy = f'(x)dx$ is the differential form of the derivative of y with respect to x

linear approximation

the linear function $L(x) = f(a) + f'(a)(x - a)$ is the linear approximation of f at $x = a$

percentage error

the relative error expressed as a percentage

propagated error

the error that results in a calculated quantity $f(x)$ resulting from a measurement error dx

relative error

given an absolute error Δq for a particular quantity, $\frac{\Delta q}{q}$ is the relative error.

tangent line approximation (linearization)

since the linear approximation of f at $x = a$ is defined using the equation of the tangent line, the linear approximation of f at $x = a$ is also known as the tangent line approximation to f at $x = a$

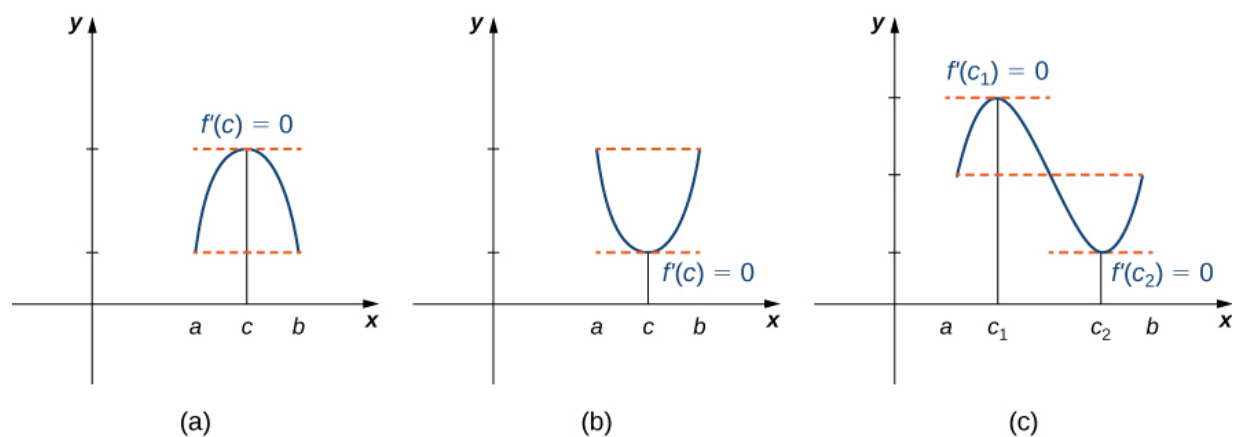
The Mean Value Theorem

- Explain the meaning of Rolle's theorem.
- Describe the significance of the Mean Value Theorem.
- State three important consequences of the Mean Value Theorem.

The **Mean Value Theorem** is one of the most important theorems in calculus. We look at some of its implications at the end of this section. First, let's start with a special case of the Mean Value Theorem, called Rolle's theorem.

Rolle's Theorem

Informally, **Rolle's theorem** states that if the outputs of a differentiable function f are equal at the endpoints of an interval, then there must be an interior point c where $f'(c) = 0$. [\[link\]](#) illustrates this theorem.



If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between a and b .

Note:

Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Proof

Let $k = f(a) = f(b)$. We consider three cases:

1. $f(x) = k$ for all $x \in (a, b)$.
2. There exists $x \in (a, b)$ such that $f(x) > k$.
3. There exists $x \in (a, b)$ such that $f(x) < k$.

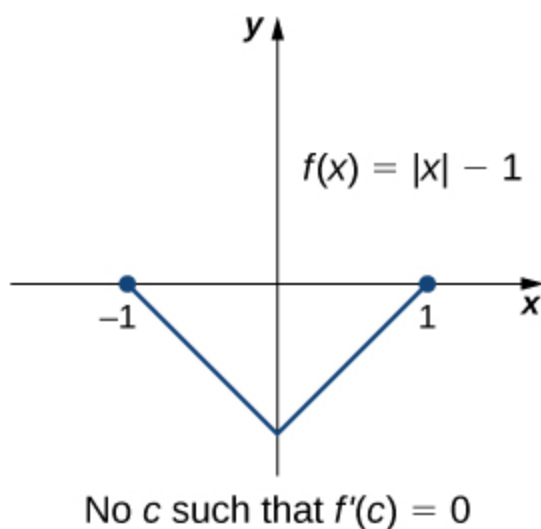
Case 1: If $f(x) = k$ for all $x \in (a, b)$, then $f'(x) = 0$ for all $x \in (a, b)$.

Case 2: Since f is a continuous function over the closed, bounded interval $[a, b]$, by the extreme value theorem, it has an absolute maximum. Also, since there is a point $x \in (a, b)$ such that $f(x) > k$, the absolute maximum is greater than k . Therefore, the absolute maximum does not occur at either endpoint. As a result, the absolute maximum must occur at an interior point $c \in (a, b)$. Because f has a maximum at an interior point c , and f is differentiable at c , by Fermat's theorem, $f'(c) = 0$.

Case 3: The case when there exists a point $x \in (a, b)$ such that $f(x) < k$ is analogous to case 2, with maximum replaced by minimum.

□

An important point about Rolle's theorem is that the differentiability of the function f is critical. If f is not differentiable, even at a single point, the result may not hold. For example, the function $f(x) = |x| - 1$ is continuous over $[-1, 1]$ and $f(-1) = 0 = f(1)$, but $f'(c) \neq 0$ for any $c \in (-1, 1)$ as shown in the following figure.



Since $f(x) = |x| - 1$ is not differentiable at $x = 0$, the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c \in (-1, 1)$ such that $f'(c) = 0$.

Let's now consider functions that satisfy the conditions of Rolle's theorem and calculate explicitly the points c where $f'(c) = 0$.

Example:

Exercise:

Problem:

Using Rolle's Theorem

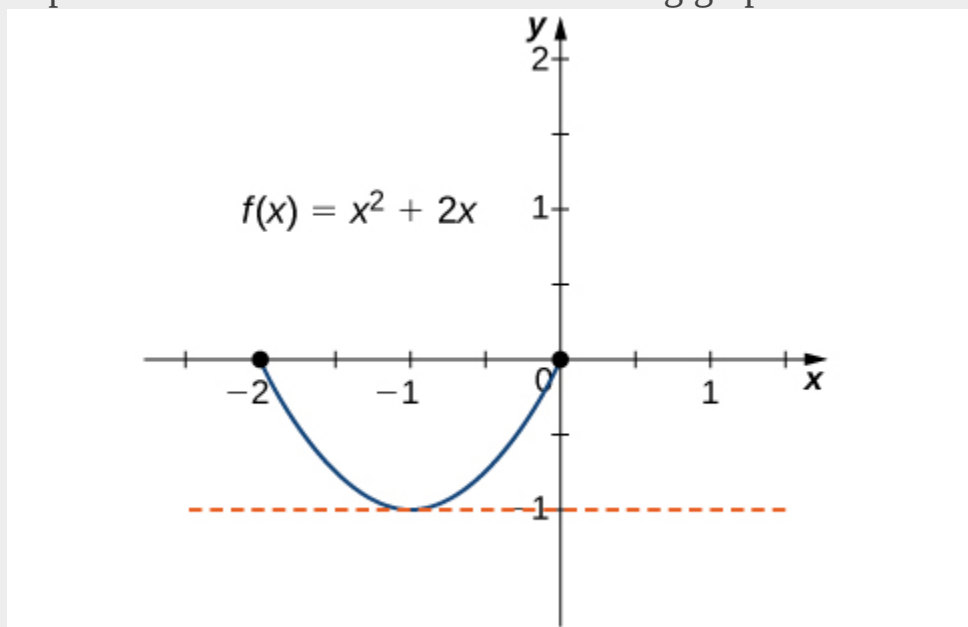
For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c) = 0$.

- a. $f(x) = x^2 + 2x$ over $[-2, 0]$

b. $f(x) = x^3 - 4x$ over $[-2, 2]$

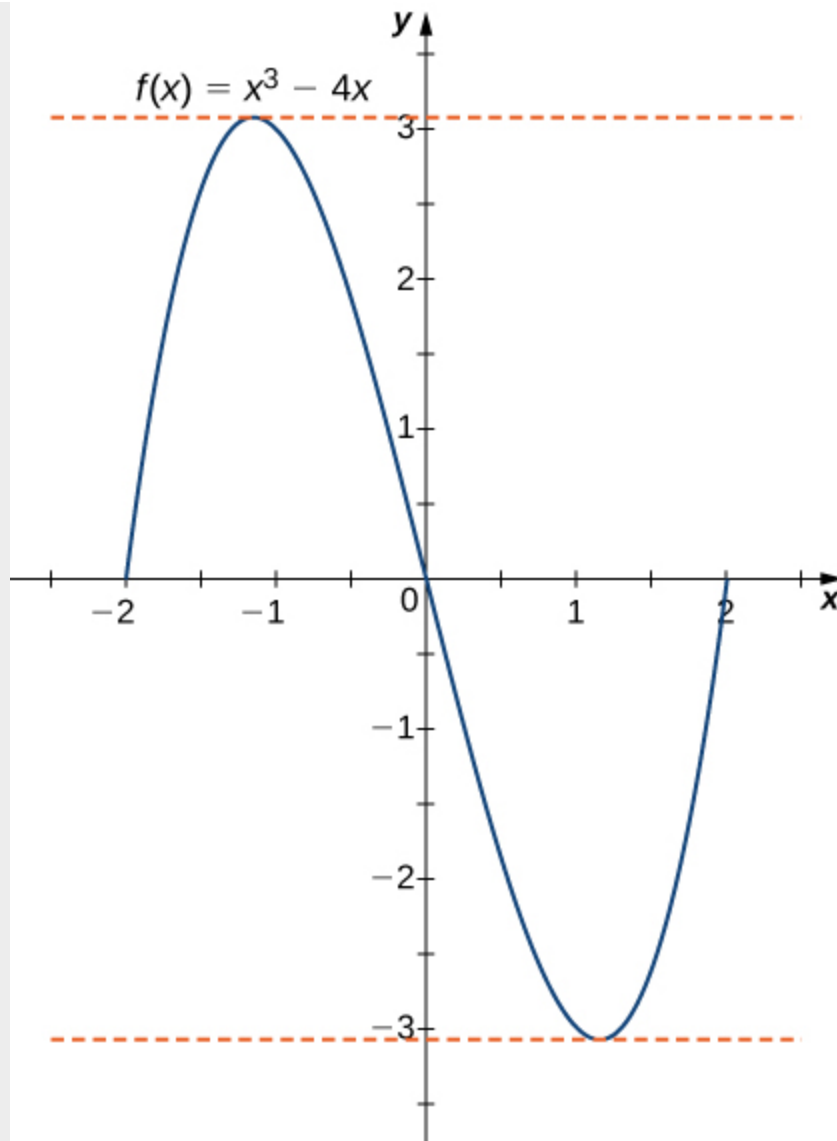
Solution:

- a. Since f is a polynomial, it is continuous and differentiable everywhere. In addition, $f(-2) = 0 = f(0)$. Therefore, f satisfies the criteria of Rolle's theorem. We conclude that there exists at least one value $c \in (-2, 0)$ such that $f'(c) = 0$. Since $f'(x) = 2x + 2 = 2(x + 1)$, we see that $f'(c) = 2(c + 1) = 0$ implies $c = -1$ as shown in the following graph.



This function is continuous and differentiable over $[-2, 0]$, $f'(c) = 0$ when $c = -1$.

- b. As in part a. f is a polynomial and therefore is continuous and differentiable everywhere. Also, $f(-2) = 0 = f(2)$. That said, f satisfies the criteria of Rolle's theorem. Differentiating, we find that $f'(x) = 3x^2 - 4$. Therefore, $f'(c) = 0$ when $x = \pm \frac{2}{\sqrt{3}}$. Both points are in the interval $[-2, 2]$, and, therefore, both points satisfy the conclusion of Rolle's theorem as shown in the following graph.



For this polynomial over $[-2, 2]$, $f'(c) = 0$ at $x = \pm 2/\sqrt{3}$.

Note:

Exercise:

Problem:

Verify that the function $f(x) = 2x^2 - 8x + 6$ defined over the interval $[1, 3]$ satisfies the conditions of Rolle's theorem. Find all points c guaranteed by Rolle's theorem.

Solution:

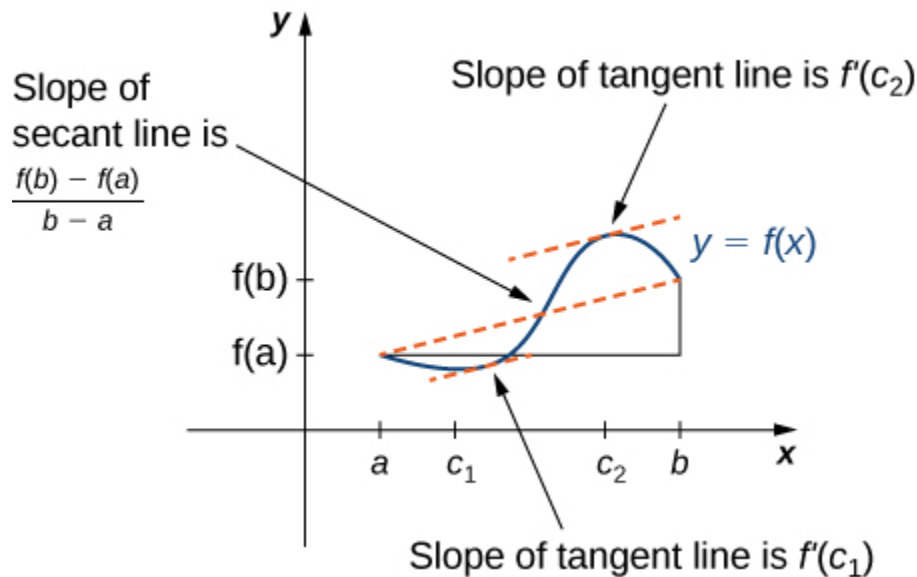
$$c = 2$$

Hint

Find all values c , where $f'(c) = 0$.

The Mean Value Theorem and Its Meaning

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f defined on a closed interval $[a, b]$ with $f(a) = f(b)$. The Mean Value Theorem generalizes Rolle's theorem by considering functions that do not necessarily have equal value at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem ([\[link\]](#)). The Mean Value Theorem states that if f is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b) , then there exists a point $c \in (a, b)$ such that the tangent line to the graph of f at c is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.



The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.

Note:

Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b) . Then, there exists at least one point $c \in (a, b)$ such that

Equation:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof

The proof follows from Rolle's theorem by introducing an appropriate function that satisfies the criteria of Rolle's theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. Since the slope of that line is

Equation:

$$\frac{f(b) - f(a)}{b - a}$$

and the line passes through the point $(a, f(a))$, the equation of that line can be written as

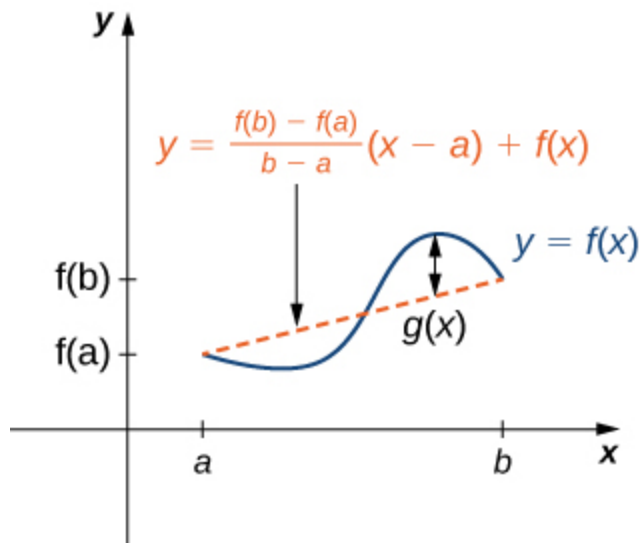
Equation:

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let $g(x)$ denote the vertical difference between the point $(x, f(x))$ and the point (x, y) on that line. Therefore,

Equation:

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a) \right].$$



The value $g(x)$ is the vertical difference between the point $(x, f(x))$ and the point (x, y) on the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Since the graph of f intersects the secant line when $x = a$ and $x = b$, we see that $g(a) = 0 = g(b)$. Since f is a differentiable function over (a, b) , g is also a differentiable function over (a, b) . Furthermore, since f is continuous over $[a, b]$, g is also continuous over $[a, b]$. Therefore, g satisfies the criteria of Rolle's theorem. Consequently, there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since

Equation:

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we see that

Equation:

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Since $g'(c) = 0$, we conclude that

Equation:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

In the next example, we show how the Mean Value Theorem can be applied to the function $f(x) = \sqrt{x}$ over the interval $[0, 9]$. The method is the same for other functions, although sometimes with more interesting consequences.

Example:

Exercise:

Problem:

Verifying that the Mean Value Theorem Applies

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find these values c guaranteed by the Mean Value Theorem.

Solution:

We know that $f(x) = \sqrt{x}$ is continuous over $[0, 9]$ and differentiable over $(0, 9)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem, and there must exist at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and

$(9, f(9))$ ([link](#)). To determine which value(s) of c are guaranteed, first calculate the derivative of f . The derivative $f'(x) = \frac{1}{(2\sqrt{x})}$. The slope of the line connecting $(0, f(0))$ and $(9, f(9))$ is given by

Equation:

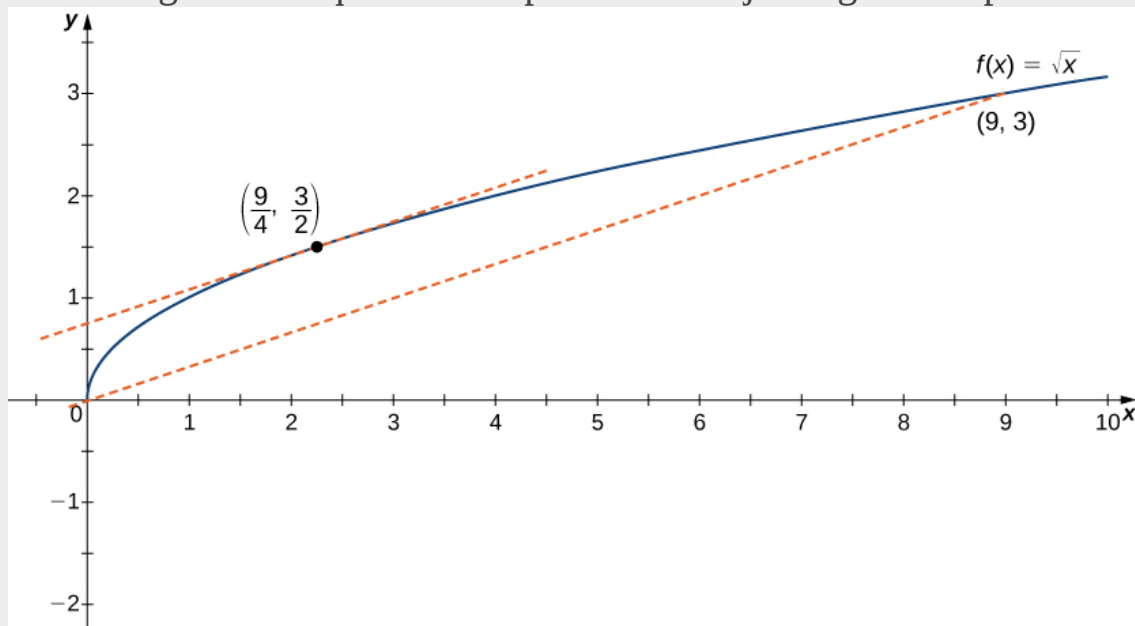
$$\frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9 - 0} = \frac{3}{9} = \frac{1}{3}.$$

We want to find c such that $f'(c) = \frac{1}{3}$. That is, we want to find c such that

Equation:

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

Solving this equation for c , we obtain $c = \frac{9}{4}$. At this point, the slope of the tangent line equals the slope of the line joining the endpoints.



The slope of the tangent line at $c = 9/4$ is the same as the slope of the line segment connecting $(0, 0)$ and $(9, 3)$.

One application that helps illustrate the Mean Value Theorem involves velocity. For example, suppose we drive a car for 1 h down a straight road with an average velocity of 45 mph. Let $s(t)$ and $v(t)$ denote the position and velocity of the car, respectively, for $0 \leq t \leq 1$ h. Assuming that the position function $s(t)$ is differentiable, we can apply the Mean Value Theorem to conclude that, at some time $c \in (0, 1)$, the speed of the car was exactly

Equation:

$$v(c) = s'(c) = \frac{s(1) - s(0)}{1 - 0} = 45 \text{ mph.}$$

Example:

Exercise:

Problem:

Mean Value Theorem and Velocity

If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t^2 + 100$.

- Determine how long it takes before the rock hits the ground.
- Find the average velocity v_{avg} of the rock for when the rock is released and the rock hits the ground.
- Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg} .

Solution:

- a. When the rock hits the ground, its position is $s(t) = 0$. Solving the equation $-16t^2 + 100 = 0$ for t , we find that $t = \pm \frac{5}{2}$ sec. Since we are only considering $t \geq 0$, the ball will hit the ground $\frac{5}{2}$ sec after it is dropped.
- b. The average velocity is given by

Equation:

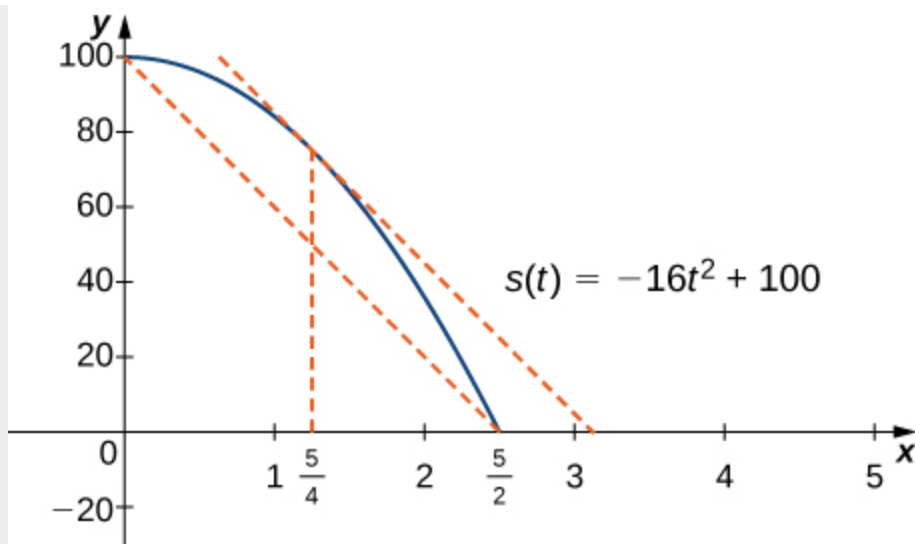
$$v_{\text{avg}} = \frac{s(5/2) - s(0)}{5/2 - 0} = \frac{1 - 100}{5/2} = -40 \text{ ft/sec.}$$

- c. The instantaneous velocity is given by the derivative of the position function. Therefore, we need to find a time t such that $v(t) = s'(t) = v_{\text{avg}} = -40$ ft/sec. Since $s(t)$ is continuous over the interval $[0, 5/2]$ and differentiable over the interval $(0, 5/2)$, by the Mean Value Theorem, there is guaranteed to be a point $c \in (0, 5/2)$ such that

Equation:

$$s'(c) = \frac{s(5/2) - s(0)}{5/2 - 0} = -40.$$

Taking the derivative of the position function $s(t)$, we find that $s'(t) = -32t$. Therefore, the equation reduces to $s'(c) = -32c = -40$. Solving this equation for c , we have $c = \frac{5}{4}$. Therefore, $\frac{5}{4}$ sec after the rock is dropped, the instantaneous velocity equals the average velocity of the rock during its free fall: -40 ft/sec.



At time $t = 5/4$ sec, the velocity of the rock is equal to its average velocity from the time it is dropped until it hits the ground.

Note:

Exercise:

Problem:

Suppose a ball is dropped from a height of 200 ft. Its position at time t is $s(t) = -16t^2 + 200$. Find the time t when the instantaneous velocity of the ball equals its average velocity.

Solution:

$$\frac{5}{2\sqrt{2}} \text{ sec}$$

Hint

First, determine how long it takes for the ball to hit the ground. Then, find the average velocity of the ball from the time it is dropped until it hits the ground.

Corollaries of the Mean Value Theorem

Let's now look at three corollaries of the Mean Value Theorem. These results have important consequences, which we use in upcoming sections.

At this point, we know the derivative of any constant function is zero. The Mean Value Theorem allows us to conclude that the converse is also true. In particular, if $f'(x) = 0$ for all x in some interval I , then $f(x)$ is constant over that interval. This result may seem intuitively obvious, but it has important implications that are not obvious, and we discuss them shortly.

Note:

Corollary 1: Functions with a Derivative of Zero

Let f be differentiable over an interval I . If $f'(x) = 0$ for all $x \in I$, then $f(x) = \text{constant}$ for all $x \in I$.

Proof

Since f is differentiable over I , f must be continuous over I . Suppose $f(x)$ is not constant for all x in I . Then there exist $a, b \in I$, where $a \neq b$ and $f(a) \neq f(b)$. Choose the notation so that $a < b$. Therefore,

Equation:

$$\frac{f(b) - f(a)}{b - a} \neq 0.$$

Since f is a differentiable function, by the Mean Value Theorem, there exists $c \in (a, b)$ such that

Equation:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Therefore, there exists $c \in I$ such that $f'(c) \neq 0$, which contradicts the assumption that $f'(x) = 0$ for all $x \in I$.

□

From [\[link\]](#), it follows that if two functions have the same derivative, they differ by, at most, a constant.

Note:

Corollary 2: Constant Difference Theorem

If f and g are differentiable over an interval I and $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some constant C .

Proof

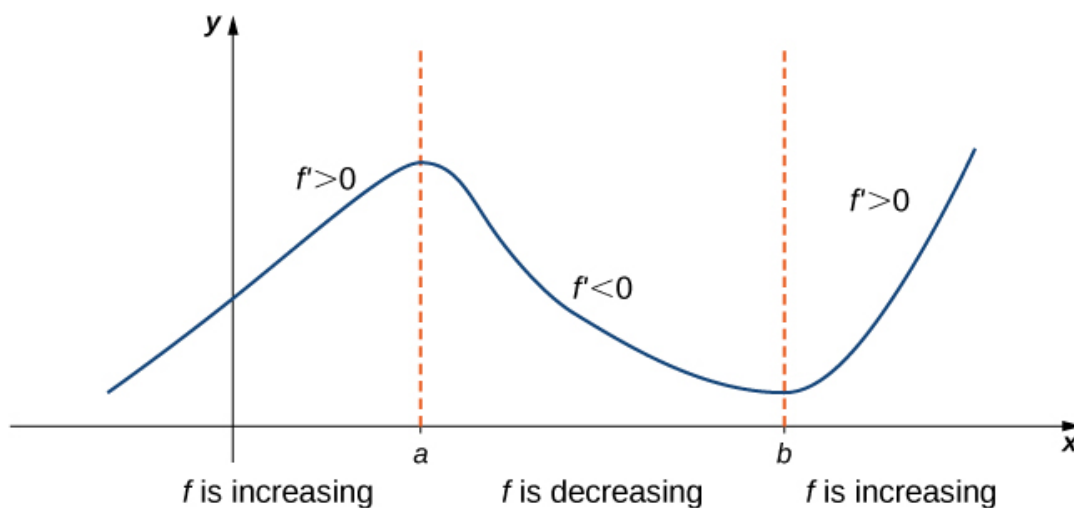
Let $h(x) = f(x) - g(x)$. Then, $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$. By Corollary 1, there is a constant C such that $h(x) = C$ for all $x \in I$. Therefore, $f(x) = g(x) + C$ for all $x \in I$.

□

The third corollary of the Mean Value Theorem discusses when a function is increasing and when it is decreasing. Recall that a function f is increasing over I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, whereas f is decreasing over I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. Using the Mean Value Theorem, we can show that if the derivative of a function is positive,

then the function is increasing; if the derivative is negative, then the function is decreasing ([\[link\]](#)). We make use of this fact in the next section, where we show how to use the derivative of a function to locate local maximum and minimum values of the function, and how to determine the shape of the graph.

This fact is important because it means that for a given function f , if there exists a function F such that $F'(x) = f(x)$; then, the only other functions that have a derivative equal to f are $F(x) + C$ for some constant C . We discuss this result in more detail later in the chapter.



If a function has a positive derivative over some interval I , then the function increases over that interval I ; if the derivative is negative over some interval I , then the function decreases over that interval I .

Note:

Corollary 3: Increasing and Decreasing Functions

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b) .

- i. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.
- ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over $[a, b]$.

Proof

We will prove i.; the proof of ii. is similar. Suppose f is not an increasing function on I . Then there exist a and b in I such that $a < b$, but $f(a) \geq f(b)$. Since f is a differentiable function over I , by the Mean Value Theorem there exists $c \in (a, b)$ such that

Equation:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f(a) \geq f(b)$, we know that $f(b) - f(a) \leq 0$. Also, $a < b$ tells us that $b - a > 0$. We conclude that

Equation:

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

However, $f'(x) > 0$ for all $x \in I$. This is a contradiction, and therefore f must be an increasing function over I .

□

Key Concepts

- If f is continuous over $[a, b]$ and differentiable over (a, b) and $f(a) = 0 = f(b)$, then there exists a point $c \in (a, b)$ such that

$f'(c) = 0$. This is Rolle's theorem.

- If f is continuous over $[a, b]$ and differentiable over (a, b) , then there exists a point $c \in (a, b)$ such that

Equation:

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This is the Mean Value Theorem.

- If $f'(x) = 0$ over an interval I , then f is constant over I .
- If two differentiable functions f and g satisfy $f'(x) = g'(x)$ over I , then $f(x) = g(x) + C$ for some constant C .
- If $f'(x) > 0$ over an interval I , then f is increasing over I . If $f'(x) < 0$ over I , then f is decreasing over I .

Exercise:

Problem:

Why do you need continuity to apply the Mean Value Theorem?
Construct a counterexample.

Exercise:

Problem:

Why do you need differentiability to apply the Mean Value Theorem?
Find a counterexample.

Solution:

One example is $f(x) = |x| + 3$, $-2 \leq x \leq 2$

Exercise:

Problem:

When are Rolle's theorem and the Mean Value Theorem equivalent?

Exercise:

Problem:

If you have a function with a discontinuity, is it still possible to have $f'(c)(b-a) = f(b) - f(a)$? Draw such an example or prove why not.

Solution:

Yes, but the Mean Value Theorem still does not apply

For the following exercises, determine over what intervals (if any) the Mean Value Theorem applies. Justify your answer.

Exercise:

Problem: $y = \sin(\pi x)$

Exercise:

Problem: $y = \frac{1}{x^3}$

Solution:

$(-\infty, 0), (0, \infty)$

Exercise:

Problem: $y = \sqrt{4 - x^2}$

Exercise:

Problem: $y = \sqrt{x^2 - 4}$

Solution:

$(-\infty, -2), (2, \infty)$

Exercise:

Problem: $y = \ln(3x - 5)$

For the following exercises, graph the functions on a calculator and draw the secant line that connects the endpoints. Estimate the number of points c such that $f'(c)(b - a) = f(b) - f(a)$.

Exercise:

Problem: [T] $y = 3x^3 + 2x + 1$ over $[-1, 1]$

Solution:

2 points

Exercise:

Problem: [T] $y = \tan\left(\frac{\pi}{4}x\right)$ over $\left[-\frac{3}{2}, \frac{3}{2}\right]$

Exercise:

Problem: [T] $y = x^2 \cos(\pi x)$ over $[-2, 2]$

Solution:

5 points

Exercise:

Problem:

[T] $y = x^6 - \frac{3}{4}x^5 - \frac{9}{8}x^4 + \frac{15}{16}x^3 + \frac{3}{32}x^2 + \frac{3}{16}x + \frac{1}{32}$ over $[-1, 1]$

For the following exercises, use the Mean Value Theorem and find all points $0 < c < 2$ such that $f(2) - f(0) = f'(c)(2 - 0)$.

Exercise:

Problem: $f(x) = x^3$

Solution:

$$c = \frac{2\sqrt{3}}{3}$$

Exercise:

Problem: $f(x) = \sin(\pi x)$

Exercise:

Problem: $f(x) = \cos(2\pi x)$

Solution:

$$c = \frac{1}{2}, 1, \frac{3}{2}$$

Exercise:

Problem: $f(x) = 1 + x + x^2$

Exercise:

Problem: $f(x) = (x - 1)^{10}$

Solution:

$$c = 1$$

Exercise:

Problem: $f(x) = (x - 1)^9$

For the following exercises, show there is no c such that $f(1) - f(-1) = f'(c)(2)$. Explain why the Mean Value Theorem does not apply over the interval $[-1, 1]$.

Exercise:

Problem: $f(x) = \left| x - \frac{1}{2} \right|$

Solution:

Not differentiable

Exercise:

Problem: $f(x) = \frac{1}{x^2}$

Exercise:

Problem: $f(x) = \sqrt{|x|}$

Solution:

Not differentiable

Exercise:

Problem:

$f(x) = \lfloor x \rfloor$ (*Hint: This is called the *floor function* and it is defined so that $f(x)$ is the largest integer less than or equal to x .*)

For the following exercises, determine whether the Mean Value Theorem applies for the functions over the given interval $[a, b]$. Justify your answer.

Exercise:

Problem: $y = e^x$ over $[0, 1]$

Solution:

Yes

Exercise:

Problem: $y = \ln(2x + 3)$ over $[-\frac{3}{2}, 0]$

Exercise:

Problem: $f(x) = \tan(2\pi x)$ over $[0, 2]$

Solution:

The Mean Value Theorem does not apply since the function is discontinuous at $x = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$.

Exercise:

Problem: $y = \sqrt{9 - x^2}$ over $[-3, 3]$

Exercise:

Problem: $y = \frac{1}{|x+1|}$ over $[0, 3]$

Solution:

Yes

Exercise:

Problem: $y = x^3 + 2x + 1$ over $[0, 6]$

Exercise:

Problem: $y = \frac{x^2+3x+2}{x}$ over $[-1, 1]$

Solution:

The Mean Value Theorem does not apply; discontinuous at $x = 0$.

Exercise:

Problem: $y = \frac{x}{\sin(\pi x)+1}$ over $[0, 1]$

Exercise:

Problem: $y = \ln(x + 1)$ over $[0, e - 1]$

Solution:

Yes

Exercise:

Problem: $y = x \sin(\pi x)$ over $[0, 2]$

Exercise:

Problem: $y = 5 + |x|$ over $[-1, 1]$

Solution:

The Mean Value Theorem does not apply; not differentiable at $x = 0$.

For the following exercises, consider the roots of the equation.

Exercise:

Problem:

Show that the equation $y = x^3 + 3x^2 + 16$ has exactly one real root.
What is it?

Exercise:

Problem:

Find the conditions for exactly one root (double root) for the equation
 $y = x^2 + bx + c$

Solution:

$$b = \pm 2\sqrt{c}$$

Exercise:

Problem:

Find the conditions for $y = e^x - b$ to have one root. Is it possible to have more than one root?

For the following exercises, use a calculator to graph the function over the interval $[a, b]$ and graph the secant line from a to b . Use the calculator to estimate all values of c as guaranteed by the Mean Value Theorem. Then, find the exact value of c , if possible, or write the final equation and use a calculator to estimate to four digits.

Exercise:

Problem: [T] $y = \tan(\pi x)$ over $[-\frac{1}{4}, \frac{1}{4}]$

Solution:

$$c = \pm \frac{1}{\pi} \cos^{-1} \left(\frac{\sqrt{\pi}}{2} \right), c = \pm 0.1533$$

Exercise:

Problem: [T] $y = \frac{1}{\sqrt{x+1}}$ over $[0, 3]$

Exercise:

Problem: [T] $y = |x^2 + 2x - 4|$ over $[-4, 0]$

Solution:

The Mean Value Theorem does not apply.

Exercise:

Problem: [T] $y = x + \frac{1}{x}$ over $[\frac{1}{2}, 4]$

Exercise:

Problem: [T] $y = \sqrt{x+1} + \frac{1}{x^2}$ over $[3, 8]$

Solution:

$$\frac{1}{2\sqrt{c+1}} - \frac{2}{c^3} = \frac{521}{2880}; c = 3.133, 5.867$$

Exercise:**Problem:**

At 10:17 a.m., you pass a police car at 55 mph that is stopped on the freeway. You pass a second police car at 55 mph at 10:53 a.m., which is located 39 mi from the first police car. If the speed limit is 60 mph, can the police cite you for speeding?

Exercise:**Problem:**

Two cars drive from one spotlight to the next, leaving at the same time and arriving at the same time. Is there ever a time when they are going the same speed? Prove or disprove.

Solution:

Yes

Exercise:**Problem:**

Show that $y = \sec^2 x$ and $y = \tan^2 x$ have the same derivative. What can you say about $y = \sec^2 x - \tan^2 x$?

Exercise:

Problem:

Show that $y = \csc^2 x$ and $y = \cot^2 x$ have the same derivative. What can you say about $y = \csc^2 x - \cot^2 x$?

Solution:

It is constant.

Glossary

mean value theorem

if f is continuous over $[a, b]$ and differentiable over (a, b) , then there exists $c \in (a, b)$ such that

Equation:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

rolle's theorem

if f is continuous over $[a, b]$ and differentiable over (a, b) , and if $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$

L'Hôpital's Rule

- Recognize when to apply L'Hôpital's rule.
- Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L'Hôpital's rule in each case.
- Describe the relative growth rates of functions.

In this section, we examine a powerful tool for evaluating limits. This tool, known as **L'Hôpital's rule**, uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value.

Applying L'Hôpital's Rule

L'Hôpital's rule can be used to evaluate limits involving the quotient of two functions. Consider **Equation:**

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}.$$

If $\lim_{x \rightarrow a} f(x) = L_1$ and $\lim_{x \rightarrow a} g(x) = L_2 \neq 0$, then

Equation:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

However, what happens if $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$? We call this one of the **indeterminate forms**, of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x \rightarrow a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

Equation:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} \text{ and } \lim_{x \rightarrow 0} \frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing **Equation:**

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 2 + 2 = 4.$$

For $\lim_{x \rightarrow 0} \frac{\sin x}{x}$ we were able to show, using a geometric argument, that

Equation:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Here we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but also, and more important, it provides us with a way to evaluate many other limits that we could not calculate previously.

The idea behind L'Hôpital's rule can be explained using local linear approximations. Consider two differentiable functions f and g such that $\lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} g(x)$ and such that $g'(a) \neq 0$. For x near a , we can write

Equation:

$$f(x) \approx f(a) + f'(a)(x - a)$$

and

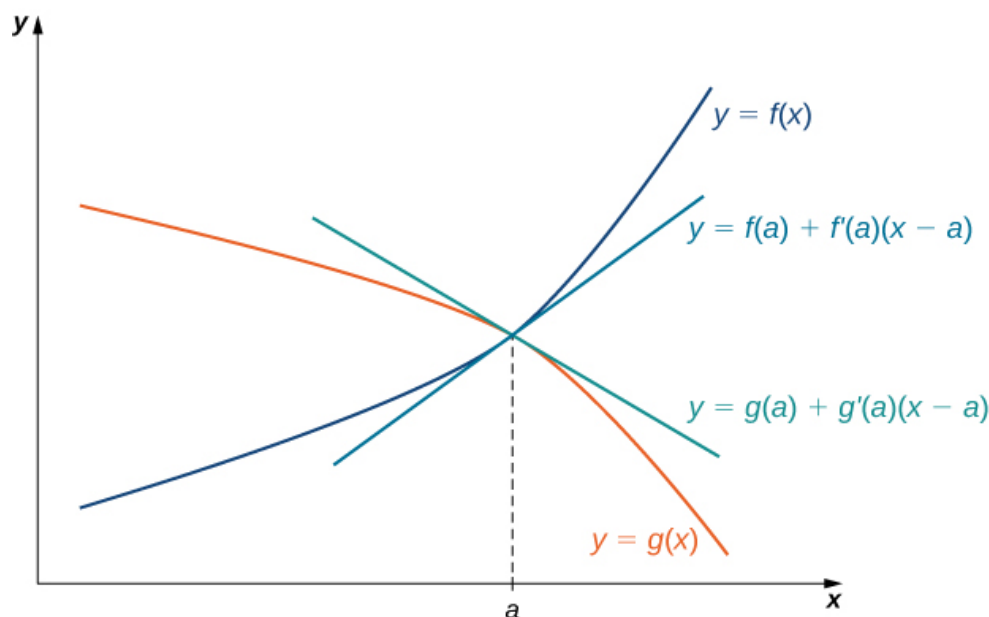
Equation:

$$g(x) \approx g(a) + g'(a)(x - a).$$

Therefore,

Equation:

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x - a)}{g(a) + g'(a)(x - a)}.$$



If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$, then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a .

Since f is differentiable at a , then f is continuous at a , and therefore $f(a) = \lim_{x \rightarrow a} f(x) = 0$.

Similarly, $g(a) = \lim_{x \rightarrow a} g(x) = 0$. If we also assume that f' and g' are continuous at $x = a$, then $f'(a) = \lim_{x \rightarrow a} f'(x)$ and $g'(a) = \lim_{x \rightarrow a} g'(x)$. Using these ideas, we conclude that

Equation:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)(x-a)}{g'(x)(x-a)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

Note that the assumption that f' and g' are continuous at a and $g'(a) \neq 0$ can be loosened. We state L'Hôpital's rule formally for the indeterminate form $\frac{0}{0}$. Also note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to represent a quotient of limits, each of which is zero.

Note:

L'Hôpital's Rule ($0/0$ Case)

Suppose f and g are differentiable functions over an open interval containing a , except possibly at a . If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$, then

Equation:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ and $-\infty$.

Proof

We provide a proof of this theorem in the special case when f, g, f' , and g' are all continuous over an open interval containing a . In that case, since $\lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} g(x)$ and f and g are continuous at a , it follows that $f(a) = 0 = g(a)$. Therefore,

Equation:

$$\begin{aligned} \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)} && \text{since } f(a) = 0 = g(a) \\ &= \lim_{x \rightarrow a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} && \text{algebra} \\ &= \frac{\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a}} && \text{limit of a quotient} \\ &= \frac{f'(a)}{g'(a)} && \text{definition of the derivative} \\ &= \frac{\lim_{x \rightarrow a} f'(x)}{\lim_{x \rightarrow a} g'(x)} && \text{continuity of } f' \text{ and } g' \\ &= \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}. && \text{limit of a quotient} \end{aligned}$$

Note that L'Hôpital's rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is important to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

□

Example:

Exercise:

Problem:

Applying L'Hôpital's Rule (0/0 Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$

- b. $\lim_{x \rightarrow 1} \frac{\sin(\pi x)}{\ln x}$
 c. $\lim_{x \rightarrow \infty} \frac{e^{1/x} - 1}{1/x}$
 d. $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^2}$

Solution:

- a. Since the numerator $1 - \cos x \rightarrow 0$ and the denominator $x \rightarrow 0$, we can apply L'Hôpital's rule to evaluate this limit. We have

Equation:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{1} \\ &= \frac{\lim_{x \rightarrow 0}(\sin x)}{\lim_{x \rightarrow 0}(1)} \\ &= \frac{0}{1} = 0. \end{aligned}$$

- b. As $x \rightarrow 1$, the numerator $\sin(\pi x) \rightarrow 0$ and the denominator $\ln(x) \rightarrow 0$. Therefore, we can apply L'Hôpital's rule. We obtain

Equation:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sin(\pi x)}{\ln x} &= \lim_{x \rightarrow 1} \frac{\pi \cos(\pi x)}{1/x} \\ &= \lim_{x \rightarrow 1} (\pi x) \cos(\pi x) \\ &= (\pi \cdot 1)(-1) = -\pi. \end{aligned}$$

- c. As $x \rightarrow \infty$, the numerator $e^{1/x} - 1 \rightarrow 0$ and the denominator $(\frac{1}{x}) \rightarrow 0$. Therefore, we can apply L'Hôpital's rule. We obtain

Equation:

$$\lim_{x \rightarrow \infty} \frac{e^{1/x} - 1}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{e^{1/x} \left(\frac{-1}{x^2} \right)}{\left(\frac{-1}{x^2} \right)} = \lim_{x \rightarrow \infty} e^{1/x} = e^0 = 1.$$

- d. As $x \rightarrow 0$, both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule. We obtain

Equation:

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^2} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as

$x \rightarrow 0$, we apply L'Hôpital's rule again. In doing so, we see that

Equation:

$$\lim_{x \rightarrow 0} \frac{\cos x - 1}{2x} = \lim_{x \rightarrow 0} \frac{-\sin x}{2} = 0.$$

Therefore, we conclude that

Equation:

$$\lim_{x \rightarrow 0} \frac{\sin x - x}{x^2} = 0.$$

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow 0} \frac{x}{\tan x}$.

Solution:

1

Hint

$$\frac{d}{dx} \tan x = \sec^2 x$$

We can also use L'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x) \rightarrow \pm\infty$ and $g(x) \rightarrow \pm\infty$. Limits of this form are classified as *indeterminate forms of type ∞/∞* . Again, note that we are not actually dividing ∞ by ∞ . Since ∞ is not a real number, that is impossible; rather, ∞/∞ is used to represent a quotient of limits, each of which is ∞ or $-\infty$.

Note:

L'Hôpital's Rule (∞/∞ Case)

Suppose f and g are differentiable functions over an open interval containing a , except possibly at a . Suppose $\lim_{x \rightarrow a} f(x) = \infty$ (or $-\infty$) and $\lim_{x \rightarrow a} g(x) = \infty$ (or $-\infty$). Then,

Equation:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a = \infty$ or $-\infty$, or the limit is one-sided.

Example:

Exercise:

Problem:

Applying L'Hôpital's Rule (∞/∞ Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x \rightarrow \infty} \frac{3x+5}{2x+1}$

b. $\lim_{x \rightarrow 0^+} \frac{\ln x}{\cot x}$

Solution:

- a. Since $3x + 5$ and $2x + 1$ are first-degree polynomials with positive leading coefficients, $\lim_{x \rightarrow \infty} (3x + 5) = \infty$ and $\lim_{x \rightarrow \infty} (2x + 1) = \infty$. Therefore, we apply L'Hôpital's rule and obtain

Equation:

$$\lim_{x \rightarrow \infty} \frac{3x + 5}{2x + 1} = \lim_{x \rightarrow \infty} \frac{3}{2} = \frac{3}{2}.$$

Note that this limit can also be calculated without invoking L'Hôpital's rule. Earlier in the chapter we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

Equation:

$$\lim_{x \rightarrow \infty} \frac{3x + 5}{2x + 1} = \lim_{x \rightarrow \infty} \frac{3 + 5/x}{2x + 1/x} = \frac{3}{2}.$$

L'Hôpital's rule provides us with an alternative means of evaluating this type of limit.

- b. Here, $\lim_{x \rightarrow 0^+} \ln x = -\infty$ and $\lim_{x \rightarrow 0^+} \cot x = \infty$. Therefore, we can apply L'Hôpital's rule and obtain

Equation:

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\cot x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-\csc^2 x} = \lim_{x \rightarrow 0^+} \frac{1}{-x \csc^2 x}.$$

Now as $x \rightarrow 0^+$, $\csc^2 x \rightarrow \infty$. Therefore, the first term in the denominator is approaching zero and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc x$ to write

Equation:

$$\lim_{x \rightarrow 0^+} \frac{1}{-x \csc^2 x} = \lim_{x \rightarrow 0^+} \frac{\sin^2 x}{-x}.$$

Now $\lim_{x \rightarrow 0^+} \sin^2 x = 0$ and $\lim_{x \rightarrow 0^+} x = 0$, so we apply L'Hôpital's rule again. We find

Equation:

$$\lim_{x \rightarrow 0^+} \frac{\sin^2 x}{-x} = \lim_{x \rightarrow 0^+} \frac{2 \sin x \cos x}{-1} = \frac{0}{-1} = 0.$$

We conclude that

Equation:

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\cot x} = 0.$$

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow \infty} \frac{\ln x}{5x}$.

Solution:

0

Hint

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

As mentioned, L'Hôpital's rule is an extremely useful tool for evaluating limits. It is important to remember, however, that to apply L'Hôpital's rule to a quotient $\frac{f(x)}{g(x)}$, it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or ∞/∞ . Consider the following example.

Example:

Exercise:

Problem:

When L'Hôpital's Rule Does Not Apply

Consider $\lim_{x \rightarrow 1} \frac{x^2 + 5}{3x + 4}$. Show that the limit cannot be evaluated by applying L'Hôpital's rule.

Solution:

Because the limits of the numerator and denominator are not both zero and are not both infinite, we cannot apply L'Hôpital's rule. If we try to do so, we get

Equation:

$$\frac{d}{dx}(x^2 + 5) = 2x$$

and

Equation:

$$\frac{d}{dx}(3x + 4) = 3.$$

At which point we would conclude erroneously that

Equation:

$$\lim_{x \rightarrow 1} \frac{x^2 + 5}{3x + 4} = \lim_{x \rightarrow 1} \frac{2x}{3} = \frac{2}{3}.$$

However, since $\lim_{x \rightarrow 1} (x^2 + 5) = 6$ and $\lim_{x \rightarrow 1} (3x + 4) = 7$, we actually have

Equation:

$$\lim_{x \rightarrow 1} \frac{x^2 + 5}{3x + 4} = \frac{6}{7}.$$

We can conclude that

Equation:

$$\lim_{x \rightarrow 1} \frac{x^2 + 5}{3x + 4} \neq \lim_{x \rightarrow 1} \frac{\frac{d}{dx}(x^2 + 5)}{\frac{d}{dx}(3x + 4)}.$$

Note:**Exercise:****Problem:**

Explain why we cannot apply L'Hôpital's rule to evaluate $\lim_{x \rightarrow 0^+} \frac{\cos x}{x}$. Evaluate $\lim_{x \rightarrow 0^+} \frac{\cos x}{x}$ by other means.

Solution:

$\lim_{x \rightarrow 0^+} \cos x = 1$. Therefore, we cannot apply L'Hôpital's rule. The limit of the quotient is ∞ .

Hint

Determine the limits of the numerator and denominator separately.

Other Indeterminate Forms

L'Hôpital's rule is very useful for evaluating limits involving the indeterminate forms $\frac{0}{0}$ and ∞/∞ . However, we can also use L'Hôpital's rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0 \cdot \infty$, $\infty - \infty$, 1^∞ , ∞^0 , and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next we realize why these are indeterminate forms and then understand how to use L'Hôpital's rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\frac{0}{0}$ or ∞/∞ .

Indeterminate Form of Type $0 \cdot \infty$

Suppose we want to evaluate $\lim_{x \rightarrow a} (f(x) \cdot g(x))$, where $f(x) \rightarrow 0$ and $g(x) \rightarrow \infty$ (or $-\infty$) as $x \rightarrow a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0 \cdot \infty$ to denote the form that arises in this situation. The expression $0 \cdot \infty$ is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x)g(x)$ as $x \rightarrow a$. For example, let n be a positive integer and consider

Equation:

$$f(x) = \frac{1}{(x^n + 1)} \text{ and } g(x) = 3x^2.$$

As $x \rightarrow \infty$, $f(x) \rightarrow 0$ and $g(x) \rightarrow \infty$. However, the limit as $x \rightarrow \infty$ of $f(x)g(x) = \frac{3x^2}{(x^n+1)}$ varies, depending on n . If $n = 2$, then $\lim_{x \rightarrow \infty} f(x)g(x) = 3$. If $n = 1$, then $\lim_{x \rightarrow \infty} f(x)g(x) = \infty$. If $n = 3$, then $\lim_{x \rightarrow \infty} f(x)g(x) = 0$. Here we consider another limit involving the indeterminate form $0 \cdot \infty$ and show how to rewrite the function as a quotient to use L'Hôpital's rule.

Example:

Exercise:

Problem:

Indeterminate Form of Type $0 \cdot \infty$

Evaluate $\lim_{x \rightarrow 0^+} x \ln x$.

Solution:

First, rewrite the function $x \ln x$ as a quotient to apply L'Hôpital's rule. If we write

Equation:

$$x \ln x = \frac{\ln x}{1/x},$$

we see that $\ln x \rightarrow -\infty$ as $x \rightarrow 0^+$ and $\frac{1}{x} \rightarrow \infty$ as $x \rightarrow 0^+$. Therefore, we can apply L'Hôpital's rule and obtain

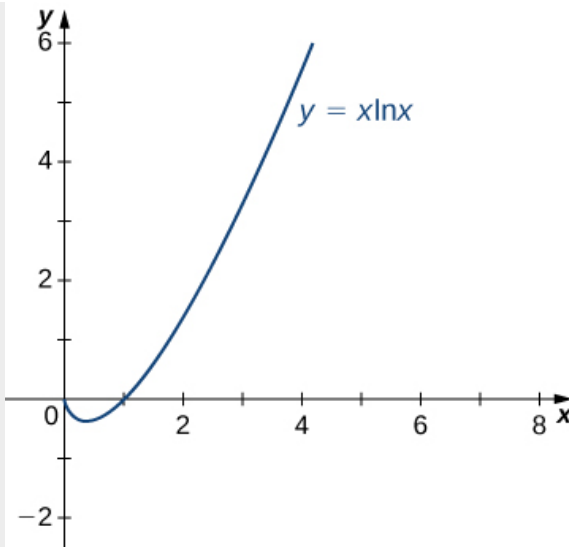
Equation:

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(1/x)} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0.$$

We conclude that

Equation:

$$\lim_{x \rightarrow 0^+} x \ln x = 0.$$



Finding the limit at $x = 0$ of the function $f(x) = x \ln x$.

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow 0} x \cot x$.

Solution:

1

Hint

Write $x \cot x = \frac{x \cos x}{\sin x}$

Indeterminate Form of Type $\infty - \infty$

Another type of indeterminate form is $\infty - \infty$. Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \rightarrow \infty$, $f(x) \rightarrow \infty$ and $g(x) \rightarrow \infty$. We are interested in $\lim_{x \rightarrow \infty} (f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \rightarrow \infty$ and $g(x) \rightarrow \infty$, we write $\infty - \infty$ to denote the form of this limit. As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own and we must do

more analysis to determine the value of the limit. For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

Equation:

$$\lim_{x \rightarrow \infty} (f(x) - g(x)) = \lim_{x \rightarrow \infty} (3x^3 - 3x^2 - 5) = \infty.$$

On the other hand, if $n = 2$, then

Equation:

$$\lim_{x \rightarrow \infty} (f(x) - g(x)) = \lim_{x \rightarrow \infty} (3x^2 - 3x^2 - 5) = -5.$$

However, if $n = 1$, then

Equation:

$$\lim_{x \rightarrow \infty} (f(x) - g(x)) = \lim_{x \rightarrow \infty} (3x - 3x^2 - 5) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply L'Hôpital's rule.

Example:

Exercise:

Problem:

Indeterminate Form of Type $\infty - \infty$

Evaluate $\lim_{x \rightarrow 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right)$.

Solution:

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x$, we have

Equation:

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2 \tan x}.$$

As $x \rightarrow 0^+$, the numerator $\tan x - x^2 \rightarrow 0$ and the denominator $x^2 \tan x \rightarrow 0$. Therefore, we can apply L'Hôpital's rule. Taking the derivatives of the numerator and the denominator, we have

Equation:

$$\lim_{x \rightarrow 0^+} \frac{(\tan x) - x^2}{x^2 \tan x} = \lim_{x \rightarrow 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x}.$$

As $x \rightarrow 0^+$, $(\sec^2 x) - 2x \rightarrow 1$ and $x^2 \sec^2 x + 2x \tan x \rightarrow 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

Equation:

$$\lim_{x \rightarrow 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x} = \infty.$$

Therefore,

Equation:

$$\lim_{x \rightarrow 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right) = \infty.$$

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{\sin x} \right)$.

Solution:

0

Hint

Rewrite the difference of fractions as a single fraction.

Another type of indeterminate form that arises when evaluating limits involves exponents. The expressions 0^0 , ∞^0 , and 1^∞ are all indeterminate forms. On their own, these expressions are meaningless because we cannot actually evaluate these expressions as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now we examine how L'Hôpital's rule can be used to evaluate limits involving these indeterminate forms.

Since L'Hôpital's rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim_{x \rightarrow a} f(x)^{g(x)}$ and we

arrive at the indeterminate form ∞^0 . (The indeterminate forms 0^0 and 1^∞ can be handled similarly.) We proceed as follows. Let

Equation:

$$y = f(x)^{g(x)}.$$

Then,

Equation:

$$\ln y = \ln \left(f(x)^{g(x)} \right) = g(x) \ln (f(x)).$$

Therefore,

Equation:

$$\lim_{x \rightarrow a} [\ln (y)] = \lim_{x \rightarrow a} [g(x) \ln (f(x))].$$

Since $\lim_{x \rightarrow a} f(x) = \infty$, we know that $\lim_{x \rightarrow a} \ln (f(x)) = \infty$. Therefore, $\lim_{x \rightarrow a} g(x) \ln (f(x))$ is of the indeterminate form $0 \cdot \infty$, and we can use the techniques discussed earlier to rewrite the expression $g(x) \ln (f(x))$ in a form so that we can apply L'Hôpital's rule. Suppose $\lim_{x \rightarrow a} g(x) \ln (f(x)) = L$, where L may be ∞ or $-\infty$. Then

Equation:

$$\lim_{x \rightarrow a} [\ln (y)] = L.$$

Since the natural logarithm function is continuous, we conclude that

Equation:

$$\ln \left(\lim_{x \rightarrow a} y \right) = L,$$

which gives us

Equation:

$$\lim_{x \rightarrow a} y = \lim_{x \rightarrow a} f(x)^{g(x)} = e^L.$$

Example:

Exercise:

Problem:

Indeterminate Form of Type ∞^0

Evaluate $\lim_{x \rightarrow \infty} x^{1/x}$.

Solution:

Let $y = x^{1/x}$. Then,

Equation:

$$\ln \left(x^{1/x} \right) = \frac{1}{x} \ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$. Applying L'Hôpital's rule, we obtain

Equation:

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0.$$

Therefore, $\lim_{x \rightarrow \infty} \ln y = 0$. Since the natural logarithm function is continuous, we conclude that

Equation:

$$\ln \left(\lim_{x \rightarrow \infty} y \right) = 0,$$

which leads to

Equation:

$$\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} \frac{\ln x}{x} = e^0 = 1.$$

Hence,

Equation:

$$\lim_{x \rightarrow \infty} x^{1/x} = 1.$$

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow \infty} x^{1/\ln(x)}$.

Solution:

e

Hint

Let $y = x^{1/\ln(x)}$ and apply the natural logarithm to both sides of the equation.

Example:

Exercise:

Problem:
Indeterminate Form of Type 0^0

Evaluate $\lim_{x \rightarrow 0^+} x^{\sin x}$.

Solution:

Let

Equation:

$$y = x^{\sin x}.$$

Therefore,

Equation:

$$\ln y = \ln(x^{\sin x}) = \sin x \ln x.$$

We now evaluate $\lim_{x \rightarrow 0^+} \sin x \ln x$. Since $\lim_{x \rightarrow 0^+} \sin x = 0$ and $\lim_{x \rightarrow 0^+} \ln x = -\infty$, we have the indeterminate form $0 \cdot \infty$. To apply L'Hôpital's rule, we need to rewrite $\sin x \ln x$ as a fraction. We could write

Equation:

$$\sin x \ln x = \frac{\sin x}{1/\ln x}$$

or

Equation:

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x}.$$

Let's consider the first option. In this case, applying L'Hôpital's rule, we would obtain
Equation:

$$\lim_{x \rightarrow 0^+} \sin x \ln x = \lim_{x \rightarrow 0^+} \frac{\sin x}{1/\ln x} = \lim_{x \rightarrow 0^+} \frac{\cos x}{-1/(x(\ln x)^2)} = \lim_{x \rightarrow 0^+} \left(-x(\ln x)^2 \cos x \right).$$

Unfortunately, we not only have another expression involving the indeterminate form $0 \cdot \infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing

Equation:

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x},$$

and applying L'Hôpital's rule, we obtain

Equation:

$$\lim_{x \rightarrow 0^+} \sin x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\csc x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x \rightarrow 0^+} \frac{-1}{x \csc x \cot x}.$$

Using the fact that $\csc x = \frac{1}{\sin x}$ and $\cot x = \frac{\cos x}{\sin x}$, we can rewrite the expression on the right-hand side as

Equation:

$$\lim_{x \rightarrow 0^+} \frac{-\sin^2 x}{x \cos x} = \lim_{x \rightarrow 0^+} \left[\frac{\sin x}{x} \cdot (-\tan x) \right] = \left(\lim_{x \rightarrow 0^+} \frac{\sin x}{x} \right) \cdot \left(\lim_{x \rightarrow 0^+} (-\tan x) \right) = 1 \cdot 0 = 0.$$

We conclude that $\lim_{x \rightarrow 0^+} \ln y = 0$. Therefore, $\ln \left(\lim_{x \rightarrow 0^+} y \right) = 0$ and we have

Equation:

$$\lim_{x \rightarrow 0^+} y = \lim_{x \rightarrow 0^+} x^{\sin x} = e^0 = 1.$$

Hence,

Equation:

$$\lim_{x \rightarrow 0^+} x^{\sin x} = 1.$$

Note:

Exercise:

Problem: Evaluate $\lim_{x \rightarrow 0^+} x^x$.

Solution:

1

Hint

Let $y = x^x$ and take the natural logarithm of both sides of the equation.

Growth Rates of Functions

Suppose the functions f and g both approach infinity as $x \rightarrow \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \rightarrow \infty$. However, as shown in the following table, the values of x^3 are growing much faster than the values of x^2 .

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = x^3$	1000	1,000,000	1,000,000,000	1,000,000,000,000

Comparing the Growth Rates of x^2 and x^3

In fact,

Equation:

$$\lim_{x \rightarrow \infty} \frac{x^3}{x^2} = \lim_{x \rightarrow \infty} x = \infty. \text{ or, equivalently, } \lim_{x \rightarrow \infty} \frac{x^2}{x^3} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \rightarrow \infty$. On the other hand, for $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$, although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \rightarrow \infty$, as shown in the following table. In fact,

Equation:

$$\lim_{x \rightarrow \infty} \frac{x^2}{3x^2 + 4x + 1} = \frac{1}{3}.$$

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = 3x^2 + 4x + 1$	341	30,401	3,004,001	300,040,001

Comparing the Growth Rates of x^2 and $3x^2 + 4x + 1$

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \rightarrow \infty$.

More generally, suppose f and g are two functions that approach infinity as $x \rightarrow \infty$. We say g grows more rapidly than f as $x \rightarrow \infty$ if

Equation:

$$\lim_{x \rightarrow \infty} \frac{g(x)}{f(x)} = \infty; \text{ or, equivalently, } \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0.$$

On the other hand, if there exists a constant $M \neq 0$ such that

Equation:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = M,$$

we say f and g grow at the same rate as $x \rightarrow \infty$.

Next we see how to use L'Hôpital's rule to compare the growth rates of power, exponential, and logarithmic functions.

Example:

Exercise:**Problem:****Comparing the Growth Rates of $\ln(x)$, x^2 , and e^x**

For each of the following pairs of functions, use L'Hôpital's rule to evaluate $\lim_{x \rightarrow \infty} \left(\frac{f(x)}{g(x)} \right)$.

- a. $f(x) = x^2$ and $g(x) = e^x$
- b. $f(x) = \ln(x)$ and $g(x) = x^2$

Solution:

- a. Since $\lim_{x \rightarrow \infty} x^2 = \infty$ and $\lim_{x \rightarrow \infty} e^x = \infty$, we can use L'Hôpital's rule to evaluate

$\lim_{x \rightarrow \infty} \left[\frac{x^2}{e^x} \right]$. We obtain

Equation:

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = \lim_{x \rightarrow \infty} \frac{2x}{e^x}.$$

Since $\lim_{x \rightarrow \infty} 2x = \infty$ and $\lim_{x \rightarrow \infty} e^x = \infty$, we can apply L'Hôpital's rule again. Since

Equation:

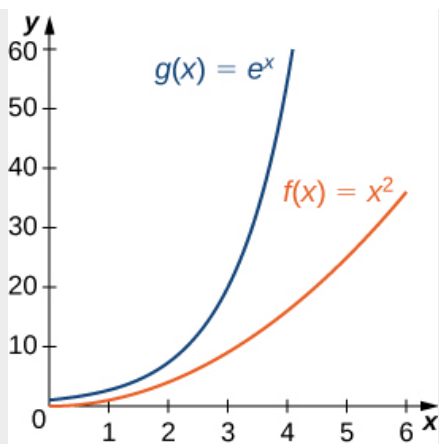
$$\lim_{x \rightarrow \infty} \frac{2x}{e^x} = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0,$$

we conclude that

Equation:

$$\lim_{x \rightarrow \infty} \frac{x^2}{e^x} = 0.$$

Therefore, e^x grows more rapidly than x^2 as $x \rightarrow \infty$ (See [\[link\]](#) and [\[link\]](#)).



An exponential function grows at a faster rate than a power function.

x	5	10	15	20
x^2	25	100	225	400
e^x	148	22,026	3,269,017	485,165,195

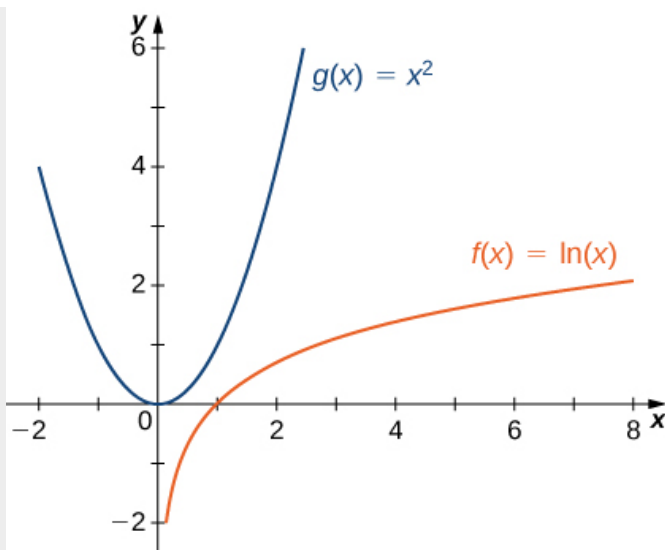
Growth rates of a power function and an exponential function.

- b. Since $\lim_{x \rightarrow \infty} \ln x = \infty$ and $\lim_{x \rightarrow \infty} x^2 = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x \rightarrow \infty} \frac{\ln x}{x^2}$. We obtain

Equation:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x^2} = \lim_{x \rightarrow \infty} \frac{1/x}{2x} = \lim_{x \rightarrow \infty} \frac{1}{2x^2} = 0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x \rightarrow \infty$ (see [\[link\]](#) and [\[link\]](#)).



A power function grows at a faster rate than a logarithmic function.

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10,000	1,000,000	100,000,000

Growth rates of a power function and a logarithmic function

Note:

Exercise:

Problem: Compare the growth rates of x^{100} and 2^x .

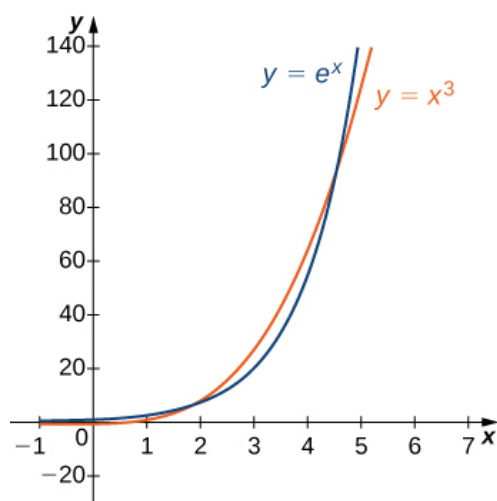
Solution:

The function 2^x grows faster than x^{100} .

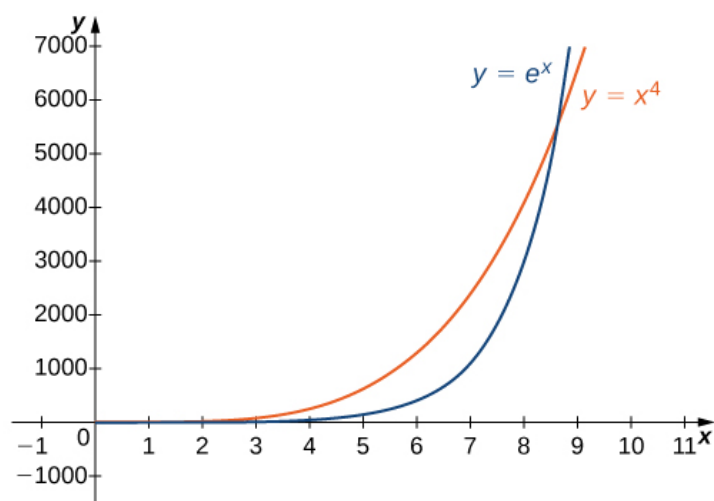
Hint

Apply L'Hôpital's rule to $x^{100}/2^x$

Using the same ideas as in [\[link\]](#)a, it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In [\[link\]](#) and [\[link\]](#), we compare e^x with x^3 and x^4 as $x \rightarrow \infty$.



(a)



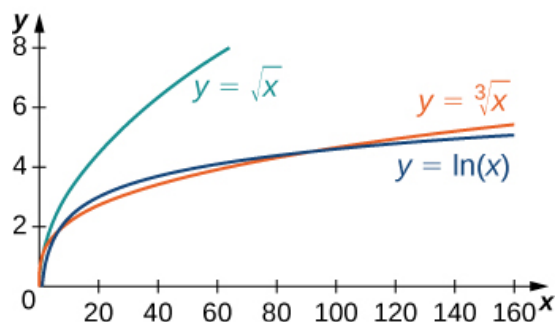
(b)

The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3 . (b) A comparison of e^x with x^4 .

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10,000	50,625	160,000
e^x	148	22,026	3,269,017	485,165,195

An exponential function grows at a faster rate than any power function

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In [\[link\]](#) and [\[link\]](#), we compare $\ln x$ with $\sqrt[3]{x}$ and \sqrt{x} .



The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \rightarrow \infty$.

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
\sqrt{x}	3.162	10	31.623	100

A logarithmic function grows at a slower rate than any root function

Key Concepts

- L'Hôpital's rule can be used to evaluate the limit of a quotient when the indeterminate form $\frac{0}{0}$ or ∞/∞ arises.
- L'Hôpital's rule can also be applied to other indeterminate forms if they can be rewritten in terms of a limit involving a quotient that has the indeterminate form $\frac{0}{0}$ or ∞/∞ .
- The exponential function e^x grows faster than any power function x^p , $p > 0$.
- The logarithmic function $\ln x$ grows more slowly than any power function x^p , $p > 0$.

For the following exercises, evaluate the limit.

Exercise:

Problem: Evaluate the limit $\lim_{x \rightarrow \infty} \frac{e^x}{x}$.

Exercise:

Problem: Evaluate the limit $\lim_{x \rightarrow \infty} \frac{e^x}{x^k}$.

Solution:

$$\infty$$

Exercise:

Problem: Evaluate the limit $\lim_{x \rightarrow \infty} \frac{\ln x}{x^k}$.

Exercise:

Problem: Evaluate the limit $\lim_{x \rightarrow a} \frac{x-a}{x^2-a^2}$, $a \neq 0$.

Solution:

$$\frac{1}{2a}$$

Exercise:

Problem: Evaluate the limit $\lim_{x \rightarrow a} \frac{x-a}{x^3-a^3}$, $a \neq 0$.

Exercise:

Problem: Evaluate the limit $\lim_{x \rightarrow a} \frac{x-a}{x^n-a^n}$, $a \neq 0$.

Solution:

$$\frac{1}{na^{n-1}}$$

For the following exercises, determine whether you can apply L'Hôpital's rule directly. Explain why or why not. Then, indicate if there is some way you can alter the limit so you can apply L'Hôpital's rule.

Exercise:

Problem: $\lim_{x \rightarrow 0^+} x^2 \ln x$

Exercise:

Problem: $\lim_{x \rightarrow \infty} x^{1/x}$

Solution:

Cannot apply directly; use logarithms

Exercise:

Problem: $\lim_{x \rightarrow 0} x^{2/x}$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{x^2}{1/x}$

Solution:

Cannot apply directly; rewrite as $\lim_{x \rightarrow 0} x^3$

Exercise:

Problem: $\lim_{x \rightarrow \infty} \frac{e^x}{x}$

For the following exercises, evaluate the limits with either L'Hôpital's rule or previously learned methods.

Exercise:

Problem: $\lim_{x \rightarrow 3} \frac{x^2-9}{x-3}$

Solution:

6

Exercise:

Problem: $\lim_{x \rightarrow 3} \frac{x^2-9}{x+3}$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{(1+x)^{-2}-1}{x}$

Solution:

-2

Exercise:

Problem: $\lim_{x \rightarrow \pi/2} \frac{\cos x}{\frac{\pi}{2}-x}$

Exercise:

Problem: $\lim_{x \rightarrow \pi} \frac{x - \pi}{\sin x}$

Solution:

$$-1$$

Exercise:

Problem: $\lim_{x \rightarrow 1} \frac{x - 1}{\sin x}$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x}$

Solution:

$$n$$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1 - nx}{x^2}$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^3}$

Solution:

$$-\frac{1}{2}$$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{x^2}$

Solution:

$$\frac{1}{2}$$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{\tan x}{\sqrt{x}}$

Exercise:

Problem: $\lim_{x \rightarrow 1} \frac{x-1}{\ln x}$

Solution:

$$1$$

Exercise:

Problem: $\lim_{x \rightarrow 0} (x+1)^{1/x}$

Exercise:

Problem: $\lim_{x \rightarrow 1} \frac{\sqrt{x} - \sqrt[3]{x}}{x-1}$

Solution:

$$\frac{1}{6}$$

Exercise:

Problem: $\lim_{x \rightarrow 0^+} x^{2x}$

Exercise:

Problem: $\lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right)$

Solution:

$$1$$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^2}$

Exercise:

Problem: $\lim_{x \rightarrow 0^+} x \ln(x^4)$

Solution:

0

Exercise:

Problem: $\lim_{x \rightarrow \infty} (x - e^x)$

Exercise:

Problem: $\lim_{x \rightarrow \infty} x^2 e^{-x}$

Solution:

0

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{3^x - 2^x}{x}$

Exercise:

Problem: $\lim_{x \rightarrow 0} \frac{1 + 1/x}{1 - 1/x}$

Solution:

-1

Exercise:

Problem: $\lim_{x \rightarrow \pi/4} (1 - \tan x) \cot x$

Exercise:

Problem: $\lim_{x \rightarrow \infty} x e^{1/x}$

Solution:

∞

Exercise:

Problem: $\lim_{x \rightarrow 0} x^{1/\cos x}$

Exercise:

Problem: $\lim_{x \rightarrow 0} x^{1/x}$

Solution:

$$1$$

Exercise:

Problem: $\lim_{x \rightarrow 0} \left(1 - \frac{1}{x}\right)^x$

Exercise:

Problem: $\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^x$

Solution:

$$\frac{1}{e}$$

For the following exercises, use a calculator to graph the function and estimate the value of the limit, then use L'Hôpital's rule to find the limit directly.

Exercise:

Problem: [T] $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$

Exercise:

Problem: [T] $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right)$

Solution:

$$0$$

Exercise:

Problem: [T] $\lim_{x \rightarrow 1} \frac{x-1}{1-\cos(\pi x)}$

Exercise:

Problem: [T] $\lim_{x \rightarrow 1} \frac{e^{(x-1)} - 1}{x-1}$

Solution:

$$1$$

Exercise:

Problem: [T] $\lim_{x \rightarrow 1} \frac{(x-1)^2}{\ln x}$

Exercise:

Problem: [T] $\lim_{x \rightarrow \pi} \frac{1 + \cos x}{\sin x}$

Solution:

0

Exercise:

Problem: [T] $\lim_{x \rightarrow 0} \left(\csc x - \frac{1}{x} \right)$

Exercise:

Problem: [T] $\lim_{x \rightarrow 0^+} \tan(x^x)$

Solution:

$\tan(1)$

Exercise:

Problem: [T] $\lim_{x \rightarrow 0^+} \frac{\ln x}{\sin x}$

Exercise:

Problem: [T] $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x}$

Solution:

2

Glossary

indeterminate forms

when evaluating a limit, the forms $\frac{0}{0}$, ∞/∞ , $0 \cdot \infty$, $\infty - \infty$, 0^0 , ∞^0 , and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is

L'Hôpital's rule

if f and g are differentiable functions over an interval a , except possibly at a , and $\lim_{x \rightarrow a} f(x) = 0 = \lim_{x \rightarrow a} g(x)$ or $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ are infinite, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}, \text{ assuming the limit on the right exists or is } \infty \text{ or } -\infty$$

Newton's Method

- Describe the steps of Newton's method.
- Explain what an iterative process means.
- Recognize when Newton's method does not work.
- Apply iterative processes to various situations.

In many areas of pure and applied mathematics, we are interested in finding solutions to an equation of the form $f(x) = 0$. For most functions, however, it is difficult—if not impossible—to calculate their zeroes explicitly. In this section, we take a look at a technique that provides a very efficient way of approximating the zeroes of functions. This technique makes use of tangent line approximations and is behind the method used often by calculators and computers to find zeroes.

Describing Newton's Method

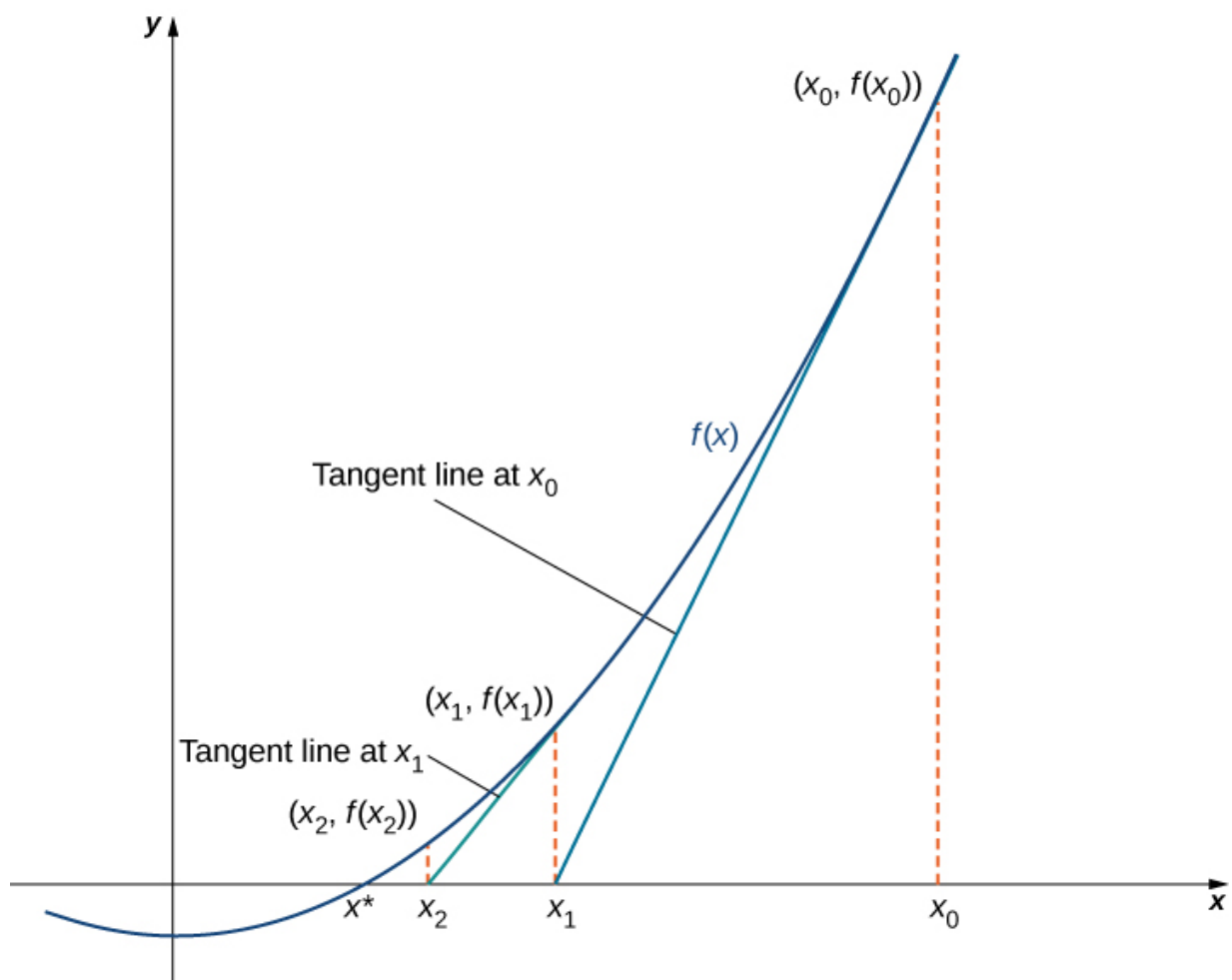
Consider the task of finding the solutions of $f(x) = 0$. If f is the first-degree polynomial $f(x) = ax + b$, then the solution of $f(x) = 0$ is given by the formula $x = -\frac{b}{a}$. If f is the second-degree polynomial $f(x) = ax^2 + bx + c$, the solutions of $f(x) = 0$ can be found by using the quadratic formula. However, for polynomials of degree 3 or more, finding roots of f becomes more complicated. Although formulas exist for third- and fourth-degree polynomials, they are quite complicated. Also, if f is a polynomial of degree 5 or greater, it is known that no such formulas exist. For example, consider the function

Equation:

$$f(x) = x^5 + 8x^4 + 4x^3 - 2x - 7.$$

No formula exists that allows us to find the solutions of $f(x) = 0$. Similar difficulties exist for nonpolynomial functions. For example, consider the task of finding solutions of $\tan(x) - x = 0$. No simple formula exists for the solutions of this equation. In cases such as these, we can use Newton's method to approximate the roots.

Newton's method makes use of the following idea to approximate the solutions of $f(x) = 0$. By sketching a graph of f , we can estimate a root of $f(x) = 0$. Let's call this estimate x_0 . We then draw the tangent line to f at x_0 . If $f'(x_0) \neq 0$, this tangent line intersects the x -axis at some point $(x_1, 0)$. Now let x_1 be the next approximation to the actual root. Typically, x_1 is closer than x_0 to an actual root. Next we draw the tangent line to f at x_1 . If $f'(x_1) \neq 0$, this tangent line also intersects the x -axis, producing another approximation, x_2 . We continue in this way, deriving a list of approximations: x_0, x_1, x_2, \dots . Typically, the numbers x_0, x_1, x_2, \dots quickly approach an actual root x^* , as shown in the following figure.



The approximations x_0, x_1, x_2, \dots approach the actual root x^* . The approximations are derived by looking at tangent lines to the graph of

f .

Now let's look at how to calculate the approximations x_0, x_1, x_2, \dots . If x_0 is our first approximation, the approximation x_1 is defined by letting $(x_1, 0)$ be the x -intercept of the tangent line to f at x_0 . The equation of this tangent line is given by

Equation:

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Therefore, x_1 must satisfy

Equation:

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

Solving this equation for x_1 , we conclude that

Equation:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, the point $(x_2, 0)$ is the x -intercept of the tangent line to f at x_1 . Therefore, x_2 satisfies the equation

Equation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, for $n > 0$, x_n satisfies

Equation:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}.$$

Next we see how to make use of this technique to approximate the root of the polynomial $f(x) = x^3 - 3x + 1$.

Example:

Exercise:

Problem:

Finding a Root of a Polynomial

Use Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find x_1, x_2, x_3, x_4 , and x_5 .

Solution:

From [\[link\]](#), we see that f has one root over the interval $(1, 2)$. Therefore $x_0 = 2$ seems like a reasonable first approximation. To find the next approximation, we use [\[link\]](#). Since $f(x) = x^3 - 3x + 1$, the derivative is $f'(x) = 3x^2 - 3$. Using [\[link\]](#) with $n = 1$ (and a calculator that displays 10 digits), we obtain

Equation:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{9} \approx 1.666666667.$$

To find the next approximation, x_2 , we use [\[link\]](#) with $n = 2$ and the value of x_1 stored on the calculator. We find that

Equation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.548611111.$$

Continuing in this way, we obtain the following results:

Equation:

$$x_1 \approx 1.666666667$$

$$x_2 \approx 1.548611111$$

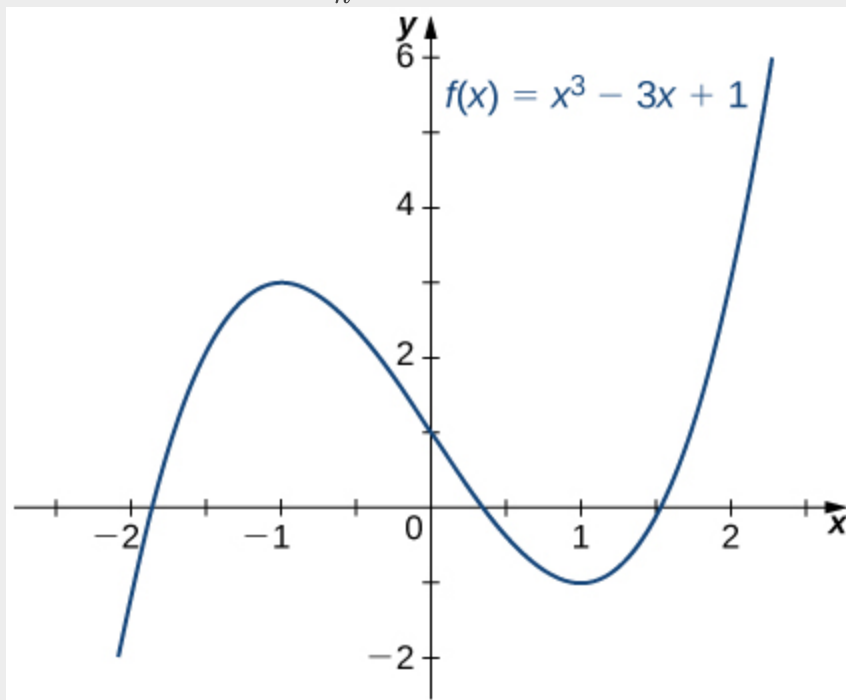
$$x_3 \approx 1.532390162$$

$$x_4 \approx 1.532088989$$

$$x_5 \approx 1.532088886$$

$$x_6 \approx 1.532088886.$$

We note that we obtained the same value for x_5 and x_6 . Therefore, any subsequent application of Newton's method will most likely give the same value for x_n .



The function $f(x) = x^3 - 3x + 1$ has one root over the interval $[1, 2]$.

Note:**Exercise:****Problem:**

Letting $x_0 = 0$, let's use Newton's method to approximate the root of $f(x) = x^3 - 3x + 1$ over the interval $[0, 1]$ by calculating x_1 and x_2 .

Solution:

$$x_1 \approx 0.33333333, x_2 \approx 0.34722222$$

Hint

Use [\[link\]](#).

Newton's method can also be used to approximate square roots. Here we show how to approximate $\sqrt{2}$. This method can be modified to approximate the square root of any positive number.

Example:**Exercise:****Problem:****Finding a Square Root**

Use Newton's method to approximate $\sqrt{2}$ ([\[link\]](#)). Let $f(x) = x^2 - 2$, let $x_0 = 2$, and calculate x_1, x_2, x_3, x_4, x_5 . (We note that since $f(x) = x^2 - 2$ has a zero at $\sqrt{2}$, the initial value $x_0 = 2$ is a reasonable choice to approximate $\sqrt{2}$.)

Solution:

For $f(x) = x^2 - 2$, $f'(x) = 2x$. From [\[link\]](#), we know that

Equation:

$$\begin{aligned}
 x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\
 &= x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}} \\
 &= \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}} \\
 &= \frac{1}{2} \left(x_{n-1} + \frac{2}{x_{n-1}} \right).
 \end{aligned}$$

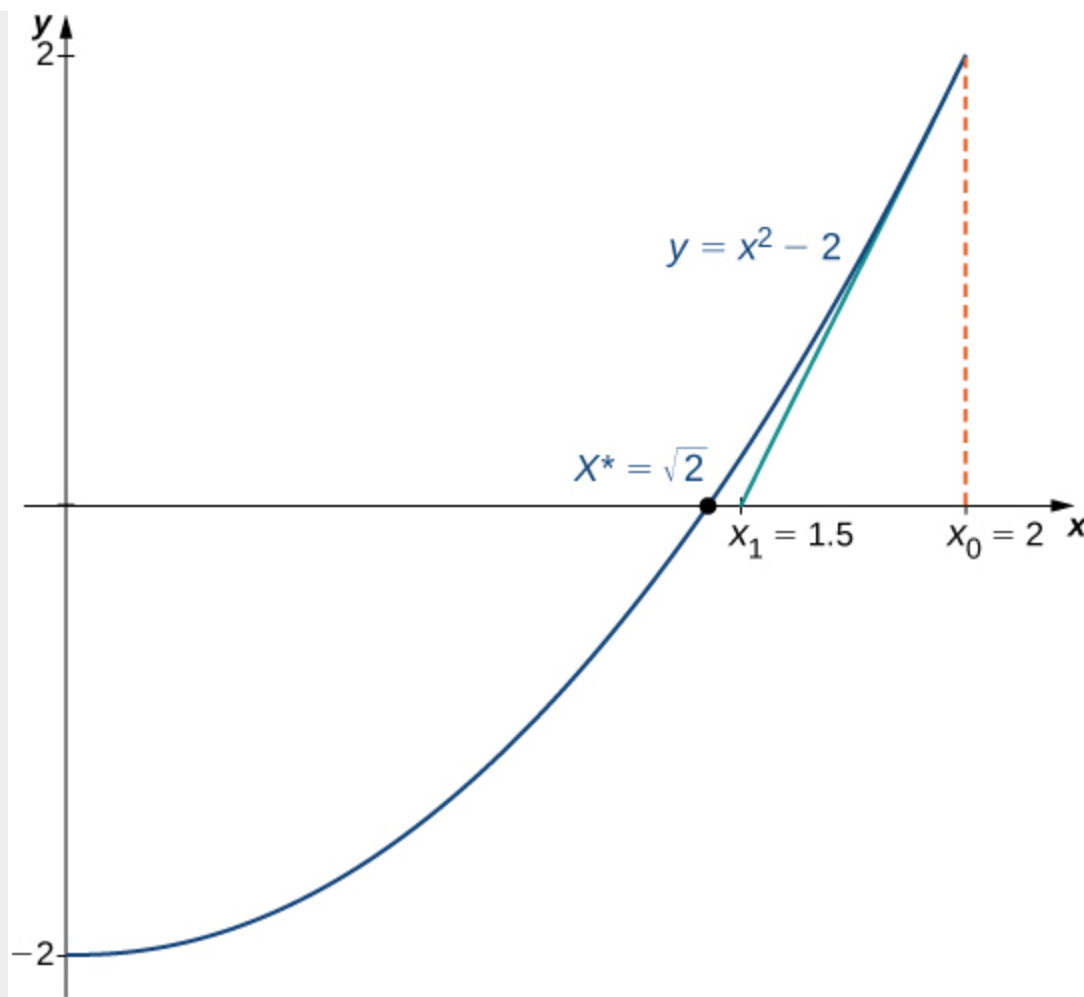
Therefore,
Equation:

$$\begin{aligned}
 x_1 &= \frac{1}{2} \left(x_0 + \frac{2}{x_0} \right) = \frac{1}{2} \left(2 + \frac{2}{2} \right) = 1.5 \\
 x_2 &= \frac{1}{2} \left(x_1 + \frac{2}{x_1} \right) = \frac{1}{2} \left(1.5 + \frac{2}{1.5} \right) \approx 1.416666667.
 \end{aligned}$$

Continuing in this way, we find that
Equation:

$$\begin{aligned}
 x_1 &= 1.5 \\
 x_2 &\approx 1.416666667 \\
 x_3 &\approx 1.414215686 \\
 x_4 &\approx 1.414213562 \\
 x_5 &\approx 1.414213562.
 \end{aligned}$$

Since we obtained the same value for x_4 and x_5 , it is unlikely that the value x_n will change on any subsequent application of Newton's method. We conclude that $\sqrt{2} \approx 1.414213562$.



We can use Newton's method to find $\sqrt{2}$.

Note:

Exercise:

Problem:

Use Newton's method to approximate $\sqrt{3}$ by letting $f(x) = x^2 - 3$ and $x_0 = 3$. Find x_1 and x_2 .

Solution:

$$x_1 = 2, x_2 = 1.75$$

Hint

For $f(x) = x^2 - 3$, [\[link\]](#) reduces to $x_n = \frac{x_{n-1}}{2} + \frac{3}{2x_{n-1}}$.

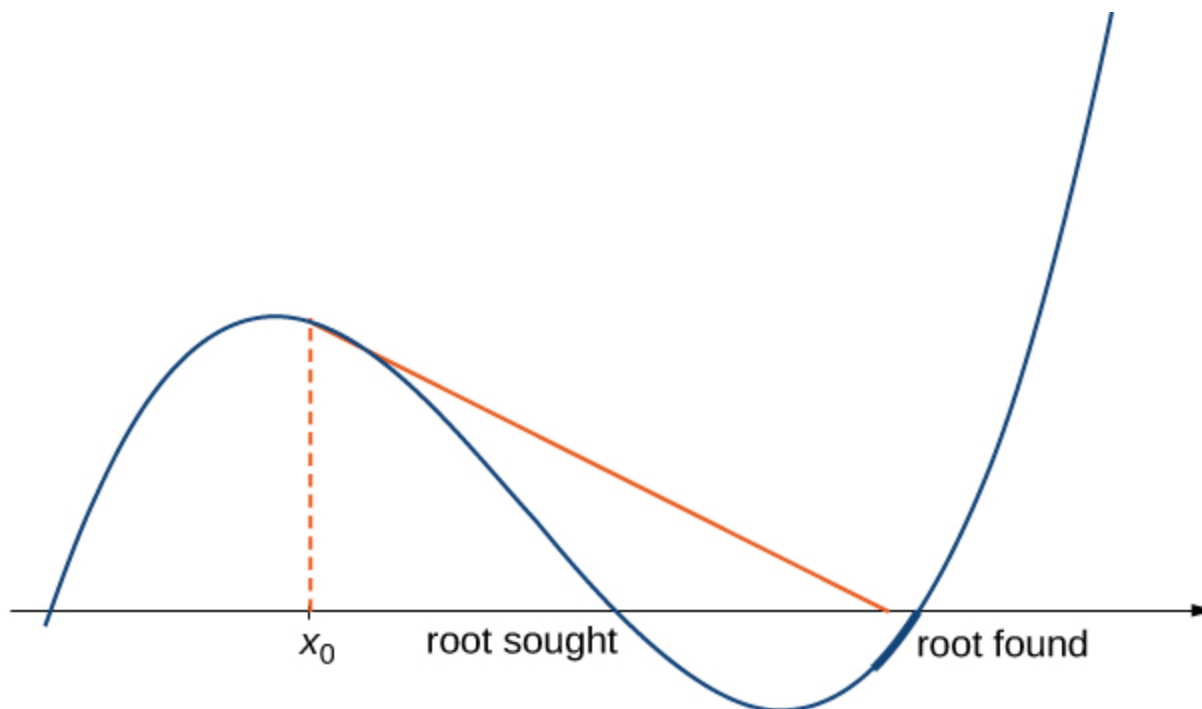
When using Newton's method, each approximation after the initial guess is defined in terms of the previous approximation by using the same formula. In particular, by defining the function $F(x) = x - \left[\frac{f(x)}{f'(x)} \right]$, we can rewrite [\[link\]](#) as $x_n = F(x_{n-1})$. This type of process, where each x_n is defined in terms of x_{n-1} by repeating the same function, is an example of an **iterative process**. Shortly, we examine other iterative processes. First, let's look at the reasons why Newton's method could fail to find a root.

Failures of Newton's Method

Typically, Newton's method is used to find roots fairly quickly. However, things can go wrong. Some reasons why Newton's method might fail include the following:

1. At one of the approximations x_n , the derivative f' is zero at x_n , but $f(x_n) \neq 0$. As a result, the tangent line of f at x_n does not intersect the x -axis. Therefore, we cannot continue the iterative process.
2. The approximations x_0, x_1, x_2, \dots may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root (see [\[link\]](#)). This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.
3. The approximations may fail to approach a root entirely. In [\[link\]](#), we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the

successive approximations continue to alternate back and forth between two values.



If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.

Example:

Exercise:

Problem:

When Newton's Method Fails

Consider the function $f(x) = x^3 - 2x + 2$. Let $x_0 = 0$. Show that the sequence x_1, x_2, \dots fails to approach a root of f .

Solution:

For $f(x) = x^3 - 2x + 2$, the derivative is $f'(x) = 3x^2 - 2$.

Therefore,

Equation:

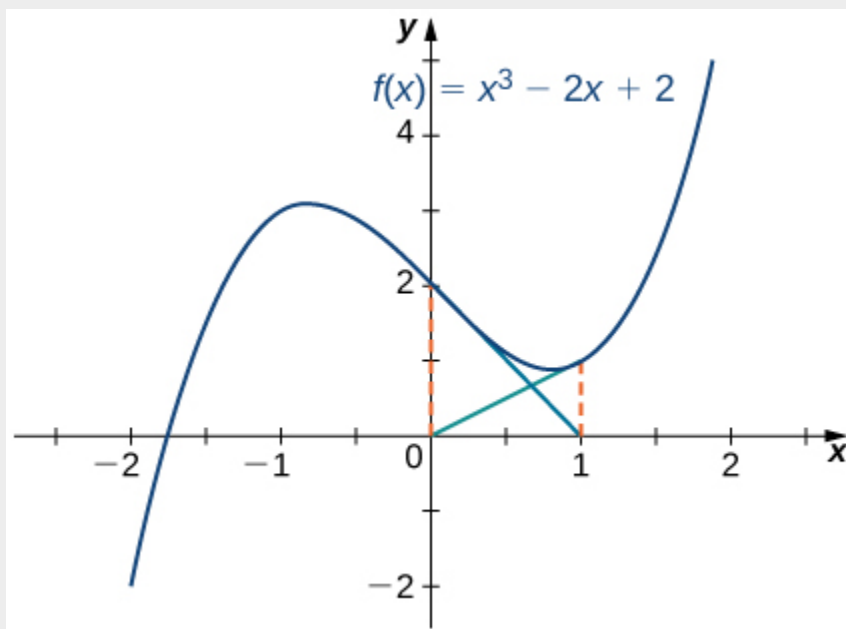
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1.$$

In the next step,

Equation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0.$$

Consequently, the numbers x_0, x_1, x_2, \dots continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2, -1]$ (see [\[link\]](#)). Fortunately, if we choose an initial approximation x_0 closer to the actual root, we can avoid this situation.



The approximations continue to alternate between 0 and 1 and never approach the root of f .

Note:

Exercise:

Problem:

For $f(x) = x^3 - 2x + 2$, let $x_0 = -1.5$ and find x_1 and x_2 .

Solution:

$$x_1 \approx -1.842105263, x_2 \approx -1.772826920$$

Hint

Use [\[link\]](#).

From [\[link\]](#), we see that Newton's method does not always work. However, when it does work, the sequence of approximations approaches the root very quickly. Discussions of how quickly the sequence of approximations approach a root found using Newton's method are included in texts on numerical analysis.

Other Iterative Processes

As mentioned earlier, Newton's method is a type of iterative process. We now look at an example of a different type of iterative process.

Consider a function F and an initial number x_0 . Define the subsequent numbers x_n by the formula $x_n = F(x_{n-1})$. This process is an iterative

process that creates a list of numbers $x_0, x_1, x_2, \dots, x_n, \dots$. This list of numbers may approach a finite number x^* as n gets larger, or it may not. In [\[link\]](#), we see an example of a function F and an initial guess x_0 such that the resulting list of numbers approaches a finite value.

Example:

Exercise:

Problem:

Finding a Limit for an Iterative Process

Let $F(x) = \frac{1}{2}x + 4$ and let $x_0 = 0$. For all $n \geq 1$, let $x_n = F(x_{n-1})$. Find the values x_1, x_2, x_3, x_4, x_5 . Make a conjecture about what happens to this list of numbers $x_1, x_2, x_3, \dots, x_n, \dots$ as $n \rightarrow \infty$. If the list of numbers x_1, x_2, x_3, \dots approaches a finite number x^* , then x^* satisfies $x^* = F(x^*)$, and x^* is called a fixed point of F .

Solution:

If $x_0 = 0$, then

Equation:

$$x_1 = \frac{1}{2}(0) + 4 = 4$$

$$x_2 = \frac{1}{2}(4) + 4 = 6$$

$$x_3 = \frac{1}{2}(6) + 4 = 7$$

$$x_4 = \frac{1}{2}(7) + 4 = 7.5$$

$$x_5 = \frac{1}{2}(7.5) + 4 = 7.75$$

$$x_6 = \frac{1}{2}(7.75) + 4 = 7.875$$

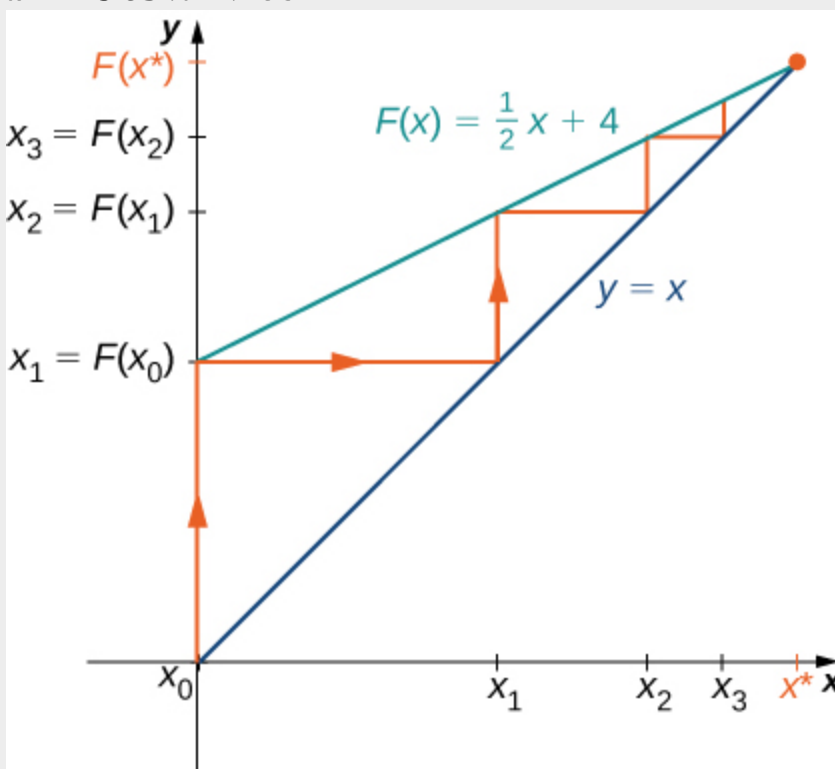
$$x_7 = \frac{1}{2}(7.875) + 4 = 7.9375$$

$$x_8 = \frac{1}{2}(7.9375) + 4 = 7.96875$$

$$x_9 = \frac{1}{2}(7.96875) + 4 = 7.984375.$$

From this list, we conjecture that the values x_n approach 8.

[\[link\]](#) provides a graphical argument that the values approach 8 as $n \rightarrow \infty$. Starting at the point (x_0, x_0) , we draw a vertical line to the point $(x_0, F(x_0))$. The next number in our list is $x_1 = F(x_0)$. We use x_1 to calculate x_2 . Therefore, we draw a horizontal line connecting (x_0, x_1) to the point (x_1, x_1) on the line $y = x$, and then draw a vertical line connecting (x_1, x_1) to the point $(x_1, F(x_1))$. The output $F(x_1)$ becomes x_2 . Continuing in this way, we could create an infinite number of line segments. These line segments are trapped between the lines $F(x) = \frac{x}{2} + 4$ and $y = x$. The line segments get closer to the intersection point of these two lines, which occurs when $x = F(x)$. Solving the equation $x = \frac{x}{2} + 4$, we conclude they intersect at $x = 8$. Therefore, our graphical evidence agrees with our numerical evidence that the list of numbers x_0, x_1, x_2, \dots approaches $x^* = 8$ as $n \rightarrow \infty$.



This iterative process approaches the value

$$x^* = 8.$$

Note:**Exercise:****Problem:**

Consider the function $F(x) = \frac{1}{3}x + 6$. Let $x_0 = 0$ and let $x_n = F(x_{n-1})$ for $n \geq 1$. Find x_1, x_2, x_3, x_4, x_5 . Make a conjecture about what happens to the list of numbers $x_1, x_2, x_3, \dots, x_n, \dots$ as $n \rightarrow \infty$.

Solution:

$$x_1 = 6, x_2 = 8, x_3 = \frac{26}{3}, x_4 = \frac{80}{9}, x_5 = \frac{242}{27}; x^* = 9$$

Hint

Consider the point where the lines $y = x$ and $y = F(x)$ intersect.

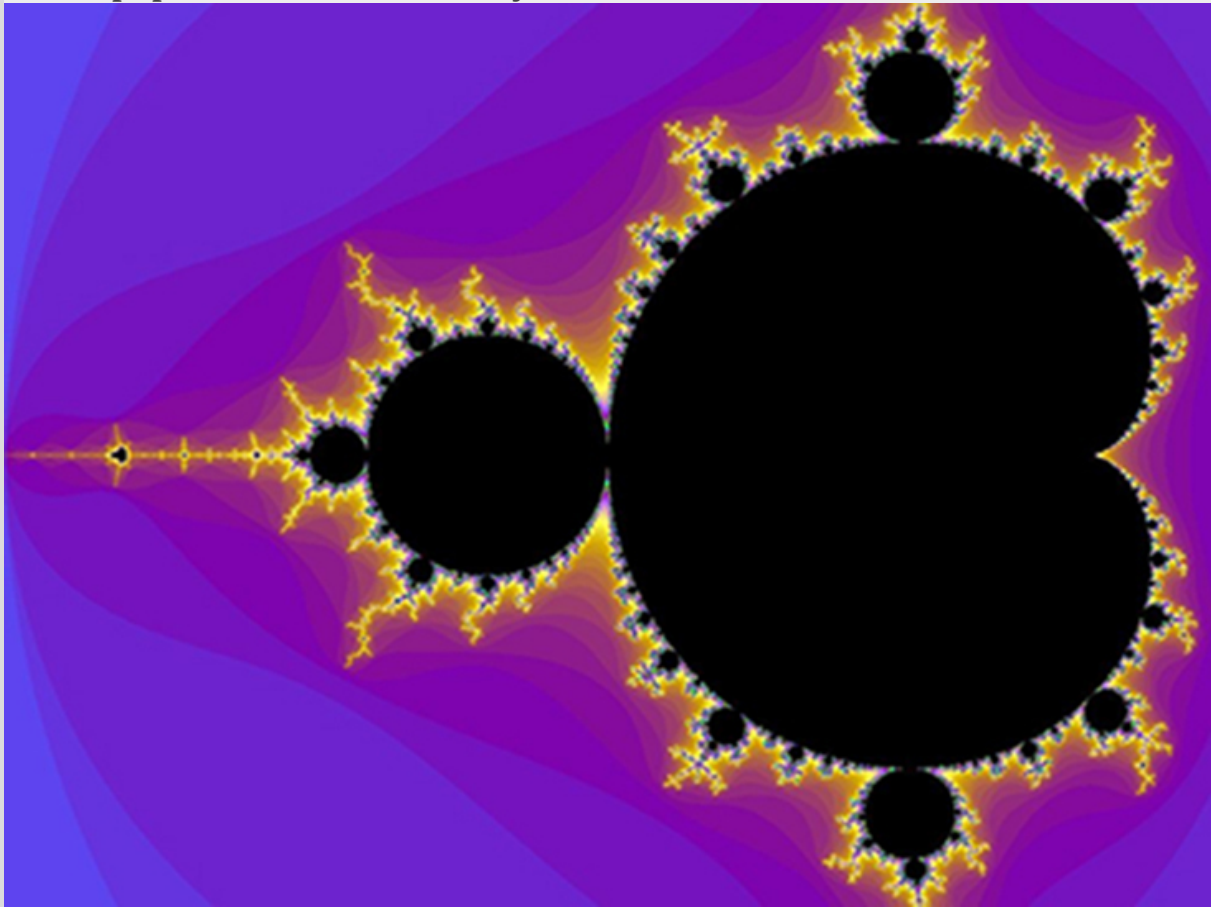
Note:**Iterative Processes and Chaos**

Iterative processes can yield some very interesting behavior. In this section, we have seen several examples of iterative processes that converge to a fixed point. We also saw in [\[link\]](#) that the iterative process bounced back and forth between two values. We call this kind of behavior a *2-cycle*. Iterative processes can converge to cycles with various periodicities, such as 2 – cycles, 4 – cycles (where the iterative process repeats a sequence of four values), 8-cycles, and so on.

Some iterative processes yield what mathematicians call *chaos*. In this case, the iterative process jumps from value to value in a seemingly random fashion and never converges or settles into a cycle. Although a

complete exploration of chaos is beyond the scope of this text, in this project we look at one of the key properties of a chaotic iterative process: sensitive dependence on initial conditions. This property refers to the concept that small changes in initial conditions can generate drastically different behavior in the iterative process.

Probably the best-known example of chaos is the Mandelbrot set (see [\[link\]](#)), named after Benoit Mandelbrot (1924–2010), who investigated its properties and helped popularize the field of chaos theory. The Mandelbrot set is usually generated by computer and shows fascinating details on enlargement, including self-replication of the set. Several colorized versions of the set have been shown in museums and can be found online and in popular books on the subject.



The Mandelbrot set is a well-known example of a set of points generated by the iterative chaotic behavior of a relatively simple function.

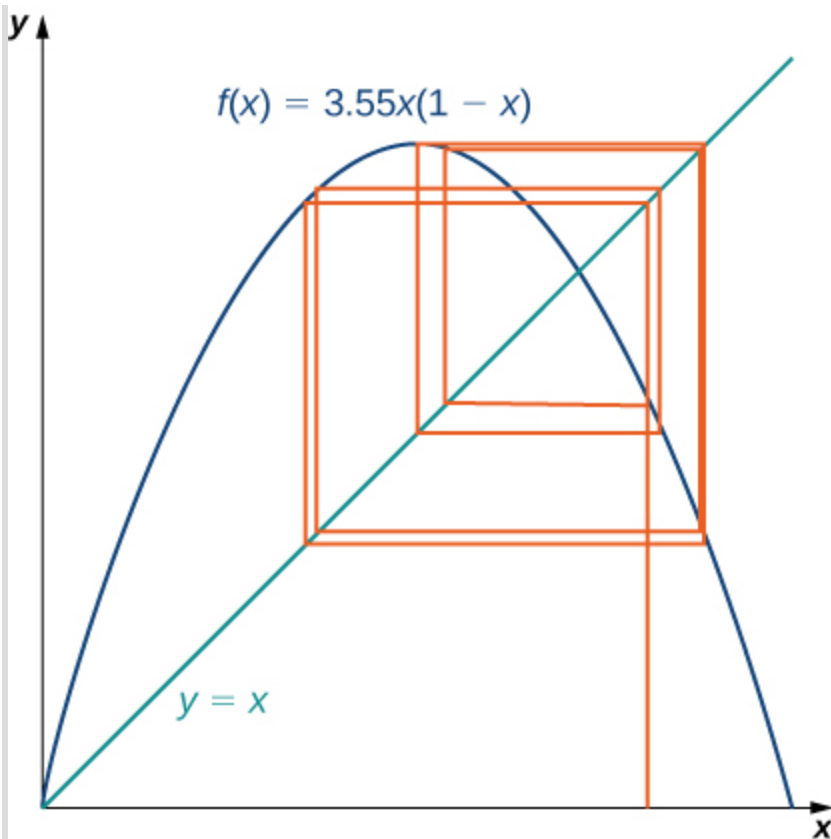
In this project we use the logistic map

Equation:

$$f(x) = rx(1 - x), \text{ where } x \in [0, 1] \text{ and } r > 0$$

as the function in our iterative process. The logistic map is a deceptively simple function; but, depending on the value of r , the resulting iterative process displays some very interesting behavior. It can lead to fixed points, cycles, and even chaos.

To visualize the long-term behavior of the iterative process associated with the logistic map, we will use a tool called a *cobweb diagram*. As we did with the iterative process we examined earlier in this section, we first draw a vertical line from the point $(x_0, 0)$ to the point $(x_0, f(x_0)) = (x_0, x_1)$. We then draw a horizontal line from that point to the point (x_1, x_1) , then draw a vertical line to $(x_1, f(x_1)) = (x_1, x_2)$, and continue the process until the long-term behavior of the system becomes apparent. [\[link\]](#) shows the long-term behavior of the logistic map when $r = 3.55$ and $x_0 = 0.2$. (The first 100 iterations are not plotted.) The long-term behavior of this iterative process is an 8-cycle.



A cobweb diagram for $f(x) = 3.55x(1 - x)$ is presented here. The sequence of values results in an 8-cycle.

1. Let $r = 0.5$ and choose $x_0 = 0.2$. Either by hand or by using a computer, calculate the first 10 values in the sequence. Does the sequence appear to converge? If so, to what value? Does it result in a cycle? If so, what kind of cycle (for example, 2-cycle, 4-cycle.)?
2. What happens when $r = 2$?
3. For $r = 3.2$ and $r = 3.5$, calculate the first 100 sequence values. Generate a cobweb diagram for each iterative process. (Several free applets are available online that generate cobweb diagrams for the logistic map.) What is the long-term behavior in each of these cases?
4. Now let $r = 4$. Calculate the first 100 sequence values and generate a cobweb diagram. What is the long-term behavior in this case?

5. Repeat the process for $r = 4$, but let $x_0 = 0.201$. How does this behavior compare with the behavior for $x_0 = 0.2$?

Key Concepts

- Newton's method approximates roots of $f(x) = 0$ by starting with an initial approximation x_0 , then uses tangent lines to the graph of f to create a sequence of approximations x_1, x_2, x_3, \dots
- Typically, Newton's method is an efficient method for finding a particular root. In certain cases, Newton's method fails to work because the list of numbers x_0, x_1, x_2, \dots does not approach a finite value or it approaches a value other than the root sought.
- Any process in which a list of numbers x_0, x_1, x_2, \dots is generated by defining an initial number x_0 and defining the subsequent numbers by the equation $x_n = F(x_{n-1})$ for some function F is an iterative process. Newton's method is an example of an iterative process, where the function $F(x) = x - \left[\frac{f(x)}{f'(x)} \right]$ for a given function f .

For the following exercises, write Newton's formula as $x_{n+1} = F(x_n)$ for solving $f(x) = 0$.

Exercise:

Problem: $f(x) = x^2 + 1$

Exercise:

Problem: $f(x) = x^3 + 2x + 1$

Solution:

$$F(x_n) = x_n - \frac{x_n^3 + 2x_n + 1}{3x_n^2 + 2}$$

Exercise:

Problem: $f(x) = \sin x$

Exercise:

Problem: $f(x) = e^x$

Solution:

$$F(x_n) = x_n - \frac{e^{x_n}}{e^{x_n}}$$

Exercise:

Problem: $f(x) = x^3 + 3xe^x$

For the following exercises, solve $f(x) = 0$ using the iteration $x_{n+1} = x_n - cf(x_n)$, which differs slightly from Newton's method. Find a c that works and a c that fails to converge, with the exception of $c = 0$.

Exercise:

Problem: $f(x) = x^2 - 4$, with $x_0 = 0$

Solution:

$$|c| > 0.5 \text{ fails, } |c| \leq 0.5 \text{ works}$$

Exercise:

Problem: $f(x) = x^2 - 4x + 3$, with $x_0 = 2$

Exercise:

Problem: What is the value of “ c ” for Newton's method?

Solution:

$$c = \frac{1}{f'(x_n)}$$

For the following exercises, start at

a. $x_0 = 0.6$ and

b. $x_0 = 2$.

Compute x_1 and x_2 using the specified iterative method.

Exercise:

Problem: $x_{n+1} = x_n^2 - \frac{1}{2}$

Exercise:

Problem: $x_{n+1} = 2x_n(1 - x_n)$

Solution:

a. $x_1 = \frac{12}{25}$, $x_2 = \frac{312}{625}$; b. $x_1 = -4$, $x_2 = -40$

Exercise:

Problem: $x_{n+1} = \sqrt{x_n}$

Exercise:

Problem: $x_{n+1} = \frac{1}{\sqrt{x_n}}$

Solution:

a. $x_1 = 1.291$, $x_2 = 0.8801$; b. $x_1 = 0.7071$, $x_2 = 1.189$

Exercise:

Problem: $x_{n+1} = 3x_n(1 - x_n)$

Exercise:

Problem: $x_{n+1} = x_n^2 + x_n - 2$

Solution:

a. $x_1 = -\frac{26}{25}, x_2 = -\frac{1224}{625}$; b. $x_1 = 4, x_2 = 18$

Exercise:

Problem: $x_{n+1} = \frac{1}{2}x_n - 1$

Exercise:

Problem: $x_{n+1} = |x_n|$

Solution:

a. $x_1 = \frac{6}{10}, x_2 = \frac{6}{10}$; b. $x_1 = 2, x_2 = 2$

For the following exercises, solve to four decimal places using Newton's method and a computer or calculator. Choose any initial guess x_0 that is not the exact root.

Exercise:

Problem: $x^2 - 10 = 0$

Exercise:

Problem: $x^4 - 100 = 0$

Solution:

3.1623 or -3.1623

Exercise:

Problem: $x^2 - x = 0$

Exercise:

Problem: $x^3 - x = 0$

Solution:

$0, -1$ or 1

Exercise:

Problem: $x + 5 \cos(x) = 0$

Exercise:

Problem: $x + \tan(x) = 0$, choose $x_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution:

0

Exercise:

Problem: $\frac{1}{1-x} = 2$

Exercise:

Problem: $1 + x + x^2 + x^3 + x^4 = 2$

Solution:

0.5188 or -1.2906

Exercise:

Problem: $x^3 + (x + 1)^3 = 10^3$

Exercise:

Problem: $x = \sin^2(x)$

Solution:

0

For the following exercises, use Newton's method to find the fixed points of the function where $f(x) = x$; round to three decimals.

Exercise:

Problem: $\sin x$

Exercise:

Problem: $\tan(x)$ on $x = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Solution:

4.493

Exercise:

Problem: $e^x - 2$

Exercise:

Problem: $\ln(x) + 2$

Solution:

0.159, 3.146

Newton's method can be used to find maxima and minima of functions in addition to the roots. In this case apply Newton's method to the derivative function $f'(x)$ to find its roots, instead of the original function. For the following exercises, consider the formulation of the method.

Exercise:**Problem:**

To find candidates for maxima and minima, we need to find the critical points $f'(x) = 0$. Show that to solve for the critical points of a function $f(x)$, Newton's method is given by $x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$.

Exercise:**Problem:**

What additional restrictions are necessary on the function f ?

Solution:

We need f to be twice continuously differentiable.

For the following exercises, use Newton's method to find the location of the local minima and/or maxima of the following functions; round to three decimals.

Exercise:

Problem: Minimum of $f(x) = x^2 + 2x + 4$

Exercise:

Problem: Minimum of $f(x) = 3x^3 + 2x^2 - 16$

Solution:

$$x = 0$$

Exercise:

Problem: Minimum of $f(x) = x^2 e^x$

Exercise:

Problem: Maximum of $f(x) = x + \frac{1}{x}$

Solution:

$$x = -1$$

Exercise:

Problem: Maximum of $f(x) = x^3 + 10x^2 + 15x - 2$

Exercise:

Problem: Maximum of $f(x) = \frac{\sqrt{x} - \sqrt[3]{x}}{x}$

Solution:

$$x = 5.619$$

Exercise:

Problem:

Minimum of $f(x) = x^2 \sin x$, closest non-zero minimum to $x = 0$

Exercise:

Problem: Minimum of $f(x) = x^4 + x^3 + 3x^2 + 12x + 6$

Solution:

$$x = -1.326$$

For the following exercises, use the specified method to solve the equation.
If it does not work, explain why it does not work.

Exercise:

Problem: Newton's method, $x^2 + 2 = 0$

Exercise:

Problem: Newton's method, $0 = e^x$

Solution:

There is no solution to the equation.

Exercise:

Problem: Newton's method, $0 = 1 + x^2$ starting at $x_0 = 0$

Exercise:

Problem: Solving $x_{n+1} = -x_n^3$ starting at $x_0 = -1$

Solution:

It enters a cycle.

For the following exercises, use the secant method, an alternative iterative method to Newton's method. The formula is given by

Equation:

$$x_n = x_{n-1} - f(x_{n-1}) \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}.$$

Exercise:**Problem:**

Find a root to $0 = x^2 - x - 3$ accurate to three decimal places.

Exercise:**Problem:**

Find a root to $0 = \sin x + 3x$ accurate to four decimal places.

Solution:

0

Exercise:

Problem: Find a root to $0 = e^x - 2$ accurate to four decimal places.

Exercise:

Problem:

Find a root to $\ln(x + 2) = \frac{1}{2}$ accurate to four decimal places.

Solution:

−0.3513

Exercise:

Problem:

Why would you use the secant method over Newton's method? What are the necessary restrictions on f ?

For the following exercises, use both Newton's method and the secant method to calculate a root for the following equations. Use a calculator or computer to calculate how many iterations of each are needed to reach within three decimal places of the exact answer. For the secant method, use the first guess from Newton's method.

Exercise:

Problem: $f(x) = x^2 + 2x + 1, x_0 = 1$

Solution:

Newton: 11 iterations, secant: 16 iterations

Exercise:

Problem: $f(x) = x^2, x_0 = 1$

Exercise:

Problem: $f(x) = \sin x, x_0 = 1$

Solution:

Newton: three iterations, secant: six iterations

Exercise:

Problem: $f(x) = e^x - 1, x_0 = 2$

Exercise:

Problem: $f(x) = x^3 + 2x + 4, x_0 = 0$

Solution:

Newton: five iterations, secant: eight iterations

In the following exercises, consider Kepler's equation regarding planetary orbits, $M = E - \varepsilon \sin(E)$, where M is the mean anomaly, E is eccentric anomaly, and ε measures eccentricity.

Exercise:

Problem:

Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{\pi}{3}$ and the eccentricity of the orbit $\varepsilon = 0.25$; round to three decimals.

Exercise:

Problem:

Use Newton's method to solve for the eccentric anomaly E when the mean anomaly $M = \frac{3\pi}{2}$ and the eccentricity of the orbit $\varepsilon = 0.8$; round to three decimals.

Solution:

$$E = 4.071$$

The following two exercises consider a bank investment. The initial investment is \$10,000. After 25 years, the investment has tripled to \$30,000.

Exercise:**Problem:**

Use Newton's method to determine the interest rate if the interest was compounded annually.

Exercise:**Problem:**

Use Newton's method to determine the interest rate if the interest was compounded continuously.

Solution:

$$4.394\%$$

Exercise:**Problem:**

The cost for printing a book can be given by the equation $C(x) = 1000 + 12x + \left(\frac{1}{2}\right)x^{2/3}$. Use Newton's method to find the break-even point if the printer sells each book for \$20.

Glossary

iterative process

process in which a list of numbers $x_0, x_1, x_2, x_3, \dots$ is generated by starting with a number x_0 and defining $x_n = F(x_{n-1})$ for $n \geq 1$

Newton's method

method for approximating roots of $f(x) = 0$; using an initial guess x_0 ; each subsequent approximation is defined by the equation

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Antiderivatives

- Find the general antiderivative of a given function.
- Explain the terms and notation used for an indefinite integral.
- State the power rule for integrals.
- Use antidifferentiation to solve simple initial-value problems.

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function f , how do we find a function with the derivative f and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. The antiderivative of a function f is a function with a derivative f . Why are we interested in antiderivatives? The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here we examine one specific example that involves rectilinear motion. In our examination in [Derivatives](#) of rectilinear motion, we showed that given a position function $s(t)$ of an object, then its velocity function $v(t)$ is the derivative of $s(t)$ —that is, $v(t) = s'(t)$. Furthermore, the acceleration $a(t)$ is the derivative of the velocity $v(t)$ —that is, $a(t) = v'(t) = s''(t)$. Now suppose we are given an acceleration function a , but not the velocity function v or the position function s . Since $a(t) = v'(t)$, determining the velocity function requires us to find an antiderivative of the acceleration function. Then, since $v(t) = s'(t)$, determining the position function requires us to find an antiderivative of the velocity function. Rectilinear motion is just one case in which the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. For now, let's look at the terminology and notation for antiderivatives, and determine the antiderivatives for several types of functions. We examine various techniques for finding antiderivatives of more complicated functions later in the text ([Introduction to Techniques of Integration](#)).

The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f , how can we find a function with derivative f ? If we can find a function F derivative f , we call F an antiderivative of f .

Note:

Definition

A function F is an **antiderivative** of the function f if

Equation:

$$F'(x) = f(x)$$

for all x in the domain of f .

Consider the function $f(x) = 2x$. Knowing the power rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F'(x) = 2x$. Are there any other antiderivatives of f ? Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x$. Therefore,

$x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives. Are there any others that are not of the form $x^2 + C$ for some constant C ? The answer is no. From Corollary 2 of the Mean Value Theorem, we know that if F and G are differentiable functions such that $F'(x) = G'(x)$, then $F(x) - G(x) = C$ for some constant C . This fact leads to the following important theorem.

Note:

General Form of an Antiderivative

Let F be an antiderivative of f over an interval I . Then,

- i. for each constant C , the function $F(x) + C$ is also an antiderivative of f over I ;
- ii. if G is an antiderivative of f over I , there is a constant C for which $G(x) = F(x) + C$ over I .

In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example:

Exercise:

Problem:

Finding Antiderivatives

For each of the following functions, find all antiderivatives.

- a. $f(x) = 3x^2$
- b. $f(x) = \frac{1}{x}$
- c. $f(x) = \cos x$
- d. $f(x) = e^x$

Solution:

- a. Because

Equation:

$$\frac{d}{dx}(x^3) = 3x^2$$

then $F(x) = x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form $x^3 + C$ for some constant C , and every function of the form $x^3 + C$ is an antiderivative of $3x^2$.

- b. Let $f(x) = \ln|x|$. For $x > 0$, $f(x) = \ln(x)$ and

Equation:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

For $x < 0$, $f(x) = \ln(-x)$ and

Equation:

$$\frac{d}{dx}(\ln(-x)) = -\frac{1}{-x} = \frac{1}{x}.$$

Therefore,

Equation:

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

Thus, $F(x) = \ln|x|$ is an antiderivative of $\frac{1}{x}$. Therefore, every antiderivative of $\frac{1}{x}$ is of the form $\ln|x| + C$ for some constant C and every function of the form $\ln|x| + C$ is an antiderivative of $\frac{1}{x}$.

c. We have

Equation:

$$\frac{d}{dx}(\sin x) = \cos x,$$

so $F(x) = \sin x$ is an antiderivative of $\cos x$. Therefore, every antiderivative of $\cos x$ is of the form $\sin x + C$ for some constant C and every function of the form $\sin x + C$ is an antiderivative of $\cos x$.

d. Since

Equation:

$$\frac{d}{dx}(e^x) = e^x,$$

then $F(x) = e^x$ is an antiderivative of e^x . Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $e^x + C$ is an antiderivative of e^x .

Note:

Exercise:

Problem: Find all antiderivatives of $f(x) = \sin x$.

Solution:

$$-\cos x + C$$

Hint

What function has a derivative of $\sin x$?

Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. These properties allow us to find antiderivatives of more complicated functions. Given a function f , we use the notation $f'(x)$ or $\frac{df}{dx}$ to denote the derivative of f . Here we introduce notation for antiderivatives. If F is an antiderivative of f , we say that $F(x) + C$ is the most general antiderivative of f and write

Equation:

$$\int f(x)dx = F(x) + C.$$

The symbol \int is called an *integral sign*, and $\int f(x)dx$ is called the indefinite integral of f .

Note:

Definition

Given a function f , the **indefinite integral** of f , denoted

Equation:

$$\int f(x)dx,$$

is the most general antiderivative of f . If F is an antiderivative of f , then

Equation:

$$\int f(x)dx = F(x) + C.$$

The expression $f(x)$ is called the *integrand* and the variable x is the *variable of integration*.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as *integrating f* .

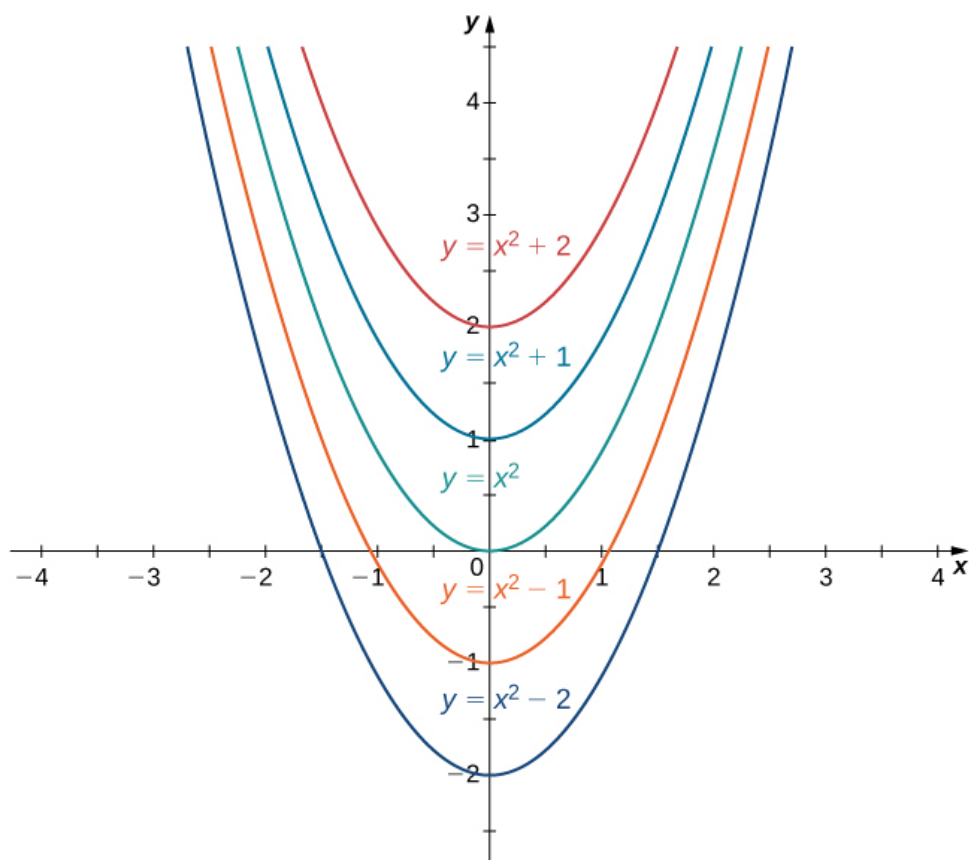
For a function f and an antiderivative F , the functions $F(x) + C$, where C is any real number, is often referred to as *the family of antiderivatives of f* . For example, since x^2 is an antiderivative of $2x$

and any antiderivative of $2x$ is of the form $x^2 + C$, we write

Equation:

$$\int 2x \, dx = x^2 + C.$$

The collection of all functions of the form $x^2 + C$, where C is any real number, is known as the *family of antiderivatives of $2x$* . [\[link\]](#) shows a graph of this family of antiderivatives.



The family of antiderivatives of $2x$ consists of all functions of the form $x^2 + C$, where C is any real number.

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1$,

Equation:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

Equation:

$$\frac{d}{dx} \left(\frac{x^{n+1}}{n+1} \right) = (n+1) \frac{x^n}{n+1} = x^n.$$

This fact is known as *the power rule for integrals*.

Note:

Power Rule for Integrals

For $n \neq -1$,

Equation:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions. A more complete list appears in [Appendix B](#).

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(k) = 0$	$\int k dx = \int kx^0 dx = kx + C$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln x + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x dx = -\cos x + C$

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + C$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C$

Integration Formulas

From the definition of indefinite integral of f , we know

Equation:

$$\int f(x) \, dx = F(x) + C$$

if and only if F is an antiderivative of f . Therefore, when claiming that

Equation:

$$\int f(x) \, dx = F(x) + C$$

it is important to check whether this statement is correct by verifying that $F'(x) = f(x)$.

Example:

Exercise:

Problem:

Verifying an Indefinite Integral

Each of the following statements is of the form $\int f(x)dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$.

a. $\int (x + e^x)dx = \frac{x^2}{2} + e^x + C$

b. $\int xe^x dx = xe^x - e^x + C$

Solution:

a. Since

Equation:

$$\frac{d}{dx} \left(\frac{x^2}{2} + e^x + C \right) = x + e^x,$$

the statement

Equation:

$$\int (x + e^x)dx = \frac{x^2}{2} + e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a sum. Furthermore, $\frac{x^2}{2}$ and e^x are antiderivatives of x and e^x , respectively, and the sum of the antiderivatives is an antiderivative of the sum. We discuss this fact again later in this section.

b. Using the product rule, we see that

Equation:

$$\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x.$$

Therefore, the statement

Equation:

$$\int xe^x dx = xe^x - e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a product. The antiderivative $xe^x - e^x$ is not a product of the antiderivatives. Furthermore, the product of antiderivatives, $x^2e^x/2$ is not an antiderivative of xe^x since

Equation:

$$\frac{d}{dx} \left(\frac{x^2 e^x}{2} \right) = x e^x + \frac{x^2 e^x}{2} \neq x e^x.$$

In general, the product of antiderivatives is not an antiderivative of a product.

Note:

Exercise:

Problem: Verify that $\int x \cos x \, dx = x \sin x + \cos x + C$.

Solution:

$$\frac{d}{dx} (x \sin x + \cos x + C) = \sin x + x \cos x - \sin x = x \cos x$$

Hint

Calculate $\frac{d}{dx} (x \sin x + \cos x + C)$.

In [\[link\]](#), we listed the indefinite integrals for many elementary functions. Let's now turn our attention to evaluating indefinite integrals for more complicated functions. For example, consider finding an antiderivative of a sum $f + g$. In [\[link\]](#)a, we showed that an antiderivative of the sum $x + e^x$ is given by the sum $\left(\frac{x^2}{2}\right) + e^x$ —that is, an antiderivative of a sum is given by a sum of antiderivatives. This result was not specific to this example. In general, if F and G are antiderivatives of any functions f and g , respectively, then

Equation:

$$\frac{d}{dx} (F(x) + G(x)) = F'(x) + G'(x) = f(x) + g(x).$$

Therefore, $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$ and we have

Equation:

$$\int (f(x) + g(x)) dx = F(x) + G(x) + C.$$

Similarly,

Equation:

$$\int (f(x) - g(x)) dx = F(x) - G(x) + C.$$

In addition, consider the task of finding an antiderivative of $kf(x)$, where k is any real number. Since
Equation:

$$\frac{d}{dx}(kf(x)) = k \frac{d}{dx}F(x) = kF'(x)$$

for any real number k , we conclude that

Equation:

$$\int kf(x)dx = kF(x) + C.$$

These properties are summarized next.

Note:

Properties of Indefinite Integrals

Let F and G be antiderivatives of f and g , respectively, and let k be any real number.

Sums and Differences

Equation:

$$\int (f(x) \pm g(x))dx = F(x) \pm G(x) + C$$

Constant Multiples

Equation:

$$\int kf(x)dx = kF(x) + C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with antiderivatives that are known. Evaluating integrals involving products, quotients, or compositions is more complicated (see [\[link\]](#) for an example involving an antiderivative of a product.) We look at and address integrals involving these more complicated functions in [Introduction to Integration](#). In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example:

Exercise:

Problem:

Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

- a. $\int (5x^3 - 7x^2 + 3x + 4) dx$
- b. $\int \frac{x^2 + 4\sqrt[3]{x}}{x} dx$
- c. $\int \frac{4}{1+x^2} dx$
- d. $\int \tan x \cos x dx$

Solution:

- a. Using [\[link\]](#), we can integrate each of the four terms in the integrand separately. We obtain
Equation:

$$\int (5x^3 - 7x^2 + 3x + 4) dx = \int 5x^3 dx - \int 7x^2 dx + \int 3x dx + \int 4 dx.$$

From the second part of [\[link\]](#), each coefficient can be written in front of the integral sign, which gives

Equation:

$$\int 5x^3 dx - \int 7x^2 dx + \int 3x dx + \int 4 dx = 5 \int x^3 dx - 7 \int x^2 dx + 3 \int x dx + 4 \int 1 dx.$$

Using the power rule for integrals, we conclude that

Equation:

$$\int (5x^3 - 7x^2 + 3x + 4) dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

- b. Rewrite the integrand as
Equation:

$$\frac{x^2 + 4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x} = 0.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the power rule, we have

Equation:

$$\begin{aligned} \int \left(x + \frac{4}{x^{2/3}} \right) dx &= \int x dx + 4 \int x^{-2/3} dx \\ &= \frac{1}{2}x^2 + 4 \frac{1}{(-\frac{2}{3})+1} x^{(-2/3)+1} + C \\ &= \frac{1}{2}x^2 + 12x^{1/3} + C. \end{aligned}$$

- c. Using [\[link\]](#), write the integral as

Equation:

$$4 \int \frac{1}{1+x^2} dx.$$

Then, use the fact that $\tan^{-1}(x)$ is an antiderivative of $\frac{1}{(1+x^2)}$ to conclude that

Equation:

$$\int \frac{4}{1+x^2} dx = 4 \tan^{-1}(x) + C.$$

d. Rewrite the integrand as

Equation:

$$\tan x \cos x = \frac{\sin x}{\cos x} \cos x = \sin x.$$

Therefore,

Equation:

$$\int \tan x \cos x = \int \sin x = -\cos x + C.$$

Note:

Exercise:

Problem: Evaluate $\int (4x^3 - 5x^2 + x - 7) dx$.

Solution:

$$x^4 - \frac{5}{3}x^3 + \frac{1}{2}x^2 - 7x + C$$

Hint

Integrate each term in the integrand separately, making use of the power rule.

Initial-Value Problems

We look at techniques for integrating a large variety of functions involving products, quotients, and compositions later in the text. Here we turn to one common use for antiderivatives that arises often in many applications: solving differential equations.

A *differential equation* is an equation that relates an unknown function and one or more of its derivatives. The equation

Equation:

$$\frac{dy}{dx} = f(x)$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f . Therefore, the solutions of [\[link\]](#) are the antiderivatives of f . If F is one antiderivative of f , every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

Equation:

$$\frac{dy}{dx} = 6x^2$$

are given by

Equation:

$$y = \int 6x^2 dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

Equation:

$$\frac{dy}{dx} = f(x)$$

with the additional condition

Equation:

$$y(x_0) = y_0$$

is an example of an **initial-value problem**. The condition $y(x_0) = y_0$ is known as an *initial condition*. For example, looking for a function y that satisfies the differential equation

Equation:

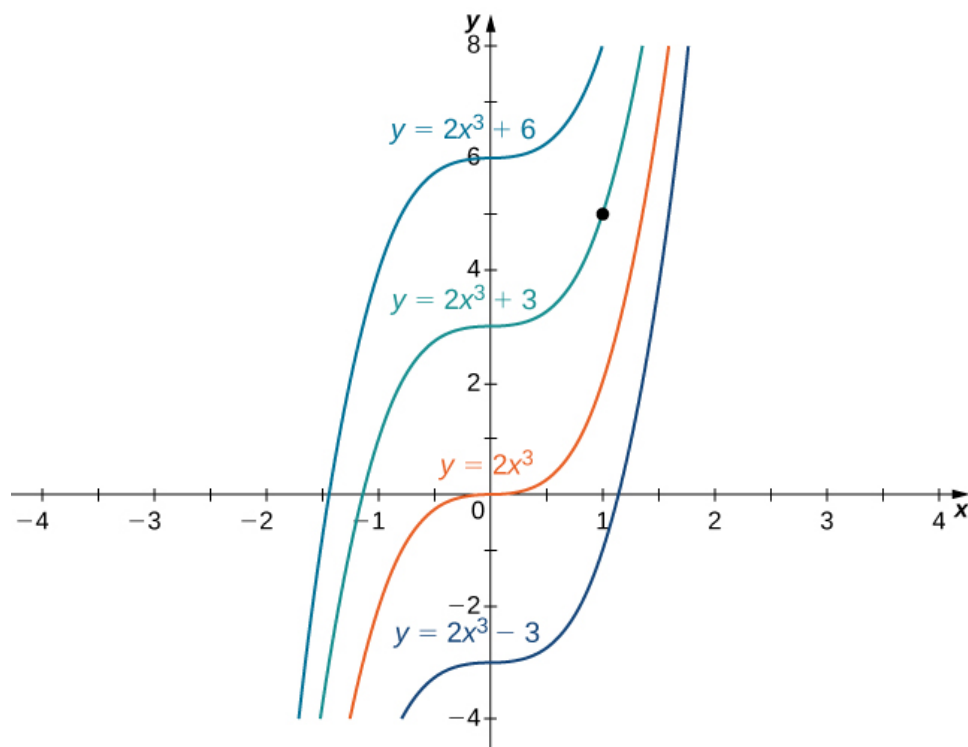
$$\frac{dy}{dx} = 6x^2$$

and the initial condition

Equation:

$$y(1) = 5$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C$, to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5$. From this equation, we see that $C = 3$, and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in the following graph.



Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5$.

Example:

Exercise:

Problem:

Solving an Initial-Value Problem

Solve the initial-value problem

Equation:

$$\frac{dy}{dx} = \sin x, y(0) = 5.$$

Solution:

First we need to solve the differential equation. If $\frac{dy}{dx} = \sin x$, then

Equation:

$$y = \int \sin(x) dx = -\cos x + C.$$

Next we need to look for a solution y that satisfies the initial condition. The initial condition $y(0) = 5$ means we need a constant C such that $-\cos x + C = 5$. Therefore,

Equation:

$$C = 5 + \cos(0) = 6.$$

The solution of the initial-value problem is $y = -\cos x + 6$.

Note:**Exercise:**

Problem: Solve the initial value problem $\frac{dy}{dx} = 3x^{-2}$, $y(1) = 2$.

Solution:

$$y = -\frac{3}{x} + 5$$

Hint

Find all antiderivatives of $f(x) = 3x^{-2}$.

Initial-value problems arise in many applications. Next we consider a problem in which a driver applies the brakes in a car. We are interested in how long it takes for the car to stop. Recall that the velocity function $v(t)$ is the derivative of a position function $s(t)$, and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the next example we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example:**Exercise:****Problem:****Decelerating Car**

A car is traveling at the rate of 88 ft/sec (60 mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/sec².

- a. How many seconds elapse before the car stops?
- b. How far does the car travel during that time?

Solution:

- a. First we introduce variables for this problem. Let t be the time (in seconds) after the brakes are first applied. Let $a(t)$ be the acceleration of the car (in feet per seconds squared) at time t . Let $v(t)$ be the velocity of the car (in feet per second) at time t . Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t .

The car is traveling at a rate of 88 ft/sec. Therefore, the initial velocity is $v(0) = 88$ ft/sec. Since the car is decelerating, the acceleration is

Equation:

$$a(t) = -15 \text{ ft/s}^2.$$

The acceleration is the derivative of the velocity,

Equation:

$$v'(t) = -15.$$

Therefore, we have an initial-value problem to solve:

Equation:

$$v'(t) = -15, v(0) = 88.$$

Integrating, we find that

Equation:

$$v(t) = -15t + C.$$

Since $v(0) = 88$, $C = 88$. Thus, the velocity function is

Equation:

$$v(t) = -15t + 88.$$

To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero. Solving $-15t + 88 = 0$, we obtain $t = \frac{88}{15}$ sec.

- b. To find how far the car travels during this time, we need to find the position of the car after $\frac{88}{15}$ sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position to be $s(0) = 0$. Therefore, we need to solve the initial-value problem

Equation:

$$s'(t) = -15t + 88, s(0) = 0.$$

Integrating, we have

Equation:

$$s(t) = -\frac{15}{2}t^2 + 88t + C.$$

Since $s(0) = 0$, the constant is $C = 0$. Therefore, the position function is

Equation:

$$s(t) = -\frac{15}{2}t^2 + 88t.$$

After $t = \frac{88}{15}$ sec, the position is $s\left(\frac{88}{15}\right) \approx 258.133$ ft.

Note:

Exercise:

Problem:

Suppose the car is traveling at the rate of 44 ft/sec. How long does it take for the car to stop?
How far will the car travel?

Solution:

2.93 sec, 64.5 ft

Hint

$$v(t) = -15t + 44.$$

Key Concepts

- If F is an antiderivative of f , then every antiderivative of f is of the form $F(x) + C$ for some constant C .
- Solving the initial-value problem

Equation:

$$\frac{dy}{dx} = f(x), y(x_0) = y_0$$

requires us first to find the set of antiderivatives of f and then to look for the particular

antiderivative that also satisfies the initial condition.

For the following exercises, show that $F(x)$ are antiderivatives of $f(x)$.

Exercise:

Problem: $F(x) = 5x^3 + 2x^2 + 3x + 1, f(x) = 15x^2 + 4x + 3$

Solution:

$$F'(x) = 15x^2 + 4x + 3$$

Exercise:

Problem: $F(x) = x^2 + 4x + 1, f(x) = 2x + 4$

Exercise:

Problem: $F(x) = x^2 e^x, f(x) = e^x (x^2 + 2x)$

Solution:

$$F'(x) = 2xe^x + x^2 e^x$$

Exercise:

Problem: $F(x) = \cos x, f(x) = -\sin x$

Exercise:

Problem: $F(x) = e^x, f(x) = e^x$

Solution:

$$F'(x) = e^x$$

For the following exercises, find the antiderivative of the function.

Exercise:

Problem: $f(x) = \frac{1}{x^2} + x$

Exercise:

Problem: $f(x) = e^x - 3x^2 + \sin x$

Solution:

$$F(x) = e^x - x^3 - \cos(x) + C$$

Exercise:

Problem: $f(x) = e^x + 3x - x^2$

Exercise:

Problem: $f(x) = x - 1 + 4\sin(2x)$

Solution:

$$F(x) = \frac{x^2}{2} - x - 2\cos(2x) + C$$

For the following exercises, find the antiderivative $F(x)$ of each function $f(x)$.

Exercise:

Problem: $f(x) = 5x^4 + 4x^5$

Exercise:

Problem: $f(x) = x + 12x^2$

Solution:

$$F(x) = \frac{1}{2}x^2 + 4x^3 + C$$

Exercise:

Problem: $f(x) = \frac{1}{\sqrt{x}}$

Exercise:

Problem: $f(x) = (\sqrt{x})^3$

Solution:

$$F(x) = \frac{2}{5}(\sqrt{x})^5 + C$$

Exercise:

Problem: $f(x) = x^{1/3} + (2x)^{1/3}$

Exercise:

Problem: $f(x) = \frac{x^{1/3}}{x^{2/3}}$

Solution:

$$F(x) = \frac{3}{2}x^{2/3} + C$$

Exercise:

Problem: $f(x) = 2\sin(x) + \sin(2x)$

Exercise:

Problem: $f(x) = \sec^2(x) + 1$

Solution:

$$F(x) = x + \tan(x) + C$$

Exercise:

Problem: $f(x) = \sin x \cos x$

Exercise:

Problem: $f(x) = \sin^2(x) \cos(x)$

Solution:

$$F(x) = \frac{1}{3} \sin^3(x) + C$$

Exercise:

Problem: $f(x) = 0$

Exercise:

Problem: $f(x) = \frac{1}{2} \csc^2(x) + \frac{1}{x^2}$

Solution:

$$F(x) = -\frac{1}{2} \cot(x) - \frac{1}{x} + C$$

Exercise:

Problem: $f(x) = \csc x \cot x + 3x$

Exercise:

Problem: $f(x) = 4 \csc x \cot x - \sec x \tan x$

Solution:

$$F(x) = -\sec x - 4 \csc x + C$$

Exercise:

Problem: $f(x) = 8 \sec x (\sec x - 4 \tan x)$

Exercise:

Problem: $f(x) = \frac{1}{2}e^{-4x} + \sin x$

Solution:

$$F(x) = -\frac{1}{8}e^{-4x} - \cos x + C$$

For the following exercises, evaluate the integral.

Exercise:

Problem: $\int (-1)dx$

Exercise:

Problem: $\int \sin x \, dx$

Solution:

$$-\cos x + C$$

Exercise:

Problem: $\int (4x + \sqrt{x}) \, dx$

Exercise:

Problem: $\int \frac{3x^2 + 2}{x^2} \, dx$

Solution:

$$3x - \frac{2}{x} + C$$

Exercise:

Problem: $\int (\sec x \tan x + 4x) \, dx$

Exercise:

Problem: $\int (4\sqrt{x} + \sqrt[4]{x}) \, dx$

Solution:

$$\frac{8}{3}x^{3/2} + \frac{4}{5}x^{5/4} + C$$

Exercise:

Problem: $\int (x^{-1/3} - x^{2/3}) dx$

Exercise:

Problem: $\int \frac{14x^3 + 2x + 1}{x^3} dx$

Solution:

$$14x - \frac{2}{x} - \frac{1}{2x^2} + C$$

Exercise:

Problem: $\int (e^x + e^{-x}) dx$

For the following exercises, solve the initial value problem.

Exercise:

Problem: $f'(x) = x^{-3}, f(1) = 1$

Solution:

$$f(x) = -\frac{1}{2x^2} + \frac{3}{2}$$

Exercise:

Problem: $f'(x) = \sqrt{x} + x^2, f(0) = 2$

Exercise:

Problem: $f'(x) = \cos x + \sec^2(x), f\left(\frac{\pi}{4}\right) = 2 + \frac{\sqrt{2}}{2}$

Solution:

$$f(x) = \sin x + \tan x + 1$$

Exercise:

Problem: $f'(x) = x^3 - 8x^2 + 16x + 1, f(0) = 0$

Exercise:

Problem: $f'(x) = \frac{2}{x^2} - \frac{x^2}{2}, f(1) = 0$

Solution:

$$f(x) = -\frac{1}{6}x^3 - \frac{2}{x} + \frac{13}{6}$$

For the following exercises, find two possible functions f given the second- or third-order derivatives.

Exercise:

Problem: $f''(x) = x^2 + 2$

Exercise:

Problem: $f''(x) = e^{-x}$

Solution:

Answers may vary; one possible answer is $f(x) = e^{-x}$

Exercise:

Problem: $f''(x) = 1 + x$

Exercise:

Problem: $f'''(x) = \cos x$

Solution:

Answers may vary; one possible answer is $f(x) = -\sin x$

Exercise:

Problem: $f'''(x) = 8e^{-2x} - \sin x$

Exercise:

Problem:

A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerates at a constant rate of 10 ft/sec^2 . How long before the car stops?

Solution:

5.867 sec

Exercise:

Problem: In the preceding problem, calculate how far the car travels in the time it takes to stop.

Exercise:

Problem:

You are merging onto the freeway, accelerating at a constant rate of 12 ft/sec^2 . How long does it take you to reach merging speed at 60 mph?

Solution:

7.333 sec

Exercise:

Problem: Based on the previous problem, how far does the car travel to reach merging speed?

Exercise:**Problem:**

A car company wants to ensure its newest model can stop in 8 sec when traveling at 75 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

Solution:

13.75 ft/sec^2

Exercise:**Problem:**

A car company wants to ensure its newest model can stop in less than 450 ft when traveling at 60 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

For the following exercises, find the antiderivative of the function, assuming $F(0) = 0$.

Exercise:

Problem: [T] $f(x) = x^2 + 2$

Solution:

$$F(x) = \frac{1}{3}x^3 + 2x$$

Exercise:

Problem: [T] $f(x) = 4x - \sqrt{x}$

Exercise:

Problem: [T] $f(x) = \sin x + 2x$

Solution:

$$F(x) = x^2 - \cos x + 1$$

Exercise:

Problem: [T] $f(x) = e^x$

Exercise:

Problem: [T] $f(x) = \frac{1}{(x+1)^2}$

Solution:

$$F(x) = -\frac{1}{(x+1)} + 1$$

Exercise:

Problem: [T] $f(x) = e^{-2x} + 3x^2$

For the following exercises, determine whether the statement is true or false. Either prove it is true or find a counterexample if it is false.

Exercise:

Problem: If $f(x)$ is the antiderivative of $v(x)$, then $2f(x)$ is the antiderivative of $2v(x)$.

Solution:

True

Exercise:

Problem: If $f(x)$ is the antiderivative of $v(x)$, then $f(2x)$ is the antiderivative of $v(2x)$.

Exercise:

Problem: If $f(x)$ is the antiderivative of $v(x)$, then $f(x) + 1$ is the antiderivative of $v(x) + 1$.

Solution:

False

Exercise:

Problem: If $f(x)$ is the antiderivative of $v(x)$, then $(f(x))^2$ is the antiderivative of $(v(x))^2$.

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample. Assume that $f(x)$ is continuous and differentiable unless stated otherwise.

Exercise:

Problem:

If $f(-1) = -6$ and $f(1) = 2$, then there exists at least one point $x \in [-1, 1]$ such that $f'(x) = 4$.

Solution:

True, by Mean Value Theorem

Exercise:

Problem: If $f'(c) = 0$, there is a maximum or minimum at $x = c$.

Exercise:**Problem:**

There is a function such that $f(x) < 0$, $f'(x) > 0$, and $f''(x) < 0$. (A graphical “proof” is acceptable for this answer.)

Solution:

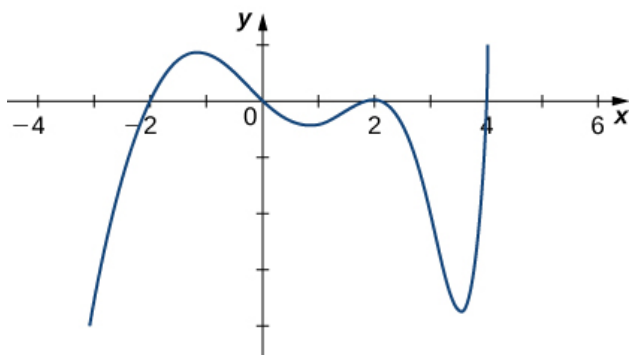
True

Exercise:**Problem:**

There is a function such that there is both an inflection point and a critical point for some value $x = a$.

Exercise:

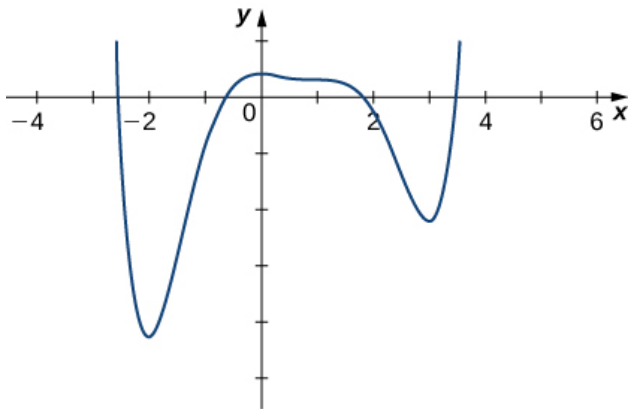
Problem: Given the graph of f' , determine where f is increasing or decreasing.

**Solution:**

Increasing: $(-2, 0) \cup (4, \infty)$, decreasing: $(-\infty, -2) \cup (0, 4)$

Exercise:

Problem: The graph of f is given below. Draw f' .



Exercise:

Problem: Find the linear approximation $L(x)$ to $y = x^2 + \tan(\pi x)$ near $x = \frac{1}{4}$.

Solution:

$$L(x) = \frac{17}{16} + \frac{1}{2}(1 + 4\pi)\left(x - \frac{1}{4}\right)$$

Exercise:

Problem: Find the differential of $y = x^2 - 5x - 6$ and evaluate for $x = 2$ with $dx = 0.1$.

Find the critical points and the local and absolute extrema of the following functions on the given interval.

Exercise:

Problem: $f(x) = x + \sin^2(x)$ over $[0, \pi]$

Solution:

Critical point: $x = \frac{3\pi}{4}$, absolute minimum: $x = 0$, absolute maximum: $x = \pi$

Exercise:

Problem: $f(x) = 3x^4 - 4x^3 - 12x^2 + 6$ over $[-3, 3]$

Determine over which intervals the following functions are increasing, decreasing, concave up, and concave down.

Exercise:

Problem: $x(t) = 3t^4 - 8t^3 - 18t^2$

Solution:

Increasing: $(-1, 0) \cup (3, \infty)$, decreasing: $(-\infty, -1) \cup (0, 3)$, concave up:
 $(-\infty, \frac{1}{3}(2 - \sqrt{13})) \cup (\frac{1}{3}(2 + \sqrt{13}), \infty)$, concave down:
 $(\frac{1}{3}(2 - \sqrt{13}), \frac{1}{3}(2 + \sqrt{13}))$

Exercise:

Problem: $y = x + \sin(\pi x)$

Exercise:

Problem: $g(x) = x - \sqrt{x}$

Solution:

Increasing: $(\frac{1}{4}, \infty)$, decreasing: $(0, \frac{1}{4})$, concave up: $(0, \infty)$, concave down: nowhere

Exercise:

Problem: $f(\theta) = \sin(3\theta)$

Evaluate the following limits.

Exercise:

Problem: $\lim_{x \rightarrow \infty} \frac{3x\sqrt{x^2+1}}{\sqrt{x^4-1}}$

Solution:

3

Exercise:

Problem: $\lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right)$

Exercise:

Problem: $\lim_{x \rightarrow 1} \frac{x-1}{\sin(\pi x)}$

Solution:

$-\frac{1}{\pi}$

Exercise:

Problem: $\lim_{x \rightarrow \infty} (3x)^{1/x}$

Use Newton's method to find the first two iterations, given the starting point.

Exercise:

Problem: $y = x^3 + 1, x_0 = 0.5$

Solution:

$$x_1 = -1, x_2 = -1$$

Exercise:

Problem: $\frac{1}{x+1} = \frac{1}{2}, x_0 = 0$

Find the antiderivatives $F(x)$ of the following functions.

Exercise:

Problem: $g(x) = \sqrt{x} - \frac{1}{x^2}$

Solution:

$$F(x) = \frac{2x^{3/2}}{3} + \frac{1}{x} + C$$

Exercise:

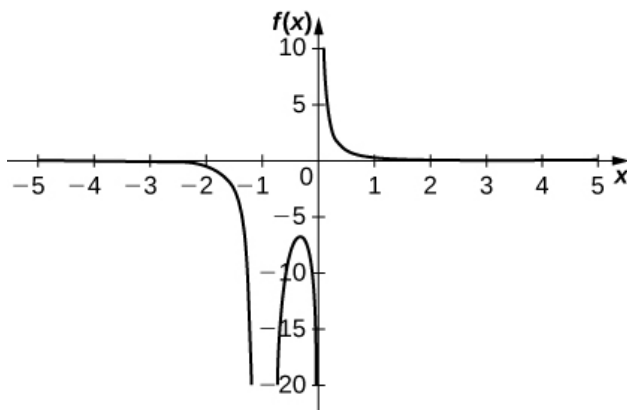
Problem: $f(x) = 2x + 6\cos x, F(\pi) = \pi^2 + 2$

Graph the following functions by hand. Make sure to label the inflection points, critical points, zeros, and asymptotes.

Exercise:

Problem: $y = \frac{1}{x(x+1)^2}$

Solution:



Inflection points: none; critical points: $x = -\frac{1}{3}$; zeros: none; vertical asymptotes: $x = -1$, $x = 0$; horizontal asymptote: $y = 0$

Exercise:

Problem: $y = x - \sqrt{4 - x^2}$

Exercise:

Problem:

A car is being compacted into a rectangular solid. The volume is decreasing at a rate of $2 \text{ m}^3/\text{sec}$. The length and width of the compactor are square, but the height is not the same length as the length and width. If the length and width walls move toward each other at a rate of 0.25 m/sec , find the rate at which the height is changing when the length and width are 2 m and the height is 1.5 m .

Solution:

The height is decreasing at a rate of 0.125 m/sec

Exercise:

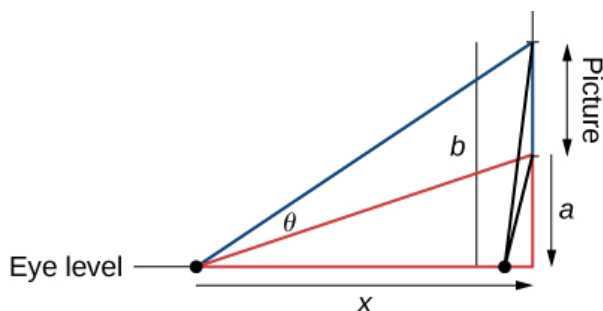
Problem:

A rocket is launched into space; its kinetic energy is given by $K(t) = \left(\frac{1}{2}\right)m(t)v(t)^2$, where K is the kinetic energy in joules, m is the mass of the rocket in kilograms, and v is the velocity of the rocket in meters/second. Assume the velocity is increasing at a rate of 15 m/sec^2 and the mass is decreasing at a rate of 10 kg/sec because the fuel is being burned. At what rate is the rocket's kinetic energy changing when the mass is 2000 kg and the velocity is 5000 m/sec ? Give your answer in mega-Joules (MJ), which is equivalent to 10^6 J .

Exercise:

Problem:

The famous Regiomontanus' problem for angle maximization was proposed during the 15th century. A painting hangs on a wall with the bottom of the painting a distance a feet above eye level, and the top b feet above eye level. What distance x (in feet) from the wall should the viewer stand to maximize the angle subtended by the painting, θ ?



Solution:

$$x = \sqrt{ab} \text{ feet}$$

Exercise:**Problem:**

An airline sells tickets from Tokyo to Detroit for \$1200. There are 500 seats available and a typical flight books 350 seats. For every \$10 decrease in price, the airline observes an additional five seats sold. What should the fare be to maximize profit? How many passengers would be onboard?

Glossary

antiderivative

a function F such that $F'(x) = f(x)$ for all x in the domain of f is an antiderivative of f

indefinite integral

the most general antiderivative of $f(x)$ is the indefinite integral of f ; we use the notation

$\int f(x)dx$ to denote the indefinite integral of f

initial value problem

a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx} = f(x)$ together with the initial condition $y(x_0) = y_0$

Introduction

class="introduction"

Iceboating
is a popular
winter sport
in parts of
the northern

United
States and
Europe.

(credit:
modification
n of work
by Carter
Brown,
Flickr)



Iceboats are a common sight on the lakes of Wisconsin and Minnesota on winter weekends. Iceboats are similar to sailboats, but they are fitted with runners, or “skates,” and are designed to run over the ice, rather than on water. Iceboats can move very quickly, and many ice boating enthusiasts are drawn to the sport because of the speed. Top iceboat racers can attain speeds up to five times the wind speed. If we know how fast an iceboat is moving, we can use integration to determine how far it travels. We revisit this question later in the chapter (see [\[link\]](#)).

Determining distance from velocity is just one of many applications of integration. In fact, integrals are used in a wide variety of mechanical and physical applications. In this chapter, we first introduce the theory behind integration and use integrals to calculate areas. From there, we develop the Fundamental Theorem of Calculus, which relates differentiation and

integration. We then study some basic integration techniques and briefly examine some applications.

Approximating Areas

- Use sigma (summation) notation to calculate sums and powers of integers.
- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area.

Archimedes was fascinated with calculating the areas of various shapes—in other words, the amount of space enclosed by the shape. He used a process that has come to be known as the *method of exhaustion*, which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area. In this process, an area bounded by curves is filled with rectangles, triangles, and shapes with exact area formulas. These areas are then summed to approximate the area of the curved region.

In this section, we develop techniques to approximate the area between a curve, defined by a function $f(x)$, and the x -axis on a closed interval $[a, b]$. Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

Let's start by introducing some notation to make the calculations easier. We then consider the case when $f(x)$ is continuous and nonnegative. Later in the chapter, we relax some of these restrictions and develop techniques that apply in more general cases.

Sigma (Summation) Notation

As mentioned, we will use shapes of known area to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter Σ , sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

Equation:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20.$$

We could probably skip writing a couple of terms and write

Equation:

$$1 + 2 + 3 + 4 + \cdots + 19 + 20,$$

which is better, but still cumbersome. With sigma notation, we write this sum as

Equation:

$$\sum_{i=1}^{20} i,$$

which is much more compact.

Typically, sigma notation is presented in the form

Equation:

$$\sum_{i=1}^n a_i$$

where a_i describes the terms to be added, and the i is called the *index*. Each term is evaluated, then we sum all the values, beginning with the value when $i = 1$ and ending with the value when $i = n$. For example, an expression like $\sum_{i=2}^7 s_i$ is interpreted as $s_2 + s_3 + s_4 + s_5 + s_6 + s_7$. Note that the index is used only to

keep track of the terms to be added; it does not factor into the calculation of the sum itself. The index is therefore called a *dummy variable*. We can use any letter we like for the index. Typically, mathematicians use i, j, k, m , and n for indices.

Let's try a couple of examples of using sigma notation.

Example:

Exercise:

Problem:

Using Sigma Notation

- Write in sigma notation and evaluate the sum of terms 3^i for $i = 1, 2, 3, 4, 5$.
- Write the sum in sigma notation:

Equation:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}.$$

Solution:

- Write

Equation:

$$\begin{aligned} \sum_{i=1}^5 3^i &= 3 + 3^2 + 3^3 + 3^4 + 3^5 \\ &= 363. \end{aligned}$$

- The denominator of each term is a perfect square. Using sigma notation, this sum can be written

as $\sum_{i=1}^5 \frac{1}{i^2}.$

Note:

Exercise:

Problem: Write in sigma notation and evaluate the sum of terms 2^i for $i = 3, 4, 5, 6$.

Solution:

$$\sum_{i=3}^6 2^i = 2^3 + 2^4 + 2^5 + 2^6 = 120$$

Hint

Use the solving steps in [\[link\]](#) as a guide.

The properties associated with the summation process are given in the following rule.

Note:

Rule: Properties of Sigma Notation

Let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m , with $1 \leq m \leq n$.

1.

Equation:

$$\sum_{i=1}^n c = nc$$

2.

Equation:

$$\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$$

3.

Equation:

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

4.

Equation:

$$\sum_{i=1}^n (a_i - b_i) = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$$

5.

Equation:

$$\sum_{i=1}^n a_i = \sum_{i=1}^m a_i + \sum_{i=m+1}^n a_i$$

Proof

We prove properties 2. and 3. here, and leave proof of the other properties to the Exercises.

2. We have

Equation:

$$\begin{aligned}\sum_{i=1}^n ca_i &= ca_1 + ca_2 + ca_3 + \cdots + ca_n \\ &= c(a_1 + a_2 + a_3 + \cdots + a_n) \\ &= c \sum_{i=1}^n a_i.\end{aligned}$$

3. We have

Equation:

$$\begin{aligned}\sum_{i=1}^n (a_i + b_i) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n) \\ &= (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n) \\ &= \sum_{i=1}^n a_i + \sum_{i=1}^n b_i.\end{aligned}$$

□

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for **sums and powers of integers**, and we use them in the next set of examples.

Note:

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

Equation:

$$\sum_{i=1}^n i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

2. The sum of consecutive integers squared is given by

Equation:

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. The sum of consecutive integers cubed is given by

Equation:

$$\sum_{i=1}^n i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Example:

Exercise:

Problem:

Evaluation Using Sigma Notation

Write using sigma notation and evaluate:

- The sum of the terms $(i - 3)^2$ for $i = 1, 2, \dots, 200$.
- The sum of the terms $(i^3 - i^2)$ for $i = 1, 2, 3, 4, 5, 6$.

Solution:

- Multiplying out $(i - 3)^2$, we can break the expression into three terms.

Equation:

$$\begin{aligned} \sum_{i=1}^{200} (i - 3)^2 &= \sum_{i=1}^{200} (i^2 - 6i + 9) \\ &= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9 \\ &= \sum_{i=1}^{200} i^2 - 6 \sum_{i=1}^{200} i + \sum_{i=1}^{200} 9 \\ &= \frac{200(200+1)(400+1)}{6} - 6 \left[\frac{200(200+1)}{2} \right] + 9(200) \\ &= 2,686,700 - 120,600 + 1800 \\ &= 2,567,900 \end{aligned}$$

- Use sigma notation property iv. and the rules for the sum of squared terms and the sum of cubed terms.

Equation:

$$\begin{aligned} \sum_{i=1}^6 (i^3 - i^2) &= \sum_{i=1}^6 i^3 - \sum_{i=1}^6 i^2 \\ &= \frac{6^2(6+1)^2}{4} - \frac{6(6+1)(2(6)+1)}{6} \\ &= \frac{1764}{4} - \frac{546}{6} \\ &= 350 \end{aligned}$$

Note:

Exercise:

Problem: Find the sum of the values of $4 + 3i$ for $i = 1, 2, \dots, 100$.

Solution:

15,550

Hint

Use the properties of sigma notation to solve the problem.

Example:

Exercise:

Problem:

Finding the Sum of the Function Values

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \dots, 10$.

Solution:

Using the formula, we have

Equation:

$$\begin{aligned}\sum_{i=0}^{10} i^3 &= \frac{(10)^2(10+1)^2}{4} \\ &= \frac{100(121)}{4} \\ &= 3025.\end{aligned}$$

Note:

Exercise:

Problem: Evaluate the sum indicated by the notation $\sum_{k=1}^{20} (2k + 1)$.

Solution:

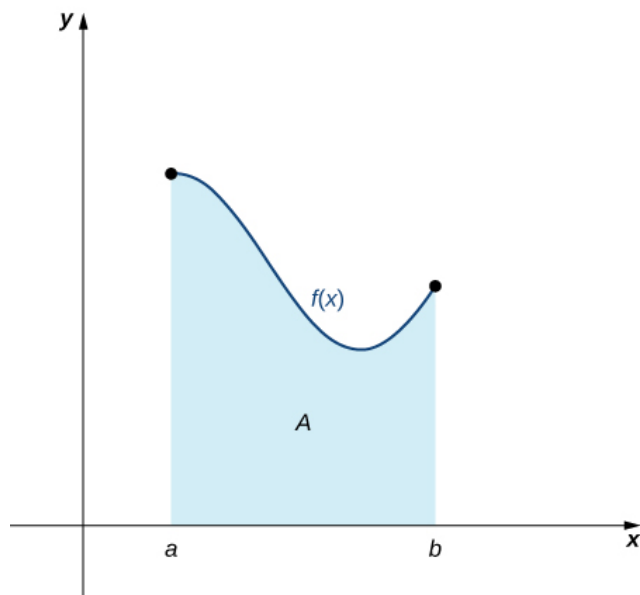
440

Hint

Use the rule on sum and powers of integers.

Approximating Area

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A bounded by $f(x)$ above, the x -axis below, the line $x = a$ on the left, and the line $x = b$ on the right ([link](#)).



An area (shaded region) bounded by the curve $f(x)$ at top, the x -axis at bottom, the line $x = a$ to the left, and the line $x = b$ at right.

How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have known area formulas, we can sum these areas and obtain a reasonable estimate of the true area. We begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\frac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \dots, x_n$ with $x_0 = a, x_n = b$, and

Equation:

$$x_i - x_{i-1} = \frac{b-a}{n}$$

for $i = 1, 2, 3, \dots, n$.

We denote the width of each subinterval with the notation Δx , so $\Delta x = \frac{b-a}{n}$ and

Equation:

$$x_i = x_0 + i\Delta x$$

for $i = 1, 2, 3, \dots, n$. This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let's define some relevant terminology.

Note:

Definition

A set of points $P = \{x_i\}$ for $i = 0, 1, 2, \dots, n$ with $a = x_0 < x_1 < x_2 < \dots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** of the interval $[a, b]$.

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

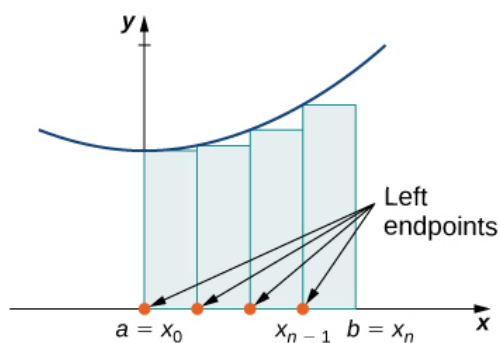
Note:

Rule: Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \dots, n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate value for A ([link](#)). We use the notation L_n to denote that this is a **left-endpoint approximation** of A using n subintervals.

Equation:

$$\begin{aligned} A \approx L_n &= f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x \\ &= \sum_{i=1}^n f(x_{i-1})\Delta x \end{aligned}$$



In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.

Note:

Rule: Right-Endpoint Approximation

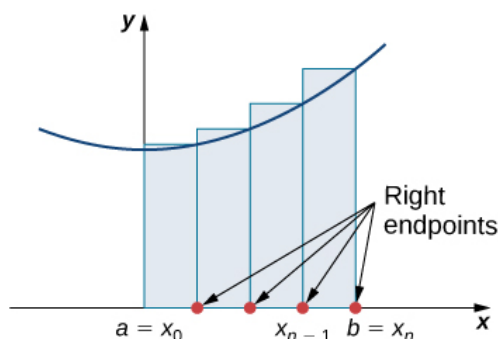
Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\Delta x$ and the approximation for A is given by

Equation:

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

$$= \sum_{i=1}^n f(x_i)\Delta x.$$

The notation R_n indicates this is a **right-endpoint approximation** for A ([\[link\]](#)).

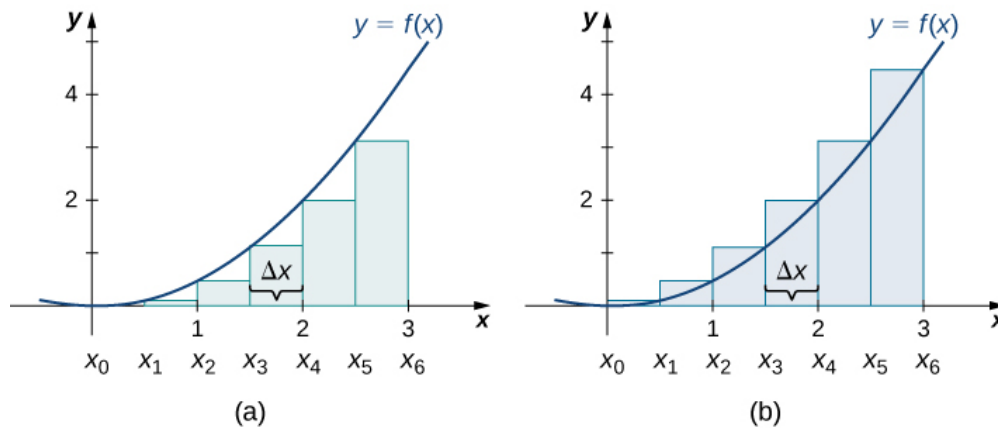


In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in [\[link\]](#).

The graphs in [\[link\]](#) represent the curve $f(x) = \frac{x^2}{2}$. In graph (a) we divide the region represented by the interval $[0, 3]$ into six subintervals, each of width 0.5. Thus, $\Delta x = 0.5$. We then form six rectangles by drawing vertical lines perpendicular to x_{i-1} , the left endpoint of each subinterval. We determine the height of each rectangle by calculating $f(x_{i-1})$ for $i = 1, 2, 3, 4, 5, 6$. The intervals are $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$, $[2, 2.5]$, $[2.5, 3]$. We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between $f(x)$ and the x -axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation. Thus,

Equation:

$$\begin{aligned}
 A \approx L_6 &= \sum_{i=1}^6 f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\
 &= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\
 &= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\
 &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\
 &= 3.4375.
 \end{aligned}$$



Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.

In [link](#)(b), we draw vertical lines perpendicular to x_i such that x_i is the right endpoint of each subinterval, and calculate $f(x_i)$ for $i = 1, 2, 3, 4, 5, 6$. We multiply each $f(x_i)$ by Δx to find the rectangular areas, and then add them. This is a right-endpoint approximation of the area under $f(x)$. Thus,

Equation:

$$\begin{aligned}
 A \approx R_6 &= \sum_{i=1}^6 f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\
 &= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\
 &= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\
 &= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\
 &= 5.6875.
 \end{aligned}$$

Example:

Exercise:

Problem:

Approximating the Area Under a Curve

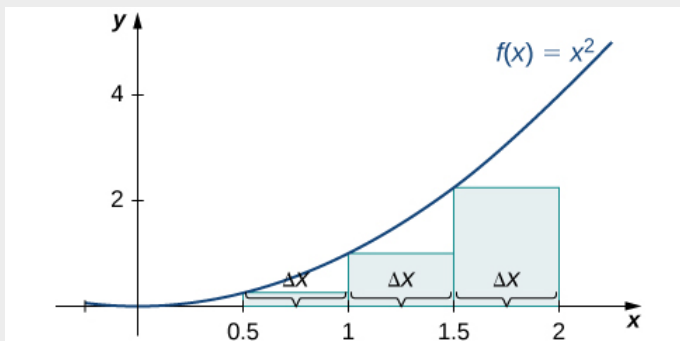
Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Solution:

First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \frac{(2-0)}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$ are shown in [\[link\]](#). Using a left-endpoint approximation, the heights are $f(0) = 0$, $f(0.5) = 0.25$, $f(1) = 1$, $f(1.5) = 2.25$. Then,

Equation:

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0(0.5) + 0.25(0.5) + 1(0.5) + 2.25(0.5) \\ &= 1.75. \end{aligned}$$

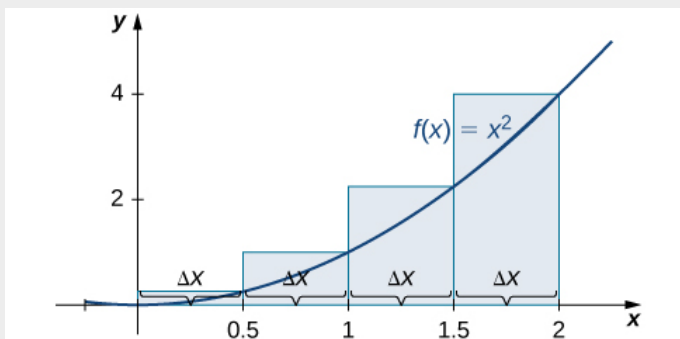


The graph shows the left-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The right-endpoint approximation is shown in [\[link\]](#). The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

Equation:

$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\ &= 3.75. \end{aligned}$$



The graph shows the right-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The left-endpoint approximation is 1.75; the right-endpoint approximation is 3.75.

Note:

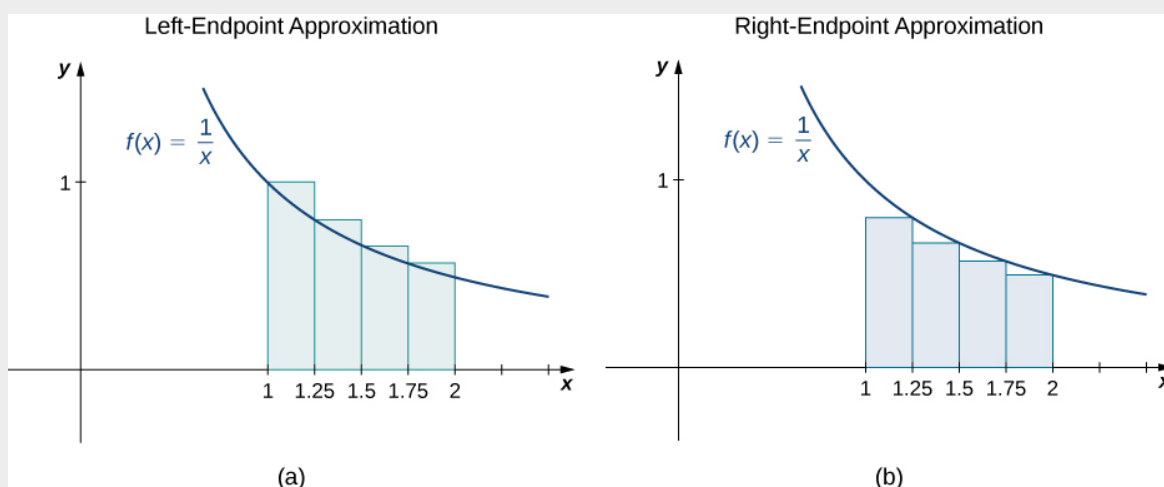
Exercise:

Problem:

Sketch left-endpoint and right-endpoint approximations for $f(x) = \frac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods.

Solution:

The left-endpoint approximation is 0.7595. The right-endpoint approximation is 0.6345. See the below [\[link\]](#).



Hint

Follow the solving strategy in [\[link\]](#) step-by-step.

Looking at [\[link\]](#) and the graphs in [\[link\]](#), we can see that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, it seems logical that if we increase the number of points in our partition, our estimate of A will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely.

We can demonstrate the improved approximation obtained through smaller intervals with an example. Let's explore the idea of increasing n , first in a left-endpoint approximation with four rectangles, then eight rectangles, and finally 32 rectangles. Then, let's do the same thing in a right-endpoint approximation, using the same sets of intervals, of the same curved region. [\[link\]](#) shows the area of the region under the curve

$f(x) = (x - 1)^3 + 4$ on the interval $[0, 2]$ using a left-endpoint approximation where $n = 4$. The width of each rectangle is

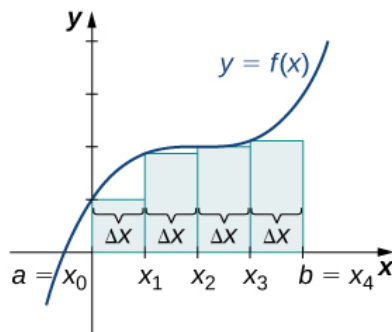
Equation:

$$\Delta x = \frac{2 - 0}{4} = \frac{1}{2}.$$

The area is approximated by the summed areas of the rectangles, or

Equation:

$$\begin{aligned} L_4 &= f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) \\ &= 7.5. \end{aligned}$$

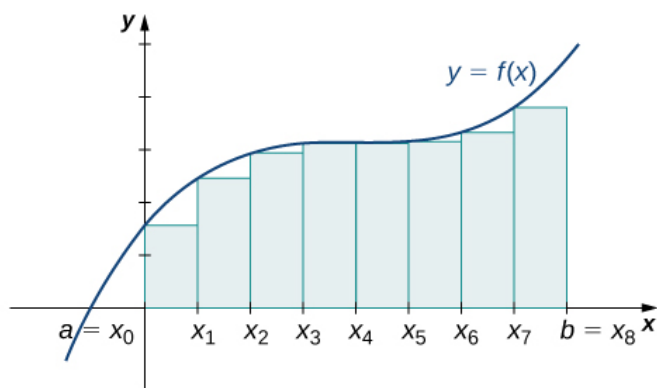


With a left-endpoint approximation and dividing the region from a to b into four equal intervals, the area under the curve is approximately equal to the sum of the areas of the rectangles.

[\[link\]](#) shows the same curve divided into eight subintervals. Comparing the graph with four rectangles in [\[link\]](#) with this graph with eight rectangles, we can see there appears to be less white space under the curve when $n = 8$. This white space is area under the curve we are unable to include using our approximation. The area of the rectangles is

Equation:

$$\begin{aligned} L_8 &= f(0)(0.25) + f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) \\ &\quad + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) \\ &= 7.75. \end{aligned}$$

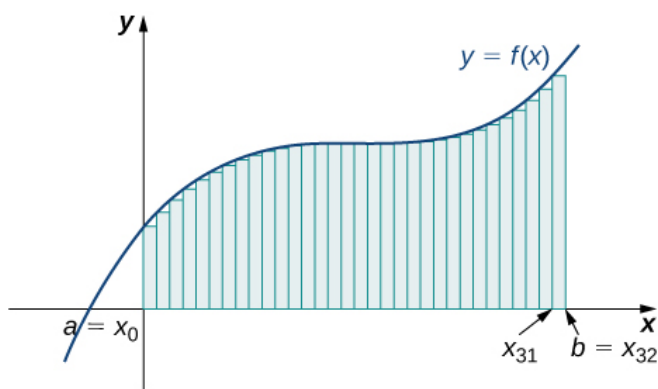


The region under the curve is divided into $n = 8$ rectangular areas of equal width for a left-endpoint approximation.

The graph in [\[link\]](#) shows the same function with 32 rectangles inscribed under the curve. There appears to be little white space left. The area occupied by the rectangles is

Equation:

$$\begin{aligned} L_{32} &= f(0)(0.0625) + f(0.0625)(0.0625) + f(0.125)(0.0625) + \cdots + f(1.9375)(0.0625) \\ &= 7.9375. \end{aligned}$$

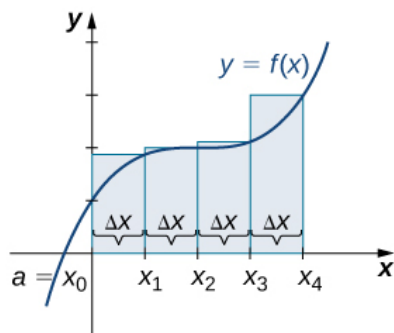


Here, 32 rectangles are inscribed under the curve for a left-endpoint approximation.

We can carry out a similar process for the right-endpoint approximation method. A right-endpoint approximation of the same curve, using four rectangles ([\[link\]](#)), yields an area

Equation:

$$\begin{aligned} R_4 &= f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) \\ &= 8.5. \end{aligned}$$

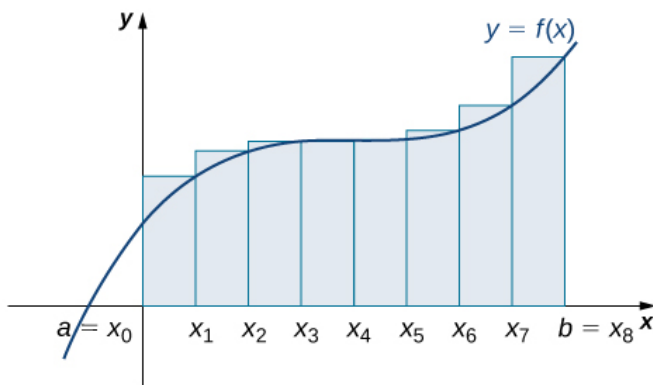


Now we divide the area under the curve into four equal subintervals for a right-endpoint approximation.

Dividing the region over the interval $[0, 2]$ into eight rectangles results in $\Delta x = \frac{2-0}{8} = 0.25$. The graph is shown in [\[link\]](#). The area is

Equation:

$$\begin{aligned} R_8 &= f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) \\ &\quad + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) + f(2)(0.25) \\ &= 8.25. \end{aligned}$$

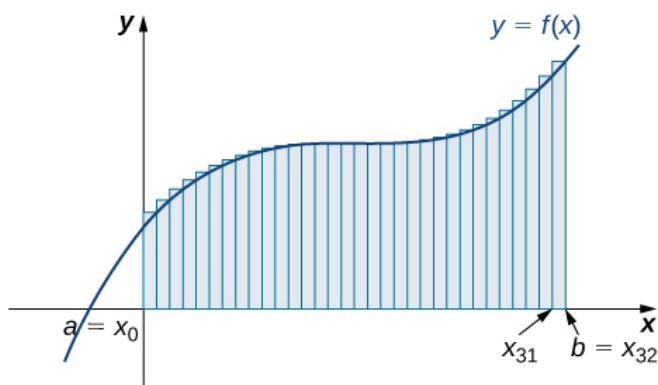


Here we use right-endpoint approximation for a region divided into eight equal subintervals.

Last, the right-endpoint approximation with $n = 32$ is close to the actual area ([\[link\]](#)). The area is approximately

Equation:

$$R_{32} = f(0.0625)(0.0625) + f(0.125)(0.0625) + f(0.1875)(0.0625) + \cdots + f(2)(0.0625) = 8.0625.$$



The region is divided into 32 equal subintervals for a right-endpoint approximation.

Based on these figures and calculations, it appears we are on the right track; the rectangles appear to approximate the area under the curve better as n gets larger. Furthermore, as n increases, both the left-endpoint and right-endpoint approximations appear to approach an area of 8 square units. [\[link\]](#) shows a numerical comparison of the left- and right-endpoint methods. The idea that the approximations of the area under the curve get better and better as n gets larger and larger is very important, and we now explore this idea in more detail.

Values of n	Approximate Area L_n	Approximate Area R_n
$n = 4$	7.5	8.5
$n = 8$	7.75	8.25
$n = 32$	7.94	8.06

Converging Values of Left- and Right-Endpoint Approximations as n Increases

Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point c_i in the subinterval $[x_{i-1}, x_i]$, and use $f(c_i)$ as the height of our rectangle. This gives us an estimate for the area of the form

Equation:

$$A \approx \sum_{i=1}^n f(x_i^*) \Delta x.$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Note:**Definition**

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be a regular partition of $[a, b]$. Let Δx be the width of each subinterval $[x_{i-1}, x_i]$ and for each i , let x_i^* be any point in $[x_{i-1}, x_i]$. A **Riemann sum** is defined for $f(x)$ as

Equation:

$$\sum_{i=1}^n f(x_i^*) \Delta x.$$

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n . We are now ready to define the area under a curve in terms of Riemann sums.

Note:**Definition**

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^n f(x_i^*) \Delta x$ be a Riemann sum for $f(x)$. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

Equation:

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

Note:

See a [graphical demonstration](#) of the construction of a Riemann sum.

Some subtleties here are worth discussing. First, note that taking the limit of a sum is a little different from taking the limit of a function $f(x)$ as x goes to infinity. Limits of sums are discussed in detail in the chapter on [Sequences and Series](#); however, for now we can assume that the computational techniques we used to compute limits of functions can also be used to calculate limits of sums.

Second, we must consider what to do if the expression converges to different limits for different choices of $\{x_i^*\}$. Fortunately, this does not happen. Although the proof is beyond the scope of this text, it can be shown that if $f(x)$ is continuous on the closed interval $[a, b]$, then $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$ exists and is unique (in other words, it does not depend on the choice of $\{x_i^*\}$).

We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for $\{x_i^*\}$. Although any choice for $\{x_i^*\}$ gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high (overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for $\{x_i^*\}$ to guarantee one result or the other.

If we want an overestimate, for example, we can choose $\{x_i^*\}$ such that for $i = 1, 2, 3, \dots, n$, $f(x_i^*) \geq f(x)$ for all $x \in [x_{i-1}, x_i]$. In other words, we choose $\{x_i^*\}$ so that for $i = 1, 2, 3, \dots, n$, $f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select $\{x_i^*\}$ in this way, then the Riemann sum $\sum_{i=1}^n f(x_i^*) \Delta x$ is called an **upper sum**. Similarly, if we want an underestimate, we can choose $\{x_i^*\}$ so that for $i = 1, 2, 3, \dots, n$, $f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

Example:

Exercise:

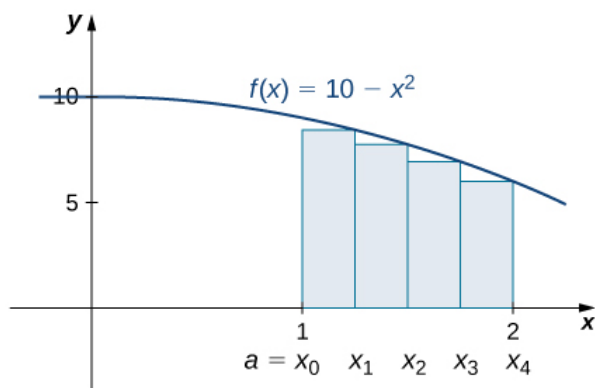
Problem:

Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.

Solution:

With $n = 4$ over the interval $[1, 2]$, $\Delta x = \frac{1}{4}$. We can list the intervals as $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, $[1.75, 2]$. Because the function is decreasing over the interval $[1, 2]$, [\[link\]](#) shows that a lower sum is obtained by using the right endpoints.



The graph of $f(x) = 10 - x^2$ is set up for a right-endpoint approximation of the area bounded by the curve and the x -axis on $[1, 2]$, and it shows a lower sum.

The Riemann sum is

Equation:

$$\begin{aligned} \sum_{k=1}^4 (10 - x^2)(0.25) &= 0.25 [10 - (1.25)^2 + 10 - (1.5)^2 + 10 - (1.75)^2 + 10 - (2)^2] \\ &= 0.25 [8.4375 + 7.75 + 6.9375 + 6] \\ &= 7.28. \end{aligned}$$

The area of 7.28 is a lower sum and an underestimate.

Note:

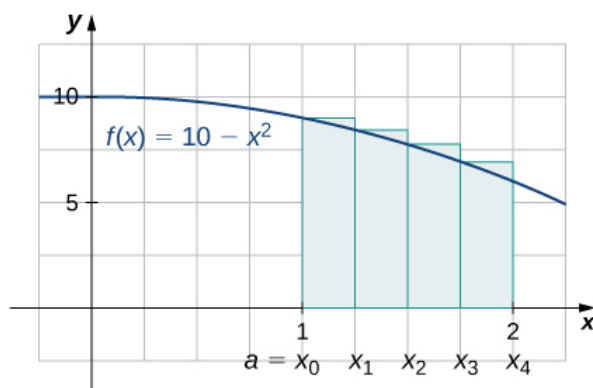
Exercise:

Problem:

- Find an upper sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$.
- Sketch the approximation.

Solution:

- Upper sum = 8.0313.
-



Hint

$f(x)$ is decreasing on $[1, 2]$, so the maximum function values occur at the left endpoints of the subintervals.

Example:

Exercise:

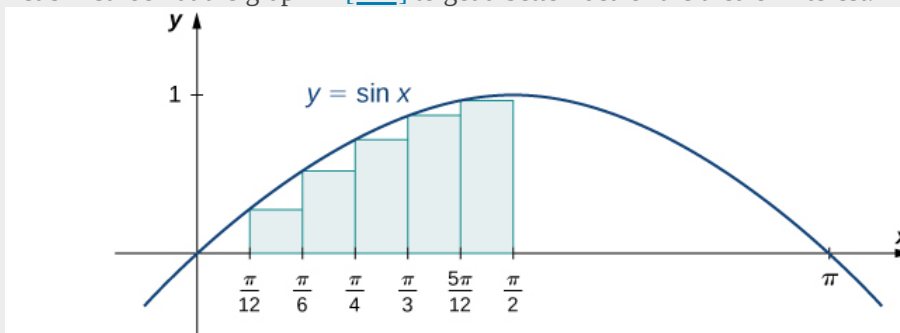
Problem:

Finding Lower and Upper Sums for $f(x) = \sin x$

Find a lower sum for $f(x) = \sin x$ over the interval $[a, b] = [0, \frac{\pi}{2}]$; let $n = 6$.

Solution:

Let's first look at the graph in [\[link\]](#) to get a better idea of the area of interest.



The graph of $y = \sin x$ is divided into six regions: $\Delta x = \frac{\pi/2}{6} = \frac{\pi}{12}$.

The intervals are $[0, \frac{\pi}{12}]$, $[\frac{\pi}{12}, \frac{\pi}{6}]$, $[\frac{\pi}{6}, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{3}]$, $[\frac{\pi}{3}, \frac{5\pi}{12}]$, and $[\frac{5\pi}{12}, \frac{\pi}{2}]$. Note that $f(x) = \sin x$ is increasing on the interval $[0, \frac{\pi}{2}]$, so a left-endpoint approximation gives us the lower sum. A left-

endpoint approximation is the Riemann sum $\sum_{i=0}^5 \sin x_i \left(\frac{\pi}{12} \right)$. We have

Equation:

$$A \approx \sin(0) \left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right) \left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{6}\right) \left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4}\right) \left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{3}\right) \left(\frac{\pi}{12}\right) + \sin\left(\frac{5\pi}{12}\right) \left(\frac{\pi}{12}\right) = 0.863.$$

Note:

Exercise:

Problem: Using the function $f(x) = \sin x$ over the interval $\left[0, \frac{\pi}{2}\right]$, find an upper sum; let $n = 6$.

Solution:

$$A \approx 1.125$$

Hint

Follow the steps from [\[link\]](#).

Key Concepts

- The use of sigma (summation) notation of the form $\sum_{i=1}^n a_i$ is useful for expressing long sums of values in compact form.
- For a continuous function defined over an interval $[a, b]$, the process of dividing the interval into n equal parts, extending a rectangle to the graph of the function, calculating the areas of the series of rectangles, and then summing the areas yields an approximation of the area of that region.
- The width of each rectangle is $\Delta x = \frac{b-a}{n}$.
- Riemann sums are expressions of the form $\sum_{i=1}^n f(x_i^*) \Delta x$, and can be used to estimate the area under the curve $y = f(x)$. Left- and right-endpoint approximations are special kinds of Riemann sums where the values of $\{x_i^*\}$ are chosen to be the left or right endpoints of the subintervals, respectively.
- Riemann sums allow for much flexibility in choosing the set of points $\{x_i^*\}$ at which the function is evaluated, often with an eye to obtaining a lower sum or an upper sum.

Key Equations

- Properties of Sigma Notation**

$$\sum_{i=1}^n c = nc$$

$$\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$$

$$\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$$

$$\sum_{i=1}^n (a_i - b_i) = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$$

$$\sum_{i=1}^n a_i = \sum_{i=1}^m a_i + \sum_{i=m+1}^n a_i$$

• **Sums and Powers of Integers**

$$\sum_{i=1}^n i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^n i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

• **Left-Endpoint Approximation**

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^n f(x_{i-1})\Delta x$$

• **Right-Endpoint Approximation**

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^n f(x_i)\Delta x$$

Exercise:

Problem: State whether the given sums are equal or unequal.

a. $\sum_{i=1}^{10} i$ and $\sum_{k=1}^{10} k$

b. $\sum_{i=1}^{10} i$ and $\sum_{i=6}^{15} (i - 5)$

c. $\sum_{i=1}^{10} i(i - 1)$ and $\sum_{j=0}^9 (j + 1)j$

d. $\sum_{i=1}^{10} i(i - 1)$ and $\sum_{k=1}^{10} (k^2 - k)$

Solution:

a. They are equal; both represent the sum of the first 10 whole numbers. b. They are equal; both represent the sum of the first 10 whole numbers. c. They are equal by substituting $j = i - 1$. d. They are equal; the first sum factors the terms of the second.

In the following exercises, use the rules for sums of powers of integers to compute the sums.

Exercise:

Problem: $\sum_{i=5}^{10} i$

Exercise:

Problem: $\sum_{i=5}^{10} i^2$

Solution:

$$385 - 30 = 355$$

Suppose that $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$. In the following exercises, compute the sums.

Exercise:

Problem: $\sum_{i=1}^{100} (a_i + b_i)$

Exercise:

Problem: $\sum_{i=1}^{100} (a_i - b_i)$

Solution:

$$15 - (-12) = 27$$

Exercise:

Problem: $\sum_{i=1}^{100} (3a_i - 4b_i)$

Exercise:

Problem: $\sum_{i=1}^{100} (5a_i + 4b_i)$

Solution:

$$5(15) + 4(-12) = 27$$

In the following exercises, use summation properties and formulas to rewrite and evaluate the sums.

Exercise:

Problem: $\sum_{k=1}^{20} 100(k^2 - 5k + 1)$

Exercise:

Problem: $\sum_{j=1}^{50} (j^2 - 2j)$

Solution:

$$\sum_{j=1}^{50} j^2 - 2 \sum_{j=1}^{50} j = \frac{(50)(51)(101)}{6} - \frac{2(50)(51)}{2} = 40,375$$

Exercise:

Problem: $\sum_{j=11}^{20} (j^2 - 10j)$

Exercise:

Problem: $\sum_{k=1}^{25} [(2k)^2 - 100k]$

Solution:

$$4 \sum_{k=1}^{25} k^2 - 100 \sum_{k=1}^{25} k = \frac{4(25)(26)(51)}{9} - 50(25)(26) = -10,400$$

Let L_n denote the left-endpoint sum using n subintervals and let R_n denote the corresponding right-endpoint sum. In the following exercises, compute the indicated left and right sums for the given functions on the indicated interval.

Exercise:

Problem: L_4 for $f(x) = \frac{1}{x-1}$ on $[2, 3]$

Exercise:

Problem: R_4 for $g(x) = \cos(\pi x)$ on $[0, 1]$

Solution:

$$R_4 = 0.25$$

Exercise:

Problem: L_6 for $f(x) = \frac{1}{x(x-1)}$ on $[2, 5]$

Exercise:

Problem: R_6 for $f(x) = \frac{1}{x(x-1)}$ on $[2, 5]$

Solution:

$$R_6 = 0.372$$

Exercise:

Problem: R_4 for $\frac{1}{x^2+1}$ on $[-2, 2]$

Exercise:

Problem: L_4 for $\frac{1}{x^2+1}$ on $[-2, 2]$

Solution:

$$L_4 = 2.20$$

Exercise:

Problem: R_4 for $x^2 - 2x + 1$ on $[0, 2]$

Exercise:

Problem: L_8 for $x^2 - 2x + 1$ on $[0, 2]$

Solution:

$$L_8 = 0.6875$$

Exercise:

Problem:

Compute the left and right Riemann sums— L_4 and R_4 , respectively—for $f(x) = (2 - |x|)$ on $[-2, 2]$. Compute their average value and compare it with the area under the graph of f .

Exercise:

Problem:

Compute the left and right Riemann sums— L_6 and R_6 , respectively—for $f(x) = (3 - |3 - x|)$ on $[0, 6]$. Compute their average value and compare it with the area under the graph of f .

Solution:

$L_6 = 9.000 = R_6$. The graph of f is a triangle with area 9.

Exercise:

Problem:

Compute the left and right Riemann sums— L_4 and R_4 , respectively—for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ and compare their values.

Exercise:

Problem:

Compute the left and right Riemann sums— L_6 and R_6 , respectively—for $f(x) = \sqrt{9 - (x - 3)^2}$ on $[0, 6]$ and compare their values.

Solution:

$L_6 = 13.12899 = R_6$. They are equal.

Express the following endpoint sums in sigma notation but do not evaluate them.

Exercise:

Problem: L_{30} for $f(x) = x^2$ on $[1, 2]$

Exercise:

Problem: L_{10} for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$

Solution:

$$L_{10} = \frac{4}{10} \sum_{i=1}^{10} \sqrt{4 - \left(-2 + 4\frac{(i-1)}{10}\right)^2}$$

Exercise:

Problem: R_{20} for $f(x) = \sin x$ on $[0, \pi]$

Exercise:

Problem: R_{100} for $\ln x$ on $[1, e]$

Solution:

$$R_{100} = \frac{e-1}{100} \sum_{i=1}^{100} \ln \left(1 + (e-1)\frac{i}{100} \right)$$

In the following exercises, graph the function then use a calculator or a computer program to evaluate the following left and right endpoint sums. Is the area under the curve between the left and right endpoint sums?

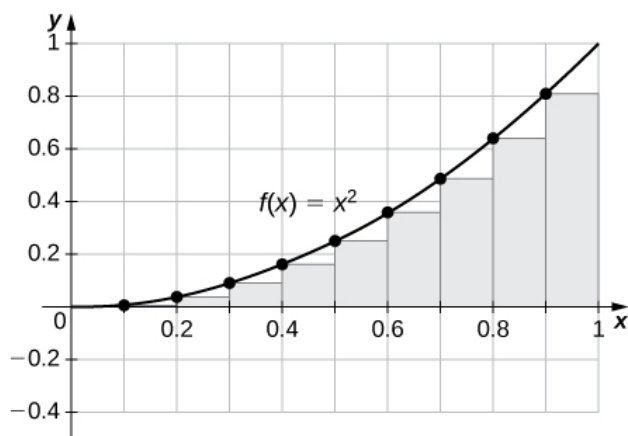
Exercise:

Problem: [T] L_{100} and R_{100} for $y = x^2 - 3x + 1$ on the interval $[-1, 1]$

Exercise:

Problem: [T] L_{100} and R_{100} for $y = x^2$ on the interval $[0, 1]$

Solution:



$R_{100} = 0.33835$, $L_{100} = 0.32835$. The plot shows that the left Riemann sum is an underestimate because the function is increasing. Similarly, the right Riemann sum is an overestimate. The area lies between the left and right Riemann sums. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

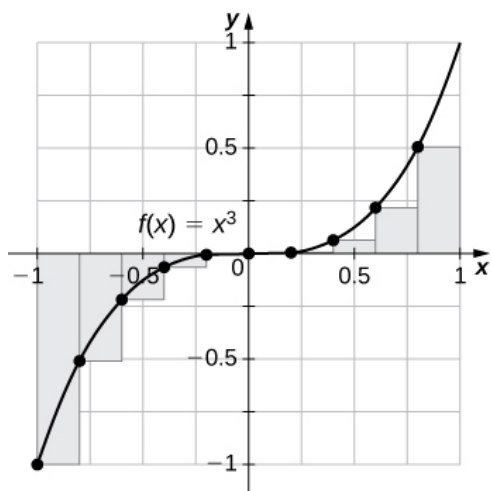
Exercise:

Problem: [T] L_{50} and R_{50} for $y = \frac{x+1}{x^2-1}$ on the interval $[2, 4]$

Exercise:

Problem: [T] L_{100} and R_{100} for $y = x^3$ on the interval $[-1, 1]$

Solution:



$L_{100} = -0.02$, $R_{100} = 0.02$. The left endpoint sum is an underestimate because the function is increasing. Similarly, a right endpoint approximation is an overestimate. The area lies between the left and right endpoint estimates.

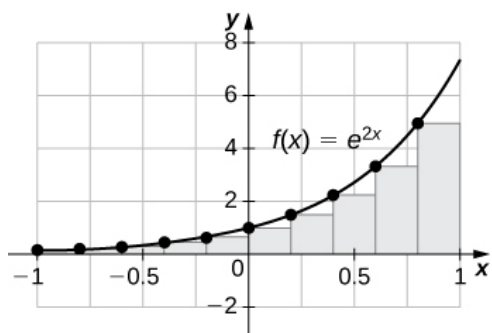
Exercise:

Problem: [T] L_{50} and R_{50} for $y = \tan(x)$ on the interval $[0, \frac{\pi}{4}]$

Exercise:

Problem: [T] L_{100} and R_{100} for $y = e^{2x}$ on the interval $[-1, 1]$

Solution:



$L_{100} = 3.555$, $R_{100} = 3.670$. The plot shows that the left Riemann sum is an underestimate because the function is increasing. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

Exercise:

Problem:

Let t_j denote the time that it took Tejay van Garteren to ride the j th stage of the Tour de France in 2014.

If there were a total of 21 stages, interpret $\sum_{j=1}^{21} t_j$.

Exercise:

Problem: Let r_j denote the total rainfall in Portland on the j th day of the year in 2009. Interpret $\sum_{j=1}^{31} r_j$.

Solution:

The sum represents the cumulative rainfall in January 2009.

Exercise:

Problem:

Let d_j denote the hours of daylight and δ_j denote the increase in the hours of daylight from day $j - 1$ to day j in Fargo, North Dakota, on the j th day of the year. Interpret $d_1 + \sum_{j=2}^{365} \delta_j$.

Exercise:

Problem:

To help get in shape, Joe gets a new pair of running shoes. If Joe runs 1 mi each day in week 1 and adds $\frac{1}{10}$ mi to his daily routine each week, what is the total mileage on Joe's shoes after 25 weeks?

Solution:

The total mileage is $7 \times \sum_{i=1}^{25} \left(1 + \frac{(i-1)}{10}\right) = 7 \times 25 + \frac{7}{10} \times 12 \times 25 = 385$ mi.

Exercise:**Problem:**

The following table gives approximate values of the average annual atmospheric rate of increase in carbon dioxide (CO₂) each decade since 1960, in parts per million (ppm). Estimate the total increase in atmospheric CO₂ between 1964 and 2013.

Decade	Ppm/y
1964–1973	1.07
1974–1983	1.34
1984–1993	1.40
1994–2003	1.87
2004–2013	2.07

Average Annual Atmospheric CO₂ Increase, 1964–2013 *Source:*
<http://www.esrl.noaa.gov/gmd/ccgg/trends/>.

Exercise:**Problem:**

The following table gives the approximate increase in sea level in inches over 20 years starting in the given year. Estimate the net change in mean sea level from 1870 to 2010.

Starting Year	20-Year Change
1870	0.3

Starting Year	20-Year Change
1890	1.5
1910	0.2
1930	2.8
1950	0.7
1970	1.1
1990	1.5

Approximate 20-Year Sea Level Increases, 1870–1990
Source:
<http://link.springer.com/article/10.1007%2Fs10712-011-9119-1>

Solution:

Add the numbers to get 8.1-in. net increase.

Exercise:

Problem:

The following table gives the approximate increase in dollars in the average price of a gallon of gas per decade since 1950. If the average price of a gallon of gas in 2010 was \$2.60, what was the average price of a gallon of gas in 1950?

Starting Year	10-Year Change
1950	0.03
1960	0.05
1970	0.86
1980	−0.03
1990	0.29
2000	1.12

Approximate 10-Year Gas Price Increases, 1950–2000
Source:
http://epb.lbl.gov/homepages/Rick_Diamond/docs/lbnl55011-trends.pdf.

Exercise:

Problem:

The following table gives the percent growth of the U.S. population beginning in July of the year indicated. If the U.S. population was 281,421,906 in July 2000, estimate the U.S. population in July 2010.

Year	% Change/Year
2000	1.12
2001	0.99
2002	0.93
2003	0.86
2004	0.93
2005	0.93
2006	0.97
2007	0.96
2008	0.95
2009	0.88

Annual Percentage Growth of U.S. Population, 2000–2009Source: <http://www.census.gov/popest/data>.

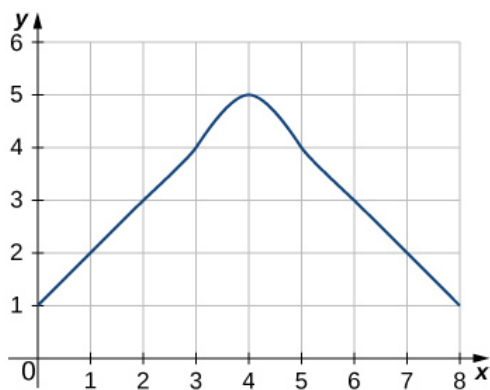
(Hint: To obtain the population in July 2001, multiply the population in July 2000 by 1.0112 to get 284,573,831.)

Solution:

309,389,957

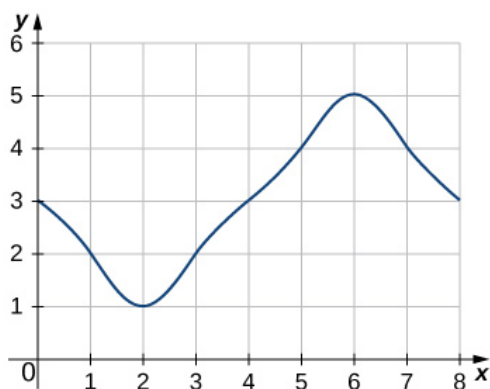
In the following exercises, estimate the areas under the curves by computing the left Riemann sums, L_8 .

Exercise:**Problem:**



Exercise:

Problem:

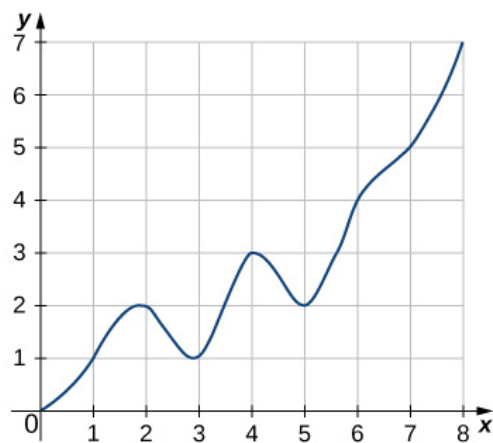


Solution:

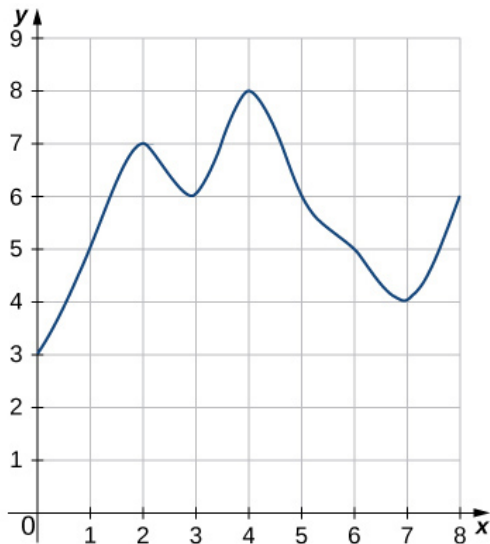
$$L_8 = 3 + 2 + 1 + 2 + 3 + 4 + 5 + 4 = 24$$

Exercise:

Problem:



Exercise:

Problem:**Solution:**

$$L_8 = 3 + 5 + 7 + 6 + 8 + 6 + 5 + 4 = 44$$

Exercise:**Problem:**

[T] Use a computer algebra system to compute the Riemann sum, L_N , for $N = 10, 30, 50$ for $f(x) = \sqrt{1-x^2}$ on $[-1, 1]$.

Exercise:**Problem:**

[T] Use a computer algebra system to compute the Riemann sum, L_N , for $N = 10, 30, 50$ for $f(x) = \frac{1}{\sqrt{1+x^2}}$ on $[-1, 1]$.

Solution:

$$L_{10} \approx 1.7604, L_{30} \approx 1.7625, L_{50} \approx 1.76265$$

Exercise:**Problem:**

[T] Use a computer algebra system to compute the Riemann sum, L_N , for $N = 10, 30, 50$ for $f(x) = \sin^2 x$ on $[0, 2\pi]$. Compare these estimates with π .

In the following exercises, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$. How do these estimates compare with the exact answers, which you can find via geometry?

Exercise:

Problem: [T] $y = \cos(\pi x)$ on the interval $[0, 1]$

Solution:

$R_1 = -1, L_1 = 1, R_{10} = -0.1, L_{10} = 0.1, L_{100} = 0.01$, and $R_{100} = -0.1$. By symmetry of the graph, the exact area is zero.

Exercise:

Problem: [T] $y = 3x + 2$ on the interval $[3, 5]$

In the following exercises, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$.

Exercise:

Problem: [T] $y = x^4 - 5x^2 + 4$ on the interval $[-2, 2]$, which has an exact area of $\frac{32}{15}$

Solution:

$R_1 = 0, L_1 = 0, R_{10} = 2.4499, L_{10} = 2.4499, R_{100} = 2.1365, L_{100} = 2.1365$

Exercise:

Problem: [T] $y = \ln x$ on the interval $[1, 2]$, which has an exact area of $2 \ln(2) - 1$

Exercise:**Problem:**

Explain why, if $f(a) \geq 0$ and f is increasing on $[a, b]$, that the left endpoint estimate is a lower bound for the area below the graph of f on $[a, b]$.

Solution:

If $[c, d]$ is a subinterval of $[a, b]$ under one of the left-endpoint sum rectangles, then the area of the rectangle contributing to the left-endpoint estimate is $f(c)(d - c)$. But, $f(c) \leq f(x)$ for $c \leq x \leq d$, so the area under the graph of f between c and d is $f(c)(d - c)$ plus the area below the graph of f but above the horizontal line segment at height $f(c)$, which is positive. As this is true for each left-endpoint sum interval, it follows that the left Riemann sum is less than or equal to the area below the graph of f on $[a, b]$.

Exercise:**Problem:**

Explain why, if $f(b) \geq 0$ and f is decreasing on $[a, b]$, that the left endpoint estimate is an upper bound for the area below the graph of f on $[a, b]$.

Exercise:

Problem: Show that, in general, $R_N - L_N = (b - a) \times \frac{f(b) - f(a)}{N}$.

Solution:

$$L_N = \frac{b-a}{N} \sum_{i=1}^N f\left(a + (b-a)\frac{i-1}{N}\right) = \frac{b-a}{N} \sum_{i=0}^{N-1} f\left(a + (b-a)\frac{i}{N}\right) \text{ and}$$

$$R_N = \frac{b-a}{N} \sum_{i=1}^N f\left(a + (b-a)\frac{i}{N}\right). \text{ The left sum has a term corresponding to } i = 0 \text{ and the right sum}$$

has a term corresponding to $i = N$. In $R_N - L_N$, any term corresponding to $i = 1, 2, \dots, N-1$ occurs once with a plus sign and once with a minus sign, so each such term cancels and one is left with

$$R_N - L_N = \frac{b-a}{N} \left(f\left(a + (b-a)\frac{N}{N}\right) - \left(f\left(a + (b-a)\frac{0}{N}\right)\right)\right) = \frac{b-a}{N} (f(b) - f(a)).$$

Exercise:

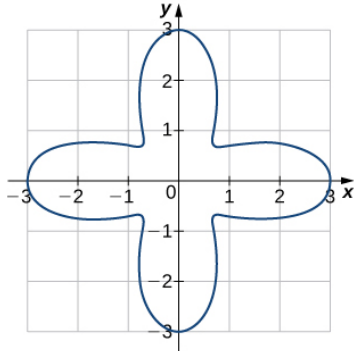
Problem:

Explain why, if f is increasing on $[a, b]$, the error between either L_N or R_N and the area A below the graph of f is at most $(b-a)\frac{f(b)-f(a)}{N}$.

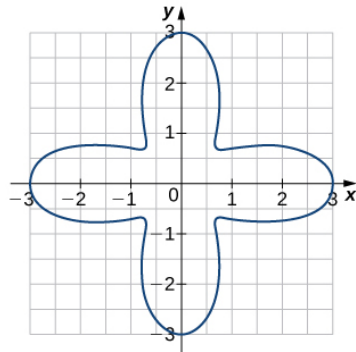
Exercise:

Problem: For each of the three graphs:

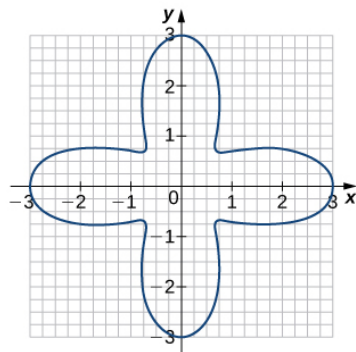
- Obtain a lower bound $L(A)$ for the area enclosed by the curve by adding the areas of the squares *enclosed completely* by the curve.
- Obtain an upper bound $U(A)$ for the area by adding to $L(A)$ the areas $B(A)$ of the squares *enclosed partially* by the curve.



Graph 1



Graph 2



Graph 3

Solution:

Graph 1: a. $L(A) = 0, B(A) = 20$; b. $U(A) = 20$. Graph 2: a. $L(A) = 9$; b. $B(A) = 11, U(A) = 20$. Graph 3: a. $L(A) = 11.0$; b. $B(A) = 4.5, U(A) = 15.5$.

Exercise:

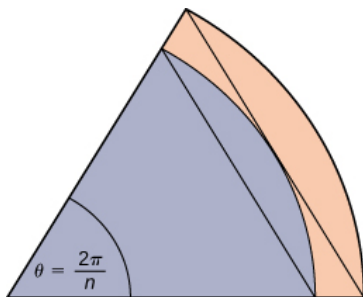
Problem:

In the previous exercise, explain why $L(A)$ gets no smaller while $U(A)$ gets no larger as the squares are subdivided into four boxes of equal area.

Exercise:

Problem:

A unit circle is made up of n wedges equivalent to the inner wedge in the figure. The base of the inner triangle is 1 unit and its height is $\sin\left(\frac{\pi}{n}\right)$. The base of the outer triangle is $B = \cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right)$ and the height is $H = B\sin\left(\frac{2\pi}{n}\right)$. Use this information to argue that the area of a unit circle is equal to π .

**Solution:**

Let A be the area of the unit circle. The circle encloses n congruent triangles each of area $\frac{\sin(\frac{2\pi}{n})}{2}$, so $\frac{n}{2} \sin\left(\frac{2\pi}{n}\right) \leq A$. Similarly, the circle is contained inside n congruent triangles each of area $\frac{BH}{2} = \frac{1}{2} \left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right) \right) \sin\left(\frac{2\pi}{n}\right)$, so $A \leq \frac{n}{2} \sin\left(\frac{2\pi}{n}\right) \left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right) \right)$. As $n \rightarrow \infty$, $\frac{n}{2} \sin\left(\frac{2\pi}{n}\right) = \frac{\pi \sin(\frac{2\pi}{n})}{(\frac{2\pi}{n})} \rightarrow \pi$, so we conclude $\pi \leq A$. Also, as $n \rightarrow \infty$, $\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right) \rightarrow 1$, so we also have $A \leq \pi$. By the squeeze theorem for limits, we conclude that $A = \pi$.

Glossary

left-endpoint approximation

an approximation of the area under a curve computed by using the left endpoint of each subinterval to calculate the height of the vertical sides of each rectangle

lower sum

a sum obtained by using the minimum value of $f(x)$ on each subinterval

partition

a set of points that divides an interval into subintervals

regular partition

a partition in which the subintervals all have the same width

riemann sum

an estimate of the area under the curve of the form $A \approx \sum_{i=1}^n f(x_i^*) \Delta x$

right-endpoint approximation

the right-endpoint approximation is an approximation of the area of the rectangles under a curve using the right endpoint of each subinterval to construct the vertical sides of each rectangle

sigma notation

(also, **summation notation**) the Greek letter sigma (Σ) indicates addition of the values; the values of the index above and below the sigma indicate where to begin the summation and where to end it

upper sum

a sum obtained by using the maximum value of $f(x)$ on each subinterval

The Definite Integral

- State the definition of the definite integral.
- Explain the terms integrand, limits of integration, and variable of integration.
- Explain when a function is integrable.
- Describe the relationship between the definite integral and net area.
- Use geometry and the properties of definite integrals to evaluate them.
- Calculate the average value of a function.

In the preceding section we defined the area under a curve in terms of Riemann sums:

Equation:

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x.$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, real-world problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

Note:

Definition

If $f(x)$ is a function defined on an interval $[a, b]$, the **definite integral** of f from a to b is given by

Equation:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x,$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an **integrable function**.

The integral symbol in the previous definition should look familiar. We have seen similar notation in the chapter on [Applications of Derivatives](#), where we used the indefinite integral symbol (without the a and b above and below) to represent an antiderivative. Although the notation for indefinite integrals may look similar to the notation for a definite integral, they are not the same. A definite integral is a number. An indefinite integral is a family of functions. Later in this chapter we examine how these concepts are related. However, close attention should always be paid to notation so we know whether we're working with a definite integral or an indefinite integral.

Integral notation goes back to the late seventeenth century and is one of the contributions of Gottfried Wilhelm Leibniz, who is often considered to be the codiscoverer of calculus, along with Isaac Newton. The integration symbol \int is an elongated S, suggesting sigma or summation. On a definite integral, above and below the summation symbol are the boundaries of the interval, $[a, b]$. The numbers a and b are x -values and

are called the **limits of integration**; specifically, a is the lower limit and b is the upper limit. To clarify, we are using the word *limit* in two different ways in the context of the definite integral. First, we talk about the limit of a sum as $n \rightarrow \infty$. Second, the boundaries of the region are called the *limits of integration*.

We call the function $f(x)$ the **integrand**, and the dx indicates that $f(x)$ is a function with respect to x , called the **variable of integration**. Note that, like the index in a sum, the variable of integration is a dummy variable, and has no impact on the computation of the integral. We could use any variable we like as the variable of integration:

Equation:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$$

Previously, we discussed the fact that if $f(x)$ is continuous on $[a, b]$, then the limit $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$ exists and is unique. This leads to the following theorem, which we state without proof.

Note:

Continuous Functions Are Integrable

If $f(x)$ is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Functions that are not continuous on $[a, b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions with a finite number of jump discontinuities on a closed interval are integrable.

It is also worth noting here that we have retained the use of a regular partition in the Riemann sums. This restriction is not strictly necessary. Any partition can be used to form a Riemann sum. However, if a nonregular partition is used to define the definite integral, it is not sufficient to take the limit as the number of subintervals goes to infinity. Instead, we must take the limit as the width of the largest subinterval goes to zero. This introduces a little more complex notation in our limits and makes the calculations more difficult without really gaining much additional insight, so we stick with regular partitions for the Riemann sums.

Example:

Exercise:

Problem:

Evaluating an Integral Using the Definition

Use the definition of the definite integral to evaluate $\int_0^2 x^2 dx$. Use a right-endpoint approximation to generate the Riemann sum.

Solution:

We first want to set up a Riemann sum. Based on the limits of integration, we have $a = 0$ and $b = 2$. For $i = 0, 1, 2, \dots, n$, let $P = \{x_i\}$ be a regular partition of $[0, 2]$. Then

Equation:

$$\Delta x = \frac{b-a}{n} = \frac{2}{n}.$$

Since we are using a right-endpoint approximation to generate Riemann sums, for each i , we need to calculate the function value at the right endpoint of the interval $[x_{i-1}, x_i]$. The right endpoint of the interval is x_i , and since P is a regular partition,

Equation:

$$x_i = x_0 + i\Delta x = 0 + i \left[\frac{2}{n} \right] = \frac{2i}{n}.$$

Thus, the function value at the right endpoint of the interval is

Equation:

$$f(x_i) = x_i^2 = \left(\frac{2i}{n} \right)^2 = \frac{4i^2}{n^2}.$$

Then the Riemann sum takes the form

Equation:

$$\sum_{i=1}^n f(x_i)\Delta x = \sum_{i=1}^n \left(\frac{4i^2}{n^2} \right) \frac{2}{n} = \sum_{i=1}^n \frac{8i^2}{n^3} = \frac{8}{n^3} \sum_{i=1}^n i^2.$$

Using the summation formula for $\sum_{i=1}^n i^2$, we have

Equation:

$$\begin{aligned} \sum_{i=1}^n f(x_i)\Delta x &= \frac{8}{n^3} \sum_{i=1}^n i^2 \\ &= \frac{8}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] \\ &= \frac{8}{n^3} \left[\frac{2n^3+3n^2+n}{6} \right] \\ &= \frac{16n^3+24n^2+n}{6n^3} \\ &= \frac{8}{3} + \frac{4}{n} + \frac{1}{6n^2}. \end{aligned}$$

Now, to calculate the definite integral, we need to take the limit as $n \rightarrow \infty$. We get

Equation:

$$\begin{aligned}
 \int_0^2 x^2 dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x \\
 &= \lim_{n \rightarrow \infty} \left(\frac{8}{3} + \frac{4}{n} + \frac{1}{6n^2} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{8}{3} \right) + \lim_{n \rightarrow \infty} \left(\frac{4}{n} \right) + \lim_{n \rightarrow \infty} \left(\frac{1}{6n^2} \right) \\
 &= \frac{8}{3} + 0 + 0 = \frac{8}{3}.
 \end{aligned}$$

Note:

Exercise:

Problem:

Use the definition of the definite integral to evaluate $\int_0^3 (2x - 1) dx$. Use a right-endpoint approximation to generate the Riemann sum.

Solution:

6

Hint

Use the solving strategy from [\[link\]](#).

Evaluating Definite Integrals

Evaluating definite integrals this way can be quite tedious because of the complexity of the calculations. Later in this chapter we develop techniques for evaluating definite integrals *without* taking limits of Riemann sums. However, for now, we can rely on the fact that definite integrals represent the area under the curve, and we can evaluate definite integrals by using geometric formulas to calculate that area. We do this to confirm that definite integrals do, indeed, represent areas, so we can then discuss what to do in the case of a curve of a function dropping below the x -axis.

Example:

Exercise:

Problem:

Using Geometric Formulas to Calculate Definite Integrals

Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9 - (x - 3)^2} dx$.

Solution:

The function describes a semicircle with radius 3. To find

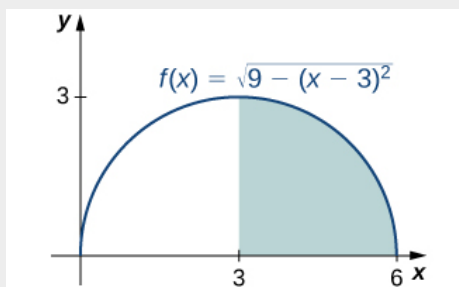
Equation:

$$\int_3^6 \sqrt{9 - (x - 3)^2} dx,$$

we want to find the area under the curve over the interval $[3, 6]$. The formula for the area of a circle is $A = \pi r^2$. The area of a semicircle is just one-half the area of a circle, or $A = \left(\frac{1}{2}\right)\pi r^2$. The shaded area in [\[link\]](#) covers one-half of the semicircle, or $A = \left(\frac{1}{4}\right)\pi r^2$. Thus,

Equation:

$$\begin{aligned} \int_3^6 \sqrt{9 - (x - 3)^2} &= \frac{1}{4}\pi(3)^2 \\ &= \frac{9}{4}\pi \\ &\approx 7.069. \end{aligned}$$



The value of the integral of the function $f(x)$ over the interval $[3, 6]$ is the area of the shaded region.

Note:

Exercise:

Problem: Use the formula for the area of a trapezoid to evaluate $\int_2^4 (2x + 3)dx$.

Solution:

18 square units

Hint

Graph the function $f(x)$ and calculate the area under the function on the interval $[2, 4]$.

Area and the Definite Integral

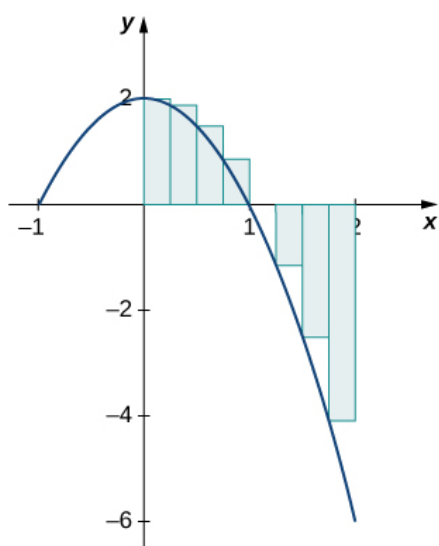
When we defined the definite integral, we lifted the requirement that $f(x)$ be nonnegative. But how do we interpret “the area under the curve” when $f(x)$ is negative?

Net Signed Area

Let us return to the Riemann sum. Consider, for example, the function $f(x) = 2 - 2x^2$ (shown in [link](#)) on the interval $[0, 2]$. Use $n = 8$ and choose $\{x_i^*\}$ as the left endpoint of each interval. Construct a rectangle on each subinterval of height $f(x_i^*)$ and width Δx . When $f(x_i^*)$ is positive, the product $f(x_i^*)\Delta x$ represents the area of the rectangle, as before. When $f(x_i^*)$ is negative, however, the product $f(x_i^*)\Delta x$ represents the *negative* of the area of the rectangle. The Riemann sum then becomes

Equation:

$$\sum_{i=1}^8 f(x_i^*)\Delta x = (\text{Area of rectangles above the } x\text{-axis}) - (\text{Area of rectangles below the } x\text{-axis})$$



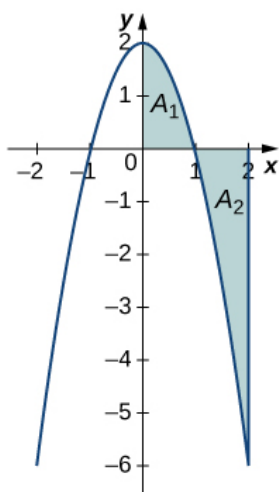
For a function that is partly negative, the Riemann sum is the area of the rectangles above the x -axis less the area of the rectangles below the x -axis.

Taking the limit as $n \rightarrow \infty$, the Riemann sum approaches the area between the curve above the x -axis and the x -axis, less the area between the curve below the x -axis and the x -axis, as shown in [link](#). Then,

Equation:

$$\begin{aligned}\int_0^2 f(x)dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i)\Delta x \\ &= A_1 - A_2.\end{aligned}$$

The quantity $A_1 - A_2$ is called the **net signed area**.



In the limit, the definite integral equals area A_1 less area A_2 , or the net signed area.

Notice that net signed area can be positive, negative, or zero. If the area above the x -axis is larger, the net signed area is positive. If the area below the x -axis is larger, the net signed area is negative. If the areas above and below the x -axis are equal, the net signed area is zero.

Example:

Exercise:

Problem:

Finding the Net Signed Area

Find the net signed area between the curve of the function $f(x) = 2x$ and the x -axis over the interval $[-3, 3]$.

Solution:

The function produces a straight line that forms two triangles: one from $x = -3$ to $x = 0$ and the other from $x = 0$ to $x = 3$ ([link](#)). Using the geometric formula for the area of a triangle, $A = \frac{1}{2}bh$, the area of triangle A_1 , above the axis, is

Equation:

$$A_1 = \frac{1}{2} 3 (6) = 9,$$

where 3 is the base and $2(3) = 6$ is the height. The area of triangle A_2 , below the axis, is

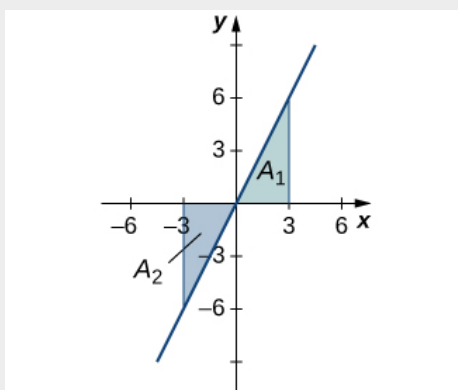
Equation:

$$A_2 = \frac{1}{2} (3) (6) = 9,$$

where 3 is the base and 6 is the height. Thus, the net area is

Equation:

$$\int_{-3}^3 2x dx = A_1 - A_2 = 9 - 9 = 0.$$



The area above the curve and below the x -axis equals the area below the curve and above the x -axis.

Analysis

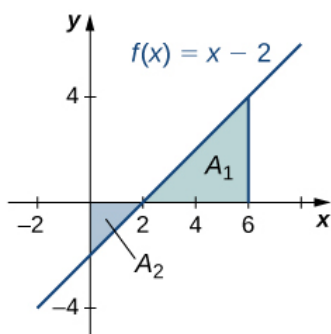
If A_1 is the area above the x -axis and A_2 is the area below the x -axis, then the net area is $A_1 - A_2$. Since the areas of the two triangles are equal, the net area is zero.

Note:

Exercise:

Problem:

Find the net signed area of $f(x) = x - 2$ over the interval $[0, 6]$, illustrated in the following image.



Solution:

6

Hint

Use the solving method described in [\[link\]](#).

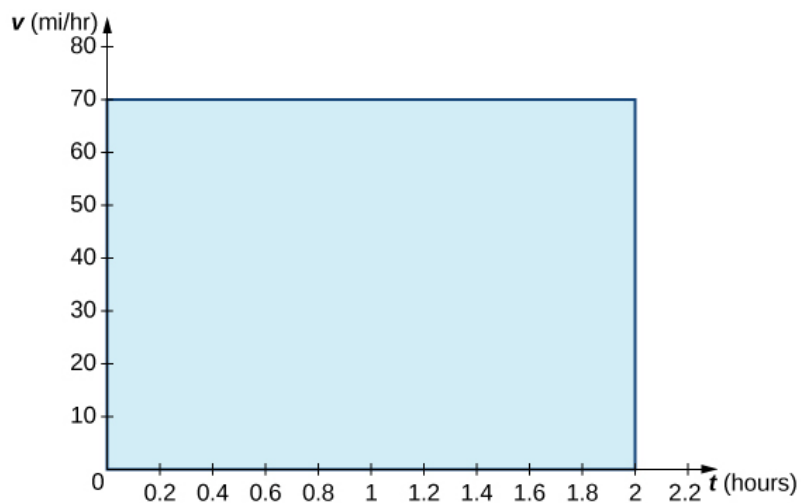
Total Area

One application of the definite integral is finding displacement when given a velocity function. If $v(t)$ represents the velocity of an object as a function of time, then the area under the curve tells us how far the object is from its original position. This is a very important application of the definite integral, and we examine it in more detail later in the chapter. For now, we're just going to look at some basics to get a feel for how this works by studying constant velocities.

When velocity is a constant, the area under the curve is just velocity times time. This idea is already very familiar. If a car travels away from its starting position in a straight line at a speed of 75 mph for 2 hours, then it is 150 mi away from its original position ([\[link\]](#)). Using integral notation, we have

Equation:

$$\int_0^2 75 dt = 150.$$



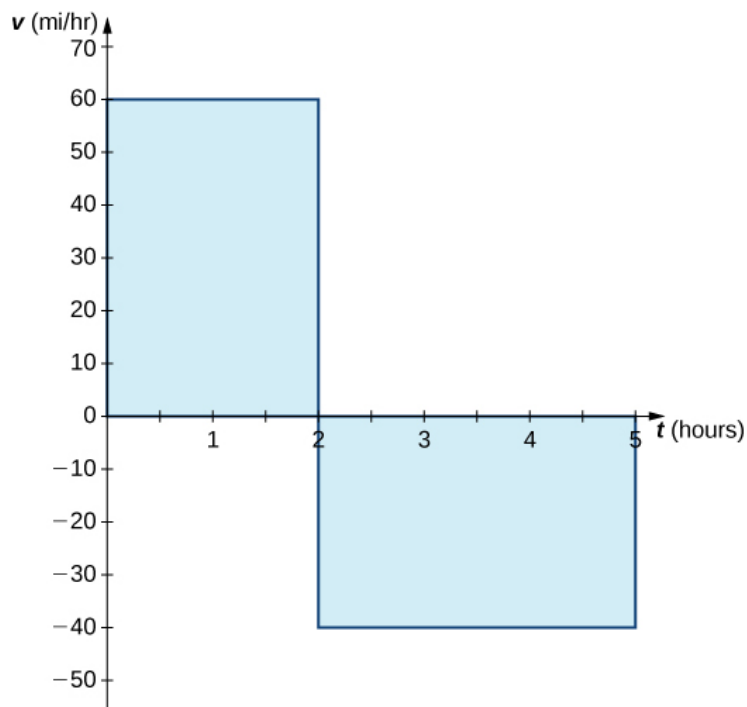
The area under the curve $v(t) = 75$ tells us how far the car is from its starting point at a given time.

In the context of displacement, net signed area allows us to take direction into account. If a car travels straight north at a speed of 60 mph for 2 hours, it is 120 mi north of its starting position. If the car then turns around and travels south at a speed of 40 mph for 3 hours, it will be back at its starting position ([link](#)). Again, using integral notation, we have

Equation:

$$\begin{aligned} \int_0^2 60 dt + \int_2^5 -40 dt &= 120 - 120 \\ &= 0. \end{aligned}$$

In this case the displacement is zero.



The area above the axis and the area below the axis are equal, so the net signed area is zero.

Suppose we want to know how far the car travels overall, regardless of direction. In this case, we want to know the area between the curve and the x -axis, regardless of whether that area is above or below the axis. This is called the **total area**.

Graphically, it is easiest to think of calculating total area by adding the areas above the axis and the areas below the axis (rather than subtracting the areas below the axis, as we did with net signed area). To accomplish this mathematically, we use the absolute value function. Thus, the total distance traveled by the car is

Equation:

$$\begin{aligned} \int_0^2 |60| dt + \int_2^5 |-40| dt &= \int_0^2 60 dt + \int_2^5 40 dt \\ &= 120 + 120 \\ &= 240. \end{aligned}$$

Bringing these ideas together formally, we state the following definitions.

Note:

Definition

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x -axis that lies *above* the axis and let A_2 represent the area between $f(x)$ and the x -axis that lies *below* the axis. Then, the **net signed area** between $f(x)$ and the x -axis is given by

Equation:

$$\int_a^b f(x)dx = A_1 - A_2.$$

The **total area** between $f(x)$ and the x -axis is given by

Equation:

$$\int_a^b |f(x)|dx = A_1 + A_2.$$

Example:

Exercise:

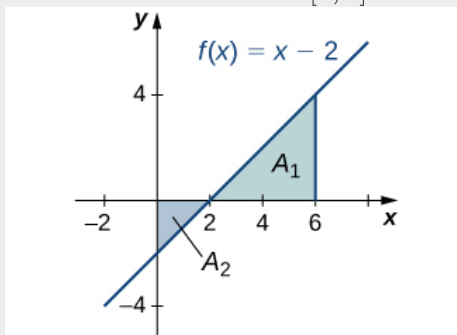
Problem:

Finding the Total Area

Find the total area between $f(x) = x - 2$ and the x -axis over the interval $[0, 6]$.

Solution:

Calculate the x -intercept as $(2, 0)$ (set $y = 0$, solve for x). To find the total area, take the area below the x -axis over the subinterval $[0, 2]$ and add it to the area above the x -axis on the subinterval $[2, 6]$ ([link](#)).



The total area between the line and the x -axis over $[0, 6]$ is A_2 plus A_1 .

We have

Equation:

$$\int_0^6 |(x - 2)|dx = A_2 + A_1.$$

Then, using the formula for the area of a triangle, we obtain

Equation:

$$A_2 = \frac{1}{2}bh = \frac{1}{2} \cdot 2 \cdot 2 = 2$$

Equation:

$$A_1 = \frac{1}{2}bh = \frac{1}{2} \cdot 4 \cdot 4 = 8.$$

The total area, then, is

Equation:

$$A_1 + A_2 = 8 + 2 = 10.$$

Note:

Exercise:

Problem: Find the total area between the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Solution:

18

Hint

Review the solving strategy in [\[link\]](#).

Properties of the Definite Integral

The properties of indefinite integrals apply to definite integrals as well. Definite integrals also have properties that relate to the limits of integration. These properties, along with the rules of integration that we examine later in this chapter, help us manipulate expressions to evaluate definite integrals.

Note:

Rule: Properties of the Definite Integral

1.

Equation:

$$\int_a^a f(x)dx = 0$$

If the limits of integration are the same, the integral is just a line and contains no area.

2.

Equation:

$$\int_b^a f(x)dx = - \int_a^b f(x)dx$$

If the limits are reversed, then place a negative sign in front of the integral.

3.

Equation:

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

The integral of a sum is the sum of the integrals.

4.

Equation:

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

The integral of a difference is the difference of the integrals.

5.

Equation:

$$\int_a^b cf(x)dx = c \int_a^b f(x)$$

for constant c . The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

6.

Equation:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Although this formula normally applies when c is between a and b , the formula holds for all values of a , b , and c , provided $f(x)$ is integrable on the largest interval.

Example:

Exercise:

Problem:

Using the Properties of the Definite Integral

Use the properties of the definite integral to express the definite integral of $f(x) = -3x^3 + 2x + 2$ over the interval $[-2, 1]$ as the sum of three definite integrals.

Solution:

Using integral notation, we have $\int_{-2}^1 (-3x^3 + 2x + 2) dx$. We apply properties 3. and 5. to get

Equation:

$$\begin{aligned}\int_{-2}^1 (-3x^3 + 2x + 2) dx &= \int_{-2}^1 -3x^3 dx + \int_{-2}^1 2x dx + \int_{-2}^1 2 dx \\ &= -3 \int_{-2}^1 x^3 dx + 2 \int_{-2}^1 x dx + \int_{-2}^1 2 dx.\end{aligned}$$

Note:

Exercise:

Problem:

Use the properties of the definite integral to express the definite integral of $f(x) = 6x^3 - 4x^2 + 2x - 3$ over the interval $[1, 3]$ as the sum of four definite integrals.

Solution:

$$6 \int_1^3 x^3 dx - 4 \int_1^3 x^2 dx + 2 \int_1^3 x dx - \int_1^3 3 dx$$

Hint

Use the solving strategy from [\[link\]](#) and the properties of definite integrals.

Example:

Exercise:

Problem:

Using the Properties of the Definite Integral

If it is known that $\int_0^8 f(x) dx = 10$ and $\int_0^5 f(x) dx = 5$, find the value of $\int_5^8 f(x) dx$.

Solution:

By property 6.,

Equation:

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

Thus,

Equation:

$$\begin{aligned}\int_0^8 f(x)dx &= \int_0^5 f(x)dx + \int_5^8 f(x)dx \\ 10 &= 5 + \int_5^8 f(x)dx \\ 5 &= \int_5^8 f(x)dx.\end{aligned}$$

Note:

Exercise:

Problem: If it is known that $\int_1^5 f(x)dx = -3$ and $\int_2^5 f(x)dx = 4$, find the value of $\int_1^2 f(x)dx$.

Solution:

-7

Hint

Use the solving strategy from [\[link\]](#) and the rule on properties of definite integrals.

Comparison Properties of Integrals

A picture can sometimes tell us more about a function than the results of computations. Comparing functions by their graphs as well as by their algebraic expressions can often give new insight into the process of integration. Intuitively, we might say that if a function $f(x)$ is above another function $g(x)$, then the area between $f(x)$ and the x -axis is greater than the area between $g(x)$ and the x -axis. This is true depending on the interval over which the comparison is made. The properties of definite integrals are valid whether $a < b$, $a = b$, or $a > b$. The following properties, however, concern only the case $a \leq b$, and are used when we want to compare the sizes of integrals.

Note:

Comparison Theorem

i. If $f(x) \geq 0$ for $a \leq x \leq b$, then

Equation:

$$\int_a^b f(x)dx \geq 0.$$

ii. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

Equation:

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

iii. If m and M are constants such that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

Equation:

$$\begin{aligned} m(b-a) &\leq \int_a^b f(x) dx \\ &\leq M(b-a). \end{aligned}$$

Example:

Exercise:

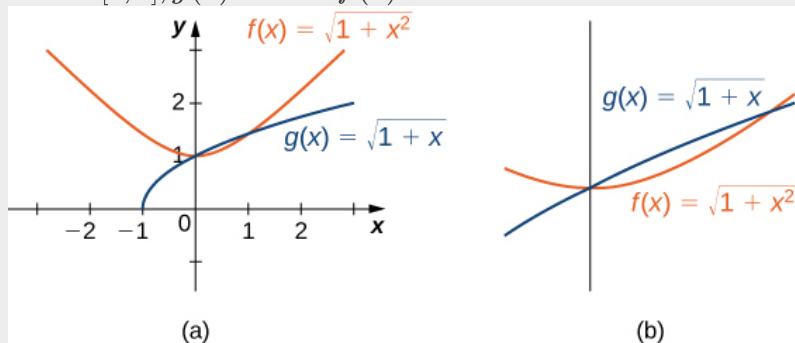
Problem:

Comparing Two Functions over a Given Interval

Compare $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0, 1]$.

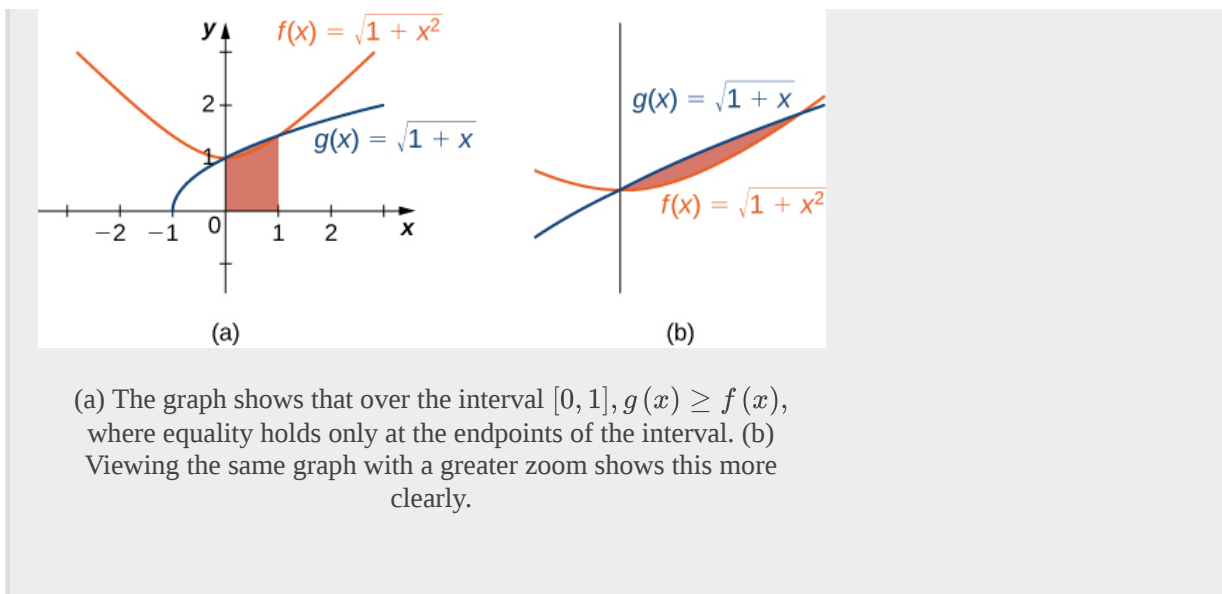
Solution:

Graphing these functions is necessary to understand how they compare over the interval $[0, 1]$. Initially, when graphed on a graphing calculator, $f(x)$ appears to be above $g(x)$ everywhere. However, on the interval $[0, 1]$, the graphs appear to be on top of each other. We need to zoom in to see that, on the interval $[0, 1]$, $g(x)$ is above $f(x)$. The two functions intersect at $x = 0$ and $x = 1$ ([link](#)).



(a) The function $f(x)$ appears above the function $g(x)$ except over the interval $[0, 1]$ (b) Viewing the same graph with a greater zoom shows this more clearly.

We can see from the graph that over the interval $[0, 1]$, $g(x) \geq f(x)$. Comparing the integrals over the specified interval $[0, 1]$, we also see that $\int_0^1 g(x) dx \geq \int_0^1 f(x) dx$ ([link](#)). The thin, red-shaded area shows just how much difference there is between these two integrals over the interval $[0, 1]$.



Average Value of a Function

We often need to find the average of a set of numbers, such as an average test grade. Suppose you received the following test scores in your algebra class: 89, 90, 56, 78, 100, and 69. Your semester grade is your average of test scores and you want to know what grade to expect. We can find the average by adding all the scores and dividing by the number of scores. In this case, there are six test scores. Thus,

Equation:

$$\frac{89 + 90 + 56 + 78 + 100 + 69}{6} = \frac{482}{6} \approx 80.33.$$

Therefore, your average test grade is approximately 80.33, which translates to a B– at most schools.

Suppose, however, that we have a function $v(t)$ that gives us the speed of an object at any time t , and we want to find the object's average speed. The function $v(t)$ takes on an infinite number of values, so we can't use the process just described. Fortunately, we can use a definite integral to find the average value of a function such as this.

Let $f(x)$ be continuous over the interval $[a, b]$ and let $[a, b]$ be divided into n subintervals of width $\Delta x = (b - a)/n$. Choose a representative x_i^* in each subinterval and calculate $f(x_i^*)$ for $i = 1, 2, \dots, n$. In other words, consider each $f(x_i^*)$ as a sampling of the function over each subinterval. The average value of the function may then be approximated as

Equation:

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n},$$

which is basically the same expression used to calculate the average of discrete values.

But we know $\Delta x = \frac{b-a}{n}$, so $n = \frac{b-a}{\Delta x}$, and we get

Equation:

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{\frac{(b-a)}{\Delta x}}.$$

Following through with the algebra, the numerator is a sum that is represented as $\sum_{i=1}^n f(x_i^*)$, and we are dividing by a fraction. To divide by a fraction, invert the denominator and multiply. Thus, an approximate value for the average value of the function is given by

Equation:

$$\begin{aligned} \frac{\sum_{i=1}^n f(x_i^*)}{\frac{(b-a)}{\Delta x}} &= \left(\frac{\Delta x}{b-a}\right) \sum_{i=1}^n f(x_i^*) \\ &= \left(\frac{1}{b-a}\right) \sum_{i=1}^n f(x_i^*) \Delta x. \end{aligned}$$

This is a Riemann sum. Then, to get the *exact* average value, take the limit as n goes to infinity. Thus, the average value of a function is given by

Equation:

$$\frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \frac{1}{b-a} \int_a^b f(x) dx.$$

Note:

Definition

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the **average value of the function** $f(x)$ (or f_{ave}) on $[a, b]$ is given by

Equation:

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Example:

Exercise:

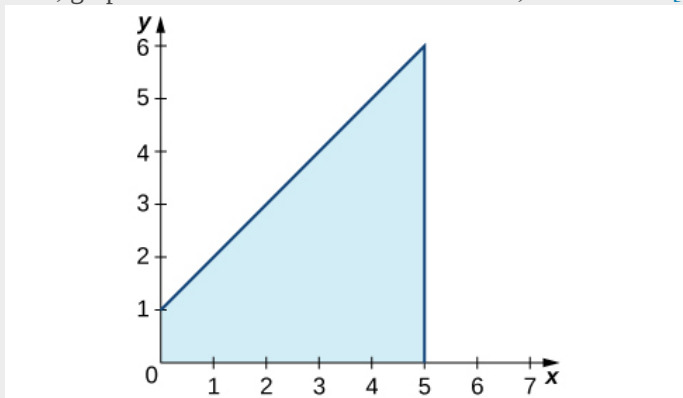
Problem:

Finding the Average Value of a Linear Function

Find the average value of $f(x) = x + 1$ over the interval $[0, 5]$.

Solution:

First, graph the function on the stated interval, as shown in [\[link\]](#).



The graph shows the area under the function $f(x) = x + 1$ over $[0, 5]$.

The region is a trapezoid lying on its side, so we can use the area formula for a trapezoid $A = \frac{1}{2}h(a + b)$, where h represents height, and a and b represent the two parallel sides. Then,

Equation:

$$\begin{aligned}\int_0^5 x + 1 dx &= \frac{1}{2}h(a + b) \\ &= \frac{1}{2} \cdot 5 \cdot (1 + 6) \\ &= \frac{35}{2}.\end{aligned}$$

Thus the average value of the function is

Equation:

$$\frac{1}{5 - 0} \int_0^5 x + 1 dx = \frac{1}{5} \cdot \frac{35}{2} = \frac{7}{2}.$$

Note:

Exercise:

Problem: Find the average value of $f(x) = 6 - 2x$ over the interval $[0, 3]$.

Solution:

3

Hint

Use the average value formula, and use geometry to evaluate the integral.

Key Concepts

- The definite integral can be used to calculate net signed area, which is the area above the x-axis less the area below the x-axis. Net signed area can be positive, negative, or zero.
- The component parts of the definite integral are the integrand, the variable of integration, and the limits of integration.
- Continuous functions on a closed interval are integrable. Functions that are not continuous may still be integrable, depending on the nature of the discontinuities.
- The properties of definite integrals can be used to evaluate integrals.
- The area under the curve of many functions can be calculated using geometric formulas.
- The average value of a function can be calculated using definite integrals.

Key Equations

- **Definite Integral**

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

- **Properties of the Definite Integral**

$$\int_a^a f(x)dx = 0$$

$$\int_b^a f(x)dx = - \int_a^b f(x)dx$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c \int_a^b f(x) \text{ for constant } c$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

In the following exercises, express the limits as integrals.

Exercise:

Problem: $\lim_{n \rightarrow \infty} \sum_{i=1}^n (x_i^*) \Delta x$ over $[1, 3]$

Exercise:

Problem: $\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(5(x_i^*)^2 - 3(x_i^*)^3 \right) \Delta x$ over $[0, 2]$

Solution:

$$\int_0^2 (5x^2 - 3x^3)dx$$

Exercise:

Problem: $\lim_{n \rightarrow \infty} \sum_{i=1}^n \sin^2(2\pi x_i^*) \Delta x$ over $[0, 1]$

Exercise:

Problem: $\lim_{n \rightarrow \infty} \sum_{i=1}^n \cos^2(2\pi x_i^*) \Delta x$ over $[0, 1]$

Solution:

$$\int_0^1 \cos^2(2\pi x) dx$$

In the following exercises, given L_n or R_n as indicated, express their limits as $n \rightarrow \infty$ as definite integrals, identifying the correct intervals.

Exercise:

Problem: $L_n = \frac{1}{n} \sum_{i=1}^n \frac{i-1}{n}$

Exercise:

Problem: $R_n = \frac{1}{n} \sum_{i=1}^n \frac{i}{n}$

Solution:

$$\int_0^1 x dx$$

Exercise:

Problem: $L_n = \frac{2}{n} \sum_{i=1}^n \left(1 + 2 \frac{i-1}{n}\right)$

Exercise:

Problem: $R_n = \frac{3}{n} \sum_{i=1}^n \left(3 + 3 \frac{i}{n}\right)$

Solution:

$$\int_3^6 x dx$$

Exercise:

Problem: $L_n = \frac{2\pi}{n} \sum_{i=1}^n 2\pi \frac{i-1}{n} \cos\left(2\pi \frac{i-1}{n}\right)$

Exercise:

Problem: $R_n = \frac{1}{n} \sum_{i=1}^n \left(1 + \frac{i}{n}\right) \log\left(\left(1 + \frac{i}{n}\right)^2\right)$

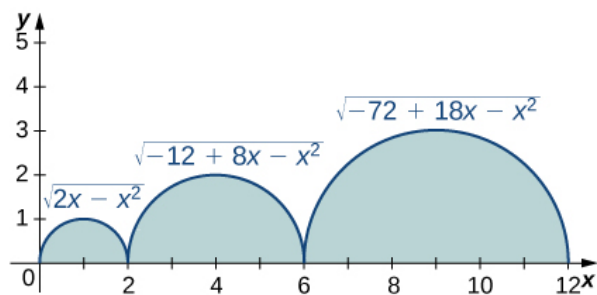
Solution:

$$\int_1^2 x \log(x^2) dx$$

In the following exercises, evaluate the integrals of the functions graphed using the formulas for areas of triangles and circles, and subtracting the areas below the x-axis.

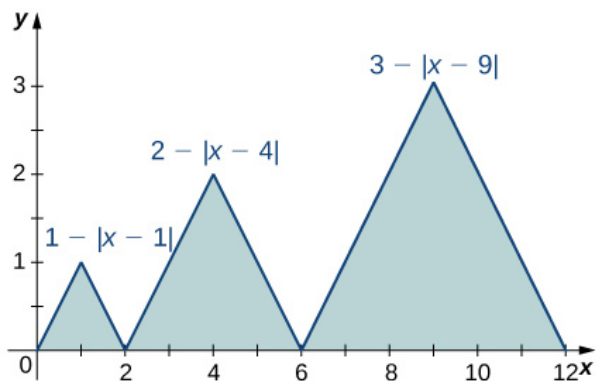
Exercise:

Problem:



Exercise:

Problem:

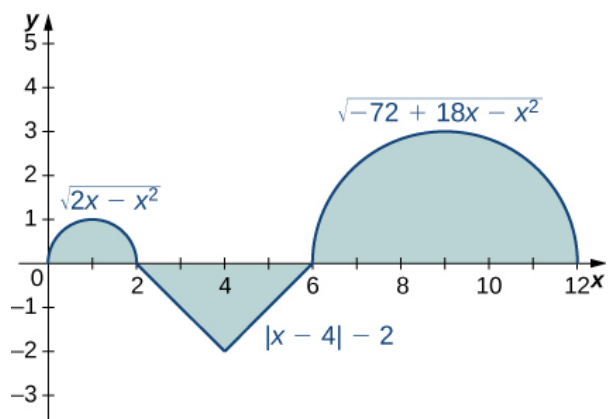


Solution:

$$1 + 2 \cdot 2 + 3 \cdot 3 = 14$$

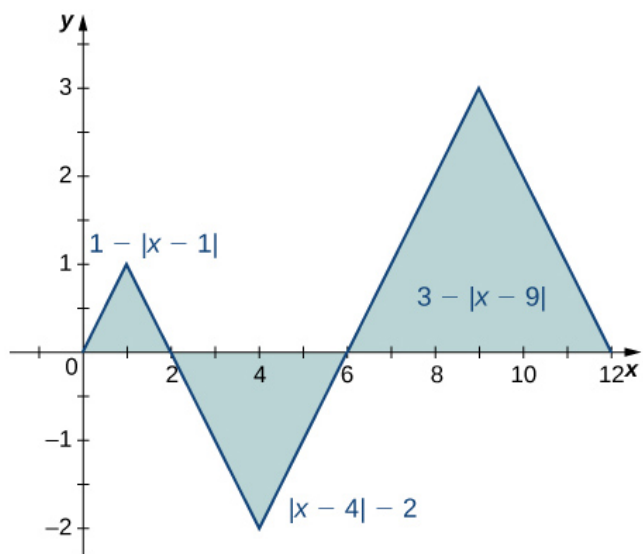
Exercise:

Problem:



Exercise:

Problem:

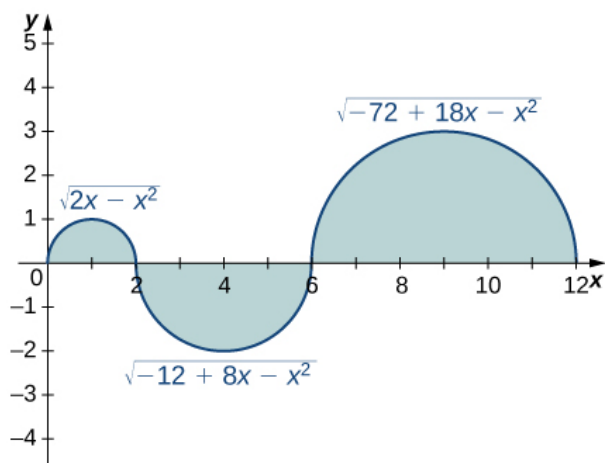


Solution:

$$1 - 4 + 9 = 6$$

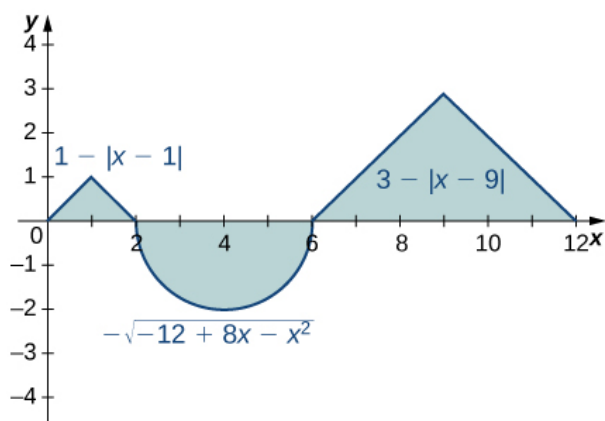
Exercise:

Problem:



Exercise:

Problem:



Solution:

$$1 - 2\pi + 9 = 10 - 2\pi$$

In the following exercises, evaluate the integral using area formulas.

Exercise:

Problem: $\int_0^3 (3 - x) dx$

Exercise:

Problem: $\int_2^3 (3 - x) dx$

Solution:

The integral is the area of the triangle, $\frac{1}{2}$

Exercise:

Problem: $\int_{-3}^3 (3 - |x|) dx$

Exercise:

Problem: $\int_0^6 (3 - |x - 3|) dx$

Solution:

The integral is the area of the triangle, 9.

Exercise:

Problem: $\int_{-2}^2 \sqrt{4 - x^2} dx$

Exercise:

Problem: $\int_1^5 \sqrt{4 - (x - 3)^2} dx$

Solution:

The integral is the area $\frac{1}{2}\pi r^2 = 2\pi$.

Exercise:

Problem: $\int_0^{12} \sqrt{36 - (x - 6)^2} dx$

Exercise:

Problem: $\int_{-2}^3 (3 - |x|) dx$

Solution:

The integral is the area of the “big” triangle less the “missing” triangle, $9 - \frac{1}{2}$.

In the following exercises, use averages of values at the left (L) and right (R) endpoints to compute the integrals of the piecewise linear functions with graphs that pass through the given list of points over the indicated intervals.

Exercise:

Problem: $\{(0, 0), (2, 1), (4, 3), (5, 0), (6, 0), (8, 3)\}$ over $[0, 8]$

Exercise:

Problem: $\{(0, 2), (1, 0), (3, 5), (5, 5), (6, 2), (8, 0)\}$ over $[0, 8]$

Solution:

$$L = 2 + 0 + 10 + 5 + 4 = 21, R = 0 + 10 + 10 + 2 + 0 = 22, \frac{L+R}{2} = 21.5$$

Exercise:

Problem: $\{(-4, -4), (-2, 0), (0, -2), (3, 3), (4, 3)\}$ over $[-4, 4]$

Exercise:

Problem: $\{(-4, 0), (-2, 2), (0, 0), (1, 2), (3, 2), (4, 0)\}$ over $[-4, 4]$

Solution:

$$L = 0 + 4 + 0 + 4 + 2 = 10, R = 4 + 0 + 2 + 4 + 0 = 10, \frac{L+R}{2} = 10$$

Suppose that $\int_0^4 f(x)dx = 5$ and $\int_0^2 f(x)dx = -3$, and $\int_0^4 g(x)dx = -1$ and $\int_0^2 g(x)dx = 2$. In the following exercises, compute the integrals.

Exercise:

Problem: $\int_0^4 (f(x) + g(x))dx$

Exercise:

Problem: $\int_2^4 (f(x) + g(x))dx$

Solution:

$$\int_2^4 f(x)dx + \int_2^4 g(x)dx = 8 - 3 = 5$$

Exercise:

Problem: $\int_0^2 (f(x) - g(x))dx$

Exercise:

Problem: $\int_2^4 (f(x) - g(x))dx$

Solution:

$$\int_2^4 f(x)dx - \int_2^4 g(x)dx = 8 + 3 = 11$$

Exercise:

Problem: $\int_0^2 (3f(x) - 4g(x))dx$

Exercise:

Problem: $\int_2^4 (4f(x) - 3g(x))dx$

Solution:

$$4 \int_2^4 f(x)dx - 3 \int_2^4 g(x)dx = 32 + 9 = 41$$

In the following exercises, use the identity $\int_{-A}^A f(x)dx = \int_{-A}^0 f(x)dx + \int_0^A f(x)dx$ to compute the integrals.

Exercise:

Problem: $\int_{-\pi}^{\pi} \frac{\sin t}{1+t^2} dt$ (*Hint:* $\sin(-t) = -\sin(t)$)

Exercise:

Problem: $\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{t}{1+\cos t} dt$

Solution:

The integrand is odd; the integral is zero.

Exercise:

Problem: $\int_1^3 (2-x)dx$ (*Hint:* Look at the graph of f .)

Exercise:

Problem: $\int_2^4 (x-3)^3 dx$ (*Hint:* Look at the graph of f .)

Solution:

The integrand is antisymmetric with respect to $x = 3$. The integral is zero.

In the following exercises, given that $\int_0^1 x dx = \frac{1}{2}$, $\int_0^1 x^2 dx = \frac{1}{3}$, and $\int_0^1 x^3 dx = \frac{1}{4}$, compute the integrals.

Exercise:

Problem: $\int_0^1 (1 + x + x^2 + x^3) dx$

Exercise:

Problem: $\int_0^1 (1 - x + x^2 - x^3) dx$

Solution:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$$

Exercise:

Problem: $\int_0^1 (1 - x)^2 dx$

Exercise:

Problem: $\int_0^1 (1 - 2x)^3 dx$

Solution:

$$\int_0^1 (1 - 6x + 12x^2 - 8x^3) dx = \left(x - 3x^2 + 4x^3 - 2x^4 \right) = \left(1 - 3 + 4 - 2 \right) \left(0 - 0 + 0 - 0 \right) = 0$$

Exercise:

Problem: $\int_0^1 \left(6x - \frac{4}{3}x^2 \right) dx$

Exercise:

Problem: $\int_0^1 (7 - 5x^3) dx$

Solution:

$$7 - \frac{5}{4} = \frac{23}{4}$$

In the following exercises, use the [comparison theorem](#).

Exercise:

Problem: Show that $\int_0^3 (x^2 - 6x + 9) dx \geq 0$.

Exercise:

Problem: Show that $\int_{-2}^3 (x-3)(x+2)dx \leq 0$.

Solution:

The integrand is negative over $[-2, 3]$.

Exercise:

Problem: Show that $\int_0^1 \sqrt{1+x^3}dx \leq \int_0^1 \sqrt{1+x^2}dx$.

Exercise:

Problem: Show that $\int_1^2 \sqrt{1+x}dx \leq \int_1^2 \sqrt{1+x^2}dx$.

Solution:

$x \leq x^2$ over $[1, 2]$, so $\sqrt{1+x} \leq \sqrt{1+x^2}$ over $[1, 2]$.

Exercise:

Problem: Show that $\int_0^{\pi/2} \sin t dt \geq \frac{\pi}{4}$. (*Hint:* $\sin t \geq \frac{2t}{\pi}$ over $[0, \frac{\pi}{2}]$)

Exercise:

Problem: Show that $\int_{-\pi/4}^{\pi/4} \cos t dt \geq \pi\sqrt{2}/4$.

Solution:

$\cos(t) \geq \frac{\sqrt{2}}{2}$. Multiply by the length of the interval to get the inequality.

In the following exercises, find the average value f_{ave} of f between a and b , and find a point c , where $f(c) = f_{\text{ave}}$.

Exercise:

Problem: $f(x) = x^2, a = -1, b = 1$

Exercise:

Problem: $f(x) = x^5, a = -1, b = 1$

Solution:

$f_{\text{ave}} = 0; c = 0$

Exercise:

Problem: $f(x) = \sqrt{4 - x^2}$, $a = 0$, $b = 2$

Exercise:

Problem: $f(x) = (3 - |x|)$, $a = -3$, $b = 3$

Solution:

$$\frac{3}{2} \text{ when } c = \pm \frac{3}{2}$$

Exercise:

Problem: $f(x) = \sin x$, $a = 0$, $b = 2\pi$

Exercise:

Problem: $f(x) = \cos x$, $a = 0$, $b = 2\pi$

Solution:

$$f_{\text{ave}} = 0; c = \frac{\pi}{2}, \frac{3\pi}{2}$$

In the following exercises, approximate the average value using Riemann sums L_{100} and R_{100} . How does your answer compare with the exact given answer?

Exercise:

Problem: [T] $y = \ln(x)$ over the interval $[1, 4]$; the exact solution is $\frac{\ln(256)}{3} - 1$.

Exercise:

Problem: [T] $y = e^{x/2}$ over the interval $[0, 1]$; the exact solution is $2(\sqrt{e} - 1)$.

Solution:

$$L_{100} = 1.294, R_{100} = 1.301; \text{ the exact average is between these values.}$$

Exercise:

Problem: [T] $y = \tan x$ over the interval $[0, \frac{\pi}{4}]$; the exact solution is $\frac{2\ln(2)}{\pi}$.

Exercise:

Problem: [T] $y = \frac{x+1}{\sqrt{4-x^2}}$ over the interval $[-1, 1]$; the exact solution is $\frac{\pi}{6}$.

Solution:

$$L_{100} \times \left(\frac{1}{2}\right) = 0.5178, R_{100} \times \left(\frac{1}{2}\right) = 0.5294$$

In the following exercises, compute the average value using the left Riemann sums L_N for $N = 1, 10, 100$. How does the accuracy compare with the given exact value?

Exercise:

Problem: [T] $y = x^2 - 4$ over the interval $[0, 2]$; the exact solution is $-\frac{8}{3}$.

Exercise:

Problem: [T] $y = xe^{x^2}$ over the interval $[0, 2]$; the exact solution is $\frac{1}{4}(e^4 - 1)$.

Solution:

$L_1 = 0, L_{10} \times \left(\frac{1}{2}\right) = 8.743493, L_{100} \times \left(\frac{1}{2}\right) = 12.861728$. The exact answer ≈ 26.799 , so L_{100} is not accurate.

Exercise:

Problem: [T] $y = \left(\frac{1}{2}\right)^x$ over the interval $[0, 4]$; the exact solution is $\frac{15}{64\ln(2)}$.

Exercise:

Problem: [T] $y = x \sin(x^2)$ over the interval $[-\pi, 0]$; the exact solution is $\frac{\cos(\pi^2)-1}{2\pi}$.

Solution:

$L_1 \times \left(\frac{1}{\pi}\right) = 1.352, L_{10} \times \left(\frac{1}{\pi}\right) = -0.1837, L_{100} \times \left(\frac{1}{\pi}\right) = -0.2956$. The exact answer ≈ -0.303 , so L_{100} is not accurate to first decimal.

Exercise:

Problem: Suppose that $A = \int_0^{2\pi} \sin^2 t dt$ and $B = \int_0^{2\pi} \cos^2 t dt$. Show that $A + B = 2\pi$ and $A = B$.

Exercise:

Problem: Suppose that $A = \int_{-\pi/4}^{\pi/4} \sec^2 t dt = \pi$ and $B = \int_{-\pi/4}^{\pi/4} \tan^2 t dt$. Show that $A - B = \frac{\pi}{2}$.

Solution:

Use $\tan^2 \theta + 1 = \sec^2 \theta$. Then, $B - A = \int_{-\pi/4}^{\pi/4} 1 dx = \frac{\pi}{2}$.

Exercise:

Problem:

Show that the average value of $\sin^2 t$ over $[0, 2\pi]$ is equal to $1/2$. Without further calculation, determine whether the average value of $\sin^2 t$ over $[0, \pi]$ is also equal to $1/2$.

Exercise:

Problem:

Show that the average value of $\cos^2 t$ over $[0, 2\pi]$ is equal to $1/2$. Without further calculation, determine whether the average value of $\cos^2(t)$ over $[0, \pi]$ is also equal to $1/2$.

Solution:

$\int_0^{2\pi} \cos^2 t dt = \pi$, so divide by the length 2π of the interval. $\cos^2 t$ has period π , so yes, it is true.

Exercise:**Problem:**

Explain why the graphs of a quadratic function (parabola) $p(x)$ and a linear function $\ell(x)$ can intersect in at most two points. Suppose that $p(a) = \ell(a)$ and $p(b) = \ell(b)$, and that $\int_a^b p(t) dt > \int_a^b \ell(t) dt$.

Explain why $\int_c^d p(t) dt > \int_c^d \ell(t) dt$ whenever $a \leq c < d \leq b$.

Exercise:**Problem:**

Suppose that parabola $p(x) = ax^2 + bx + c$ opens downward ($a < 0$) and has a vertex of $y = \frac{-b}{2a} > 0$. For which interval $[A, B]$ is $\int_A^B (ax^2 + bx + c) dx$ as large as possible?

Solution:

The integral is maximized when one uses the largest interval on which p is nonnegative. Thus, $A = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $B = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$.

Exercise:**Problem:**

Suppose $[a, b]$ can be subdivided into subintervals $a = a_0 < a_1 < a_2 < \cdots < a_N = b$ such that either $f \geq 0$ over $[a_{i-1}, a_i]$ or $f \leq 0$ over $[a_{i-1}, a_i]$. Set $A_i = \int_{a_{i-1}}^{a_i} f(t) dt$.

a. Explain why $\int_a^b f(t) dt = A_1 + A_2 + \cdots + A_N$.

b. Then, explain why $\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt$.

Exercise:**Problem:**

Suppose f and g are continuous functions such that $\int_c^d f(t) dt \leq \int_c^d g(t) dt$ for every subinterval $[c, d]$ of $[a, b]$. Explain why $f(x) \leq g(x)$ for all values of x .

Solution:

If $f(t_0) > g(t_0)$ for some $t_0 \in [a, b]$, then since $f - g$ is continuous, there is an interval containing t_0 such that $f(t) > g(t)$ over the interval $[c, d]$, and then $\int_c^d f(t) dt > \int_c^d g(t) dt$ over this interval.

Exercise:**Problem:**

Suppose the average value of f over $[a, b]$ is 1 and the average value of f over $[b, c]$ is 1 where $a < c < b$. Show that the average value of f over $[a, c]$ is also 1.

Exercise:**Problem:**

Suppose that $[a, b]$ can be partitioned, taking $a = a_0 < a_1 < \cdots < a_N = b$ such that the average value of f over each subinterval $[a_{i-1}, a_i]$ is equal to 1 for each $i = 1, \dots, N$. Explain why the average value of f over $[a, b]$ is also equal to 1.

Solution:

The integral of f over an interval is the same as the integral of the average of f over that interval. Thus,

$$\begin{aligned} \int_a^b f(t) dt &= \int_{a_0}^{a_1} f(t) dt + \int_{a_1}^{a_2} f(t) dt + \cdots + \int_{a_{N-1}}^{a_N} f(t) dt = \int_{a_0}^{a_1} 1 dt + \int_{a_1}^{a_2} 1 dt + \cdots + \int_{a_{N-1}}^{a_N} 1 dt \\ &= (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_N - a_{N-1}) = a_N - a_0 = b - a. \end{aligned}$$

Dividing through by $b - a$ gives the desired identity.

Exercise:**Problem:**

Suppose that for each i such that $1 \leq i \leq N$ one has $\int_{i-1}^i f(t) dt = i$. Show that

$$\int_0^N f(t) dt = \frac{N(N+1)}{2}.$$

Exercise:**Problem:**

Suppose that for each i such that $1 \leq i \leq N$ one has $\int_{i-1}^i f(t) dt = i^2$. Show that

$$\int_0^N f(t) dt = \frac{N(N+1)(2N+1)}{6}.$$

Solution:

$$\int_0^N f(t) dt = \sum_{i=1}^N \int_{i-1}^i f(t) dt = \sum_{i=1}^N i^2 = \frac{N(N+1)(2N+1)}{6}$$

Exercise:**Problem:**

[T] Compute the left and right Riemann sums L_{10} and R_{10} and their average $\frac{L_{10}+R_{10}}{2}$ for $f(t) = t^2$ over $[0, 1]$. Given that $\int_0^1 t^2 dt = \frac{1}{3}$, to how many decimal places is $\frac{L_{10}+R_{10}}{2}$ accurate?

Exercise:

Problem:

[T] Compute the left and right Riemann sums, L_{10} and R_{10} , and their average $\frac{L_{10}+R_{10}}{2}$ for $f(t) = (4 - t^2)$ over $[1, 2]$. Given that $\int_1^2 (4 - t^2) dt = 1.6\bar{6}$, to how many decimal places is $\frac{L_{10}+R_{10}}{2}$ accurate?

Solution:

$L_{10} = 1.815$, $R_{10} = 1.515$, $\frac{L_{10}+R_{10}}{2} = 1.665$, so the estimate is accurate to two decimal places.

Exercise:

Problem: If $\int_1^5 \sqrt{1+t^4} dt = 41.7133\dots$, what is $\int_1^5 \sqrt{1+u^4} du$?

Exercise:**Problem:**

Estimate $\int_0^1 t dt$ using the left and right endpoint sums, each with a single rectangle. How does the average of these left and right endpoint sums compare with the actual value $\int_0^1 t dt$?

Solution:

The average is $1/2$, which is equal to the integral in this case.

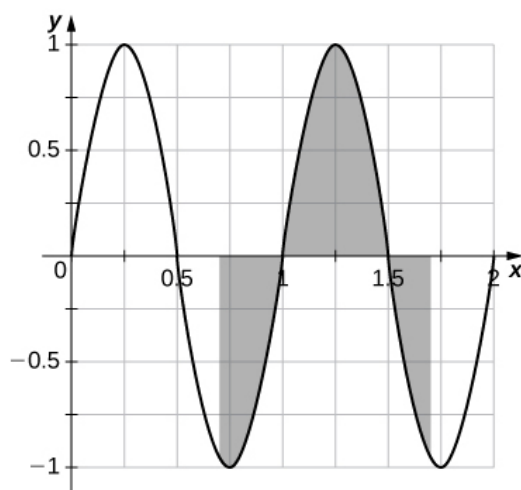
Exercise:**Problem:**

Estimate $\int_0^1 t dt$ by comparison with the area of a single rectangle with height equal to the value of t at the midpoint $t = \frac{1}{2}$. How does this midpoint estimate compare with the actual value $\int_0^1 t dt$?

Exercise:

Problem: From the graph of $\sin(2\pi x)$ shown:

- Explain why $\int_0^1 \sin(2\pi t) dt = 0$.
- Explain why, in general, $\int_a^{a+1} \sin(2\pi t) dt = 0$ for any value of a .



Solution:

a. The graph is antisymmetric with respect to $t = \frac{1}{2}$ over $[0, 1]$, so the average value is zero. b. For any value of a , the graph between $[a, a + 1]$ is a shift of the graph over $[0, 1]$, so the net areas above and below the axis do not change and the average remains zero.

Exercise:

Problem:

If f is 1-periodic ($f(t + 1) = f(t)$), odd, and integrable over $[0, 1]$, is it always true that

$$\int_0^1 f(t) dt = 0?$$

Exercise:

Problem: If f is 1-periodic and $\int_0^1 f(t) dt = A$, is it necessarily true that $\int_a^{1+a} f(t) dt = A$ for all A ?

Solution:

Yes, the integral over any interval of length 1 is the same.

Glossary

average value of a function

(or f_{ave}) the average value of a function on an interval can be found by calculating the definite integral of the function and dividing that value by the length of the interval

definite integral

a primary operation of calculus; the area between the curve and the x-axis over a given interval is a definite integral

integrable function

a function is integrable if the limit defining the integral exists; in other words, if the limit of the Riemann sums as n goes to infinity exists

integrand

the function to the right of the integration symbol; the integrand includes the function being integrated

limits of integration

these values appear near the top and bottom of the integral sign and define the interval over which the function should be integrated

net signed area

the area between a function and the x -axis such that the area below the x -axis is subtracted from the area above the x -axis; the result is the same as the definite integral of the function

total area

total area between a function and the x -axis is calculated by adding the area above the x -axis and the area below the x -axis; the result is the same as the definite integral of the absolute value of the function

variable of integration

indicates which variable you are integrating with respect to; if it is x , then the function in the integrand is followed by dx

The Fundamental Theorem of Calculus

- Describe the meaning of the Mean Value Theorem for Integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 1.
- Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 2.
- Use the Fundamental Theorem of Calculus, Part 2, to evaluate definite integrals.
- Explain the relationship between differentiation and integration.

In the previous two sections, we looked at the definite integral and its relationship to the area under the curve of a function. Unfortunately, so far, the only tools we have available to calculate the value of a definite integral are geometric area formulas and limits of Riemann sums, and both approaches are extremely cumbersome. In this section we look at some more powerful and useful techniques for evaluating definite integrals.

These new techniques rely on the relationship between differentiation and integration. This relationship was discovered and explored by both Sir Isaac Newton and Gottfried Wilhelm Leibniz (among others) during the late 1600s and early 1700s, and it is codified in what we now call the **Fundamental Theorem of Calculus**, which has two parts that we examine in this section. Its very name indicates how central this theorem is to the entire development of calculus.

Note:

Isaac Newton's contributions to mathematics and physics changed the way we look at the world. The relationships he discovered, codified as Newton's laws and the law of universal gravitation, are still taught as foundational material in physics today, and his calculus has spawned entire fields of mathematics. To learn more, read a [brief biography](#) of Newton with multimedia clips.

Before we get to this crucial theorem, however, let's examine another important theorem, the Mean Value Theorem for Integrals, which is needed to prove the Fundamental Theorem of Calculus.

The Mean Value Theorem for Integrals

The **Mean Value Theorem for Integrals** states that a continuous function on a closed interval takes on its average value at the same point in that interval. The theorem guarantees that if $f(x)$ is continuous, a point c exists in an interval $[a, b]$ such that the value of the function at c is equal to the average value of $f(x)$ over $[a, b]$. We state this theorem mathematically with the help of the formula for the average value of a function that we presented at the end of the preceding section.

Note:**The Mean Value Theorem for Integrals**

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

Equation:

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$$

This formula can also be stated as

Equation:

$$\int_a^b f(x) dx = f(c) (b-a).$$

Proof

Since $f(x)$ is continuous on $[a, b]$, by the extreme value theorem (see [Maxima and Minima](#)), it assumes minimum and maximum values— m and M , respectively—on $[a, b]$. Then, for all x in $[a, b]$, we have $m \leq f(x) \leq M$. Therefore, by the comparison theorem (see [The Definite Integral](#)), we have

Equation:

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a).$$

Dividing by $b-a$ gives us

Equation:

$$m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M.$$

Since $\frac{1}{b-a} \int_a^b f(x) dx$ is a number between m and M , and since $f(x)$ is continuous and assumes the values m and M over $[a, b]$, by the Intermediate Value Theorem (see [Continuity](#)), there is a number c over $[a, b]$ such that

Equation:

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx,$$

and the proof is complete.

□

Example:

Exercise:

Problem:

Finding the Average Value of a Function

Find the average value of the function $f(x) = 8 - 2x$ over the interval $[0, 4]$ and find c such that $f(c)$ equals the average value of the function over $[0, 4]$.

Solution:

The formula states the mean value of $f(x)$ is given by

Equation:

$$\frac{1}{4 - 0} \int_0^4 (8 - 2x) dx.$$

We can see in [\[link\]](#) that the function represents a straight line and forms a right triangle bounded by the x - and y -axes. The area of the triangle is $A = \frac{1}{2}(\text{base})(\text{height})$. We have

Equation:

$$A = \frac{1}{2}(4)(8) = 16.$$

The average value is found by multiplying the area by $1/(4 - 0)$. Thus, the average value of the function is

Equation:

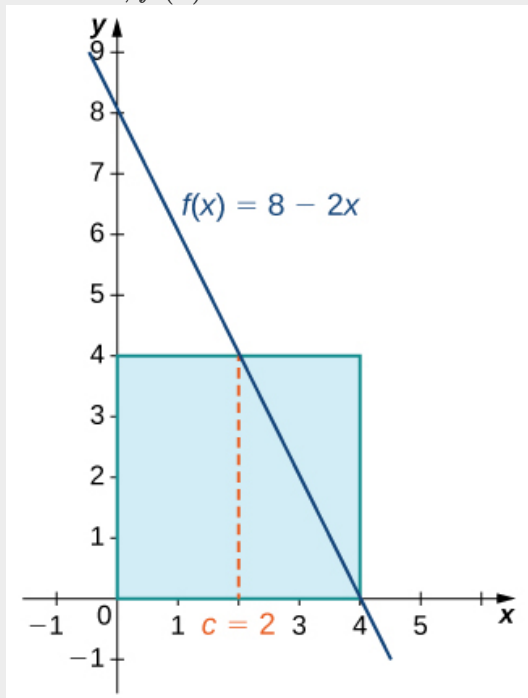
$$\frac{1}{4}(16) = 4.$$

Set the average value equal to $f(c)$ and solve for c .

Equation:

$$\begin{aligned} 8 - 2c &= 4 \\ c &= 2 \end{aligned}$$

At $c = 2$, $f(2) = 4$.



By the Mean Value Theorem, the continuous function $f(x)$ takes on its average value at c at least once over a closed interval.

Note:

Exercise:

Problem:

Find the average value of the function $f(x) = \frac{x}{2}$ over the interval $[0, 6]$ and find c such that $f(c)$ equals the average value of the function over $[0, 6]$.

Solution:

Average value $= 1.5$; $c = 3$

Hint

Use the procedures from [\[link\]](#) to solve the problem

Example:

Exercise:

Problem:

Finding the Point Where a Function Takes on Its Average Value

Given $\int_0^3 x^2 dx = 9$, find c such that $f(c)$ equals the average value of $f(x) = x^2$ over $[0, 3]$.

Solution:

We are looking for the value of c such that

Equation:

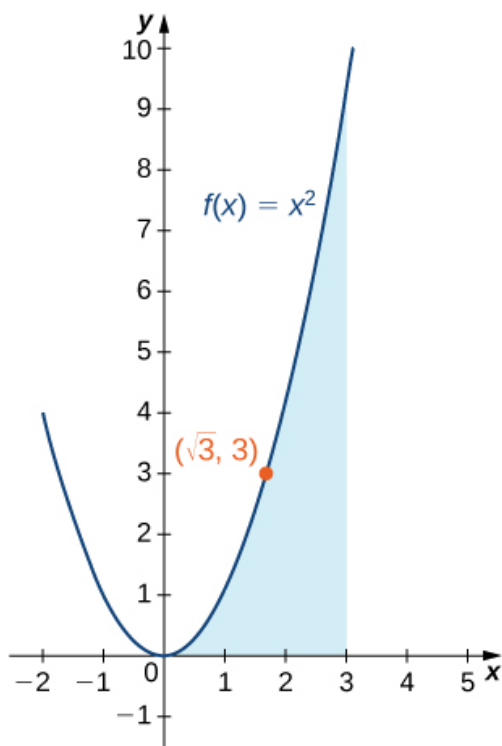
$$f(c) = \frac{1}{3-0} \int_0^3 x^2 dx = \frac{1}{3}(9) = 3.$$

Replacing $f(c)$ with c^2 , we have

Equation:

$$\begin{aligned} c^2 &= 3 \\ c &= \pm\sqrt{3}. \end{aligned}$$

Since $-\sqrt{3}$ is outside the interval, take only the positive value. Thus, $c = \sqrt{3}$ ([link](#)).



Over the interval $[0, 3]$, the function $f(x) = x^2$ takes on its average value at $c = \sqrt{3}$.

Note:

Exercise:

Problem:

Given $\int_0^3 (2x^2 - 1) dx = 15$, find c such that $f(c)$ equals the average value of $f(x) = 2x^2 - 1$ over $[0, 3]$.

Solution:

$$c = \sqrt{3}$$

Hint

Use the procedures from [\[link\]](#) to solve the problem.

Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

As mentioned earlier, the Fundamental Theorem of Calculus is an extremely powerful theorem that establishes the relationship between differentiation and integration, and gives us a way to evaluate definite integrals without using Riemann sums or calculating areas. The theorem is comprised of two parts, the first of which, the **Fundamental Theorem of Calculus, Part 1**, is stated here. Part 1 establishes the relationship between differentiation and integration.

Note:

Fundamental Theorem of Calculus, Part 1

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

Equation:

$$F(x) = \int_a^x f(t)dt,$$

then $F'(x) = f(x)$ over $[a, b]$.

Before we delve into the proof, a couple of subtleties are worth mentioning here. First, a comment on the notation. Note that we have defined a function, $F(x)$, as the definite integral of another function, $f(t)$, from the point a to the point x . At first glance, this is confusing, because we have said several times that a definite integral is a number, and here it looks like it's a function. The key here is to notice that for any particular value of x , the definite integral is a number. So the function $F(x)$ returns a number (the value of the definite integral) for each value of x .

Second, it is worth commenting on some of the key implications of this theorem. There is a reason it is called the *Fundamental* Theorem of Calculus. Not only does it establish a relationship between integration and differentiation, but also it guarantees that any integrable function has an antiderivative. Specifically, it guarantees that any continuous function has an antiderivative.

Proof

Applying the definition of the derivative, we have

Equation:

$$\begin{aligned}
F'(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t) dt - \int_a^x f(t) dt \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^{x+h} f(t) dt + \int_x^a f(t) dt \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(t) dt.
\end{aligned}$$

Looking carefully at this last expression, we see $\frac{1}{h} \int_x^{x+h} f(t) dt$ is just the average value of the function $f(x)$ over the interval $[x, x+h]$. Therefore, by [\[link\]](#), there is some number c in $[x, x+h]$ such that

Equation:

$$\frac{1}{h} \int_x^{x+h} f(x) dx = f(c).$$

In addition, since c is between x and $x+h$, c approaches x as h approaches zero. Also, since $f(x)$ is continuous, we have $\lim_{h \rightarrow 0} f(c) = \lim_{c \rightarrow x} f(c) = f(x)$. Putting all these pieces together, we have

Equation:

$$\begin{aligned}
F'(x) &= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} f(x) dx \\
&= \lim_{h \rightarrow 0} f(c) \\
&= f(x),
\end{aligned}$$

and the proof is complete.

□

Example:

Exercise:

Problem:

Finding a Derivative with the Fundamental Theorem of Calculus

Use the [\[link\]](#) to find the derivative of

Equation:

$$g(x) = \int_1^x \frac{1}{t^3 + 1} dt.$$

Solution:

According to the Fundamental Theorem of Calculus, the derivative is given by

Equation:

$$g'(x) = \frac{1}{x^3 + 1}.$$

Note:

Exercise:

Problem:

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of

$$g(r) = \int_0^r \sqrt{x^2 + 4} dx.$$

Solution:

$$g'(r) = \sqrt{r^2 + 4}$$

Hint

Follow the procedures from [\[link\]](#) to solve the problem.

Example:

Exercise:

Problem:

Using the Fundamental Theorem and the Chain Rule to Calculate Derivatives

Let $F(x) = \int_1^{\sqrt{x}} \sin t dt$. Find $F'(x)$.

Solution:

Letting $u(x) = \sqrt{x}$, we have $F(x) = \int_1^{u(x)} \sin t dt$. Thus, by the Fundamental Theorem of Calculus and the chain rule,

Equation:

$$\begin{aligned} F'(x) &= \sin(u(x)) \frac{du}{dx} \\ &= \sin(u(x)) \cdot \left(\frac{1}{2}x^{-1/2}\right) \\ &= \frac{\sin\sqrt{x}}{2\sqrt{x}}. \end{aligned}$$

Note:

Exercise:

Problem: Let $F(x) = \int_1^{x^3} \cos t dt$. Find $F'(x)$.

Solution:

$$F'(x) = 3x^2 \cos x^3$$

Hint

Use the chain rule to solve the problem.

Example:

Exercise:

Problem:

Using the Fundamental Theorem of Calculus with Two Variable Limits of Integration

Let $F(x) = \int_x^{2x} t^3 dt$. Find $F'(x)$.

Solution:

We have $F(x) = \int_x^{2x} t^3 dt$. Both limits of integration are variable, so we need to split this into two integrals. We get

Equation:

$$\begin{aligned}
 F(x) &= \int_x^{2x} t^3 dt \\
 &= \int_x^0 t^3 dt + \int_0^{2x} t^3 dt \\
 &= -\int_0^x t^3 dt + \int_0^{2x} t^3 dt.
 \end{aligned}$$

Differentiating the first term, we obtain

Equation:

$$\frac{d}{dx} \left[-\int_0^x t^3 dt \right] = -x^3.$$

Differentiating the second term, we first let $u(x) = 2x$. Then,

Equation:

$$\begin{aligned}
 \frac{d}{dx} \left[\int_0^{2x} t^3 dt \right] &= \frac{d}{dx} \left[\int_0^{u(x)} t^3 dt \right] \\
 &= (u(x))^3 \frac{du}{dx} \\
 &= (2x)^3 \cdot 2 \\
 &= 16x^3.
 \end{aligned}$$

Thus,

Equation:

$$\begin{aligned}
 F'(x) &= \frac{d}{dx} \left[-\int_0^x t^3 dt \right] + \frac{d}{dx} \left[\int_0^{2x} t^3 dt \right] \\
 &= -x^3 + 16x^3 \\
 &= 15x^3.
 \end{aligned}$$

Note:

Exercise:

Problem: Let $F(x) = \int_x^{x^2} \cos t \, dt$. Find $F'(x)$.

Solution:

$$F'(x) = 2x \cos x^2 - \cos x$$

Hint

Use the procedures from [\[link\]](#) to solve the problem.

Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

The Fundamental Theorem of Calculus, Part 2, is perhaps the most important theorem in calculus. After tireless efforts by mathematicians for approximately 500 years, new techniques emerged that provided scientists with the necessary tools to explain many phenomena. Using calculus, astronomers could finally determine distances in space and map planetary orbits. Everyday financial problems such as calculating marginal costs or predicting total profit could now be handled with simplicity and accuracy. Engineers could calculate the bending strength of materials or the three-dimensional motion of objects. Our view of the world was forever changed with calculus.

After finding approximate areas by adding the areas of n rectangles, the application of this theorem is straightforward by comparison. It almost seems too simple that the area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

Note:

The Fundamental Theorem of Calculus, Part 2

If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

Equation:

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

We often see the notation $F(x)|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated limits a and b to indicate that we should evaluate the function $F(x)$ at the

upper limit (in this case, b), and subtract the value of the function $F(x)$ evaluated at the lower limit (in this case, a).

The **Fundamental Theorem of Calculus, Part 2** (also known as the **evaluation theorem**) states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

Proof

Let $P = \{x_i\}, i = 0, 1, \dots, n$ be a regular partition of $[a, b]$. Then, we can write

Equation:

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \dots + [F(x_1) - F(x_0)] \\ &= \sum_{i=1}^n [F(x_i) - F(x_{i-1})]. \end{aligned}$$

Now, we know F is an antiderivative of f over $[a, b]$, so by the Mean Value Theorem (see [The Mean Value Theorem](#)) for $i = 0, 1, \dots, n$ we can find c_i in $[x_{i-1}, x_i]$ such that

Equation:

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\Delta x.$$

Then, substituting into the previous equation, we have

Equation:

$$F(b) - F(a) = \sum_{i=1}^n f(c_i)\Delta x.$$

Taking the limit of both sides as $n \rightarrow \infty$, we obtain

Equation:

$$\begin{aligned} F(b) - F(a) &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i)\Delta x \\ &= \int_a^b f(x)dx. \end{aligned}$$

□

Example:

Exercise:

Problem:

Evaluating an Integral with the Fundamental Theorem of Calculus

Use [\[link\]](#) to evaluate

Equation:

$$\int_{-2}^2 (t^2 - 4) dt.$$

Solution:

Recall the power rule for [Antiderivatives](#):

Equation:

$$\text{If } y = x^n, \int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

Use this rule to find the antiderivative of the function and then apply the theorem. We have

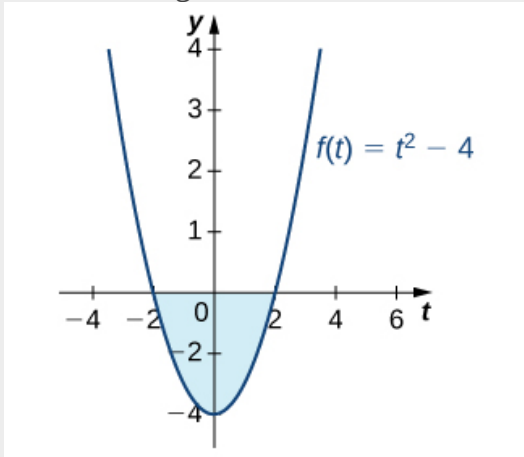
Equation:

$$\begin{aligned} \int_{-2}^2 (t^2 - 4) dt &= \left. \frac{t^3}{3} - 4t \right|_{-2}^2 \\ &= \left[\frac{(2)^3}{3} - 4(2) \right] - \left[\frac{(-2)^3}{3} - 4(-2) \right] \\ &= \left(\frac{8}{3} - 8 \right) - \left(-\frac{8}{3} + 8 \right) \\ &= \frac{8}{3} - 8 + \frac{8}{3} - 8 \\ &= \frac{16}{3} - 16 \\ &= -\frac{32}{3}. \end{aligned}$$

Analysis

Notice that we did not include the “+ C” term when we wrote the antiderivative. The reason is that, according to the Fundamental Theorem of Calculus, Part 2, *any* antiderivative works. So, for convenience, we chose the antiderivative with $C = 0$. If we had chosen another antiderivative, the constant term would have canceled out. This always happens when evaluating a definite integral.

The region of the area we just calculated is depicted in [\[link\]](#). Note that the region between the curve and the x-axis is all below the x-axis. Area is always positive, but a definite integral can still produce a negative number (a net signed area). For example, if this were a profit function, a negative number indicates the company is operating at a loss over the given interval.



The evaluation of a definite integral can produce a negative value, even though area is always positive.

Example:

Exercise:

Problem:

Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2

Evaluate the following integral using the Fundamental Theorem of Calculus, Part 2:

Equation:

$$\int_1^9 \frac{x-1}{\sqrt{x}} dx.$$

Solution:

First, eliminate the radical by rewriting the integral using rational exponents. Then, separate the numerator terms by writing each one over the denominator:

Equation:

$$\int_1^9 \frac{x-1}{x^{1/2}} dx = \int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}} \right) dx.$$

Use the properties of exponents to simplify:

Equation:

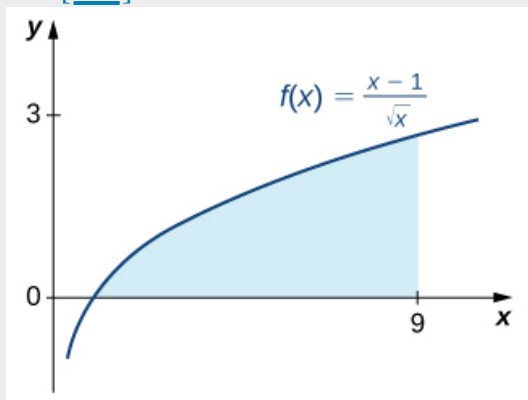
$$\int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}} \right) dx = \int_1^9 \left(x^{1/2} - x^{-1/2} \right) dx.$$

Now, integrate using the power rule:

Equation:

$$\begin{aligned} \int_1^9 \left(x^{1/2} - x^{-1/2} \right) dx &= \left(\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{1/2}}{\frac{1}{2}} \right) \Big|_1^9 \\ &= \left[\frac{(9)^{3/2}}{\frac{3}{2}} - \frac{(9)^{1/2}}{\frac{1}{2}} \right] - \left[\frac{(1)^{3/2}}{\frac{3}{2}} - \frac{(1)^{1/2}}{\frac{1}{2}} \right] \\ &= \left[\frac{2}{3} (27) - 2 (3) \right] - \left[\frac{2}{3} (1) - 2 (1) \right] \\ &= 18 - 6 - \frac{2}{3} + 2 \\ &= \frac{40}{3}. \end{aligned}$$

See [\[link\]](#).



The area under the curve from $x = 1$ to $x = 9$ can be calculated by evaluating a definite integral.

Note:

Exercise:

Problem: Use [\[link\]](#) to evaluate $\int_1^2 x^{-4} dx$.

Solution:

$$\frac{7}{24}$$

Hint

Use the power rule.

Example:

Exercise:

Problem:

A Roller-Skating Race

James and Kathy are racing on roller skates. They race along a long, straight track, and whoever has gone the farthest after 5 sec wins a prize. If James can skate at a velocity of $f(t) = 5 + 2t$ ft/sec and Kathy can skate at a velocity of $g(t) = 10 + \cos\left(\frac{\pi}{2}t\right)$ ft/sec, who is going to win the race?

Solution:

We need to integrate both functions over the interval $[0, 5]$ and see which value is bigger. For James, we want to calculate

Equation:

$$\int_0^5 (5 + 2t) dt.$$

Using the power rule, we have

Equation:

$$\begin{aligned} \int_0^5 (5 + 2t) dt &= (5t + t^2) \Big|_0^5 \\ &= (25 + 25) = 50. \end{aligned}$$

Thus, James has skated 50 ft after 5 sec. Turning now to Kathy, we want to calculate

Equation:

$$\int_0^5 10 + \cos\left(\frac{\pi}{2}t\right) dt.$$

We know $\sin t$ is an antiderivative of $\cos t$, so it is reasonable to expect that an antiderivative of $\cos\left(\frac{\pi}{2}t\right)$ would involve $\sin\left(\frac{\pi}{2}t\right)$. However, when we differentiate $\sin\left(\frac{\pi}{2}t\right)$, we get $\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)$ as a result of the chain rule, so we have to account for this additional coefficient when we integrate. We obtain

Equation:

$$\begin{aligned}\int_0^5 10 + \cos\left(\frac{\pi}{2}t\right) dt &= \left(10t + \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\right)\Big|_0^5 \\ &= \left(50 + \frac{2}{\pi}\right) - \left(0 - \frac{2}{\pi}\sin 0\right) \\ &\approx 50.6.\end{aligned}$$

Kathy has skated approximately 50.6 ft after 5 sec. Kathy wins, but not by much!

Note:

Exercise:

Problem:

Suppose James and Kathy have a rematch, but this time the official stops the contest after only 3 sec. Does this change the outcome?

Solution:

Kathy still wins, but by a much larger margin: James skates 24 ft in 3 sec, but Kathy skates 29.3634 ft in 3 sec.

Hint

Change the limits of integration from those in [\[link\]](#).

Note:

A Parachutist in Free Fall



Skydivers can adjust the velocity of their dive by changing the position of their body during the free fall. (credit: Jeremy T. Lock)

Julie is an avid skydiver. She has more than 300 jumps under her belt and has mastered the art of making adjustments to her body position in the air to control how fast she falls. If she arches her back and points her belly toward the ground, she reaches a terminal velocity of approximately 120 mph (176 ft/sec). If, instead, she orients her body with her head straight down, she falls faster, reaching a terminal velocity of 150 mph (220 ft/sec).

Since Julie will be moving (falling) in a downward direction, we assume the downward direction is positive to simplify our calculations. Julie executes her jumps from an altitude of 12,500 ft. After she exits the aircraft, she immediately starts falling at a velocity given by $v(t) = 32t$. She continues to accelerate according to this velocity function until she reaches terminal velocity. After she reaches terminal velocity, her speed remains constant until she pulls her ripcord and slows down to land.

On her first jump of the day, Julie orients herself in the slower “belly down” position (terminal velocity is 176 ft/sec). Using this information, answer the following questions.

1. How long after she exits the aircraft does Julie reach terminal velocity?
2. Based on your answer to question 1, set up an expression involving one or more integrals that represents the distance Julie falls after 30 sec.
3. If Julie pulls her ripcord at an altitude of 3000 ft, how long does she spend in a free fall?

4. Julie pulls her ripcord at 3000 ft. It takes 5 sec for her parachute to open completely and for her to slow down, during which time she falls another 400 ft. After her canopy is fully open, her speed is reduced to 16 ft/sec. Find the total time Julie spends in the air, from the time she leaves the airplane until the time her feet touch the ground.

On Julie's second jump of the day, she decides she wants to fall a little faster and orients herself in the "head down" position. Her terminal velocity in this position is 220 ft/sec.

Answer these questions based on this velocity:

5. How long does it take Julie to reach terminal velocity in this case?
6. Before pulling her ripcord, Julie reorients her body in the "belly down" position so she is not moving quite as fast when her parachute opens. If she begins this maneuver at an altitude of 4000 ft, how long does she spend in a free fall before beginning the reorientation?

Some jumpers wear "wingsuits" (see [link](#)). These suits have fabric panels between the arms and legs and allow the wearer to glide around in a free fall, much like a flying squirrel. (Indeed, the suits are sometimes called "flying squirrel suits.") When wearing these suits, terminal velocity can be reduced to about 30 mph (44 ft/sec), allowing the wearers a much longer time in the air. Wingsuit flyers still use parachutes to land; although the vertical velocities are within the margin of safety, horizontal velocities can exceed 70 mph, much too fast to land safely.



The fabric panels on the arms and legs of a wingsuit work to reduce the vertical velocity of a skydiver's fall. (credit: Richard Schneider)

Answer the following question based on the velocity in a wingsuit.

7. If Julie dons a wingsuit before her third jump of the day, and she pulls her ripcord at an altitude of 3000 ft, how long does she get to spend gliding around in the air?

Key Concepts

- The Mean Value Theorem for Integrals states that for a continuous function over a closed interval, there is a value c such that $f(c)$ equals the average value of the function. See [\[link\]](#).
- The Fundamental Theorem of Calculus, Part 1 shows the relationship between the derivative and the integral. See [\[link\]](#).
- The Fundamental Theorem of Calculus, Part 2 is a formula for evaluating a definite integral in terms of an antiderivative of its integrand. The total area under a curve can be found using this formula. See [\[link\]](#).

Key Equations

- **Mean Value Theorem for Integrals**

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$

such that $f(c) = \frac{1}{b-a} \int_a^b f(x)dx$.

- **Fundamental Theorem of Calculus Part 1**

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t)dt, \text{ then } F'(x) = f(x).$$

- **Fundamental Theorem of Calculus Part 2**

If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Exercise:

Problem:

Consider two athletes running at variable speeds $v_1(t)$ and $v_2(t)$. The runners start and finish a race at exactly the same time. Explain why the two runners must be going the same speed at some point.

Exercise:

Problem:

Two mountain climbers start their climb at base camp, taking two different routes, one steeper than the other, and arrive at the peak at exactly the same time. Is it necessarily true that, at some point, both climbers increased in altitude at the same rate?

Solution:

Yes. It is implied by the Mean Value Theorem for Integrals.

Exercise:**Problem:**

To get on a certain toll road a driver has to take a card that lists the mile entrance point. The card also has a timestamp. When going to pay the toll at the exit, the driver is surprised to receive a speeding ticket along with the toll. Explain how this can happen.

Exercise:**Problem:**

Set $F(x) = \int_1^x (1-t)dt$. Find $F'(2)$ and the average value of F' over $[1, 2]$.

Solution:

$F'(2) = -1$; average value of F' over $[1, 2]$ is $-1/2$.

In the following exercises, use the Fundamental Theorem of Calculus, Part 1, to find each derivative.

Exercise:

Problem: $\frac{d}{dx} \int_1^x e^{-t^2} dt$

Exercise:

Problem: $\frac{d}{dx} \int_1^x e^{\cos t} dt$

Solution:

$e^{\cos t}$

Exercise:

Problem: $\frac{d}{dx} \int_3^x \sqrt{9-y^2} dy$

Exercise:

Problem: $\frac{d}{dx} \int_4^x \frac{ds}{\sqrt{16-s^2}}$

Solution:

$$\frac{1}{\sqrt{16-x^2}}$$

Exercise:

Problem: $\frac{d}{dx} \int_x^{2x} t dt$

Exercise:

Problem: $\frac{d}{dx} \int_0^{\sqrt{x}} t dt$

Solution:

$$\sqrt{x} \frac{d}{dx} \sqrt{x} = \frac{1}{2}$$

Exercise:

Problem: $\frac{d}{dx} \int_0^{\sin x} \sqrt{1-t^2} dt$

Exercise:

Problem: $\frac{d}{dx} \int_{\cos x}^1 \sqrt{1-t^2} dt$

Solution:

$$-\sqrt{1-\cos^2 x} \frac{d}{dx} \cos x = |\sin x| \sin x$$

Exercise:

Problem: $\frac{d}{dx} \int_1^{\sqrt{x}} \frac{t^2}{1+t^4} dt$

Exercise:

Problem: $\frac{d}{dx} \int_1^{x^2} \frac{\sqrt{t}}{1+t} dt$

Solution:

$$2x \frac{|x|}{1+x^2}$$

Exercise:

Problem: $\frac{d}{dx} \int_0^{\ln x} e^t dt$

Exercise:

Problem: $\frac{d}{dx} \int_1^{e^2} \ln u^2 du$

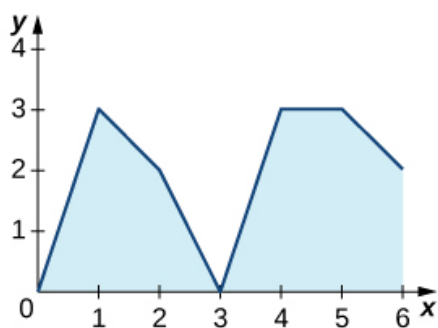
Solution:

$$\ln(e^{2x}) \frac{d}{dx} e^x = 2xe^x$$

Exercise:

Problem:

The graph of $y = \int_0^x f(t) dt$, where f is a piecewise constant function, is shown here.

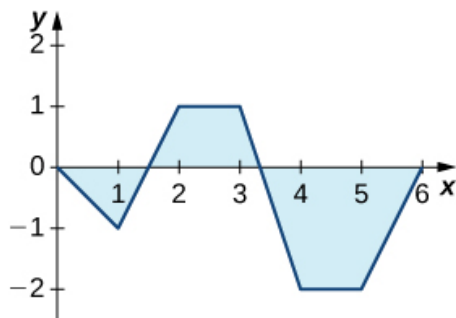


- Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?
- What are the maximum and minimum values of f ?
- What is the average value of f ?

Exercise:

Problem:

The graph of $y = \int_0^x f(t)dt$, where f is a piecewise constant function, is shown here.



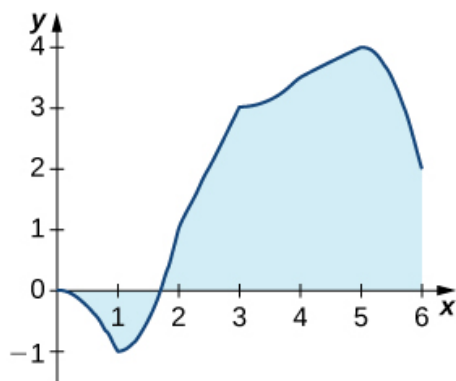
- Over which intervals is f positive? Over which intervals is it negative? Over which intervals, if any, is it equal to zero?
- What are the maximum and minimum values of f ?
- What is the average value of f ?

Solution:

a. f is positive over $[1, 2]$ and $[5, 6]$, negative over $[0, 1]$ and $[3, 4]$, and zero over $[2, 3]$ and $[4, 5]$. b. The maximum value is 2 and the minimum is -3 . c. The average value is 0.

Exercise:**Problem:**

The graph of $y = \int_0^x \ell(t)dt$, where ℓ is a piecewise linear function, is shown here.



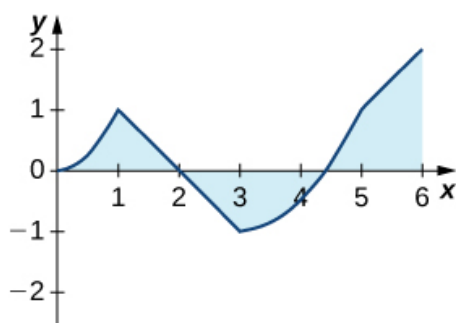
- Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?

- Over which intervals is ℓ increasing? Over which is it decreasing? Over which, if any, is it constant?
- What is the average value of ℓ ?

Exercise:

Problem:

The graph of $y = \int_0^x \ell(t) dt$, where ℓ is a piecewise linear function, is shown here.



- Over which intervals is ℓ positive? Over which intervals is it negative? Over which, if any, is it zero?
- Over which intervals is ℓ increasing? Over which is it decreasing? Over which intervals, if any, is it constant?
- What is the average value of ℓ ?

Solution:

a. ℓ is positive over $[0, 1]$ and $[3, 6]$, and negative over $[1, 3]$. b. It is increasing over $[0, 1]$ and $[3, 5]$, and it is constant over $[1, 3]$ and $[5, 6]$. c. Its average value is $\frac{1}{3}$.

In the following exercises, use a calculator to estimate the area under the curve by computing T_{10} , the average of the left- and right-endpoint Riemann sums using $N = 10$ rectangles. Then, using the Fundamental Theorem of Calculus, Part 2, determine the exact area.

Exercise:

Problem: [T] $y = x^2$ over $[0, 4]$

Exercise:

Problem: [T] $y = x^3 + 6x^2 + x - 5$ over $[-4, 2]$

Solution:

$$T_{10} = 49.08, \int_{-2}^3 x^3 + 6x^2 + x - 5 dx = 48$$

Exercise:

Problem: [T] $y = \sqrt{x^3}$ over $[0, 6]$

Exercise:

Problem: [T] $y = \sqrt{x} + x^2$ over $[1, 9]$

Solution:

$$T_{10} = 260.836, \int_1^9 (\sqrt{x} + x^2) dx = 260$$

Exercise:

Problem: [T] $\int (\cos x - \sin x) dx$ over $[0, \pi]$

Exercise:

Problem: [T] $\int \frac{4}{x^2} dx$ over $[1, 4]$

Solution:

$$T_{10} = 3.058, \int_1^4 \frac{4}{x^2} dx = 3$$

In the following exercises, evaluate each definite integral using the Fundamental Theorem of Calculus, Part 2.

Exercise:

Problem: $\int_{-1}^2 (x^2 - 3x) dx$

Exercise:

Problem: $\int_{-2}^3 (x^2 + 3x - 5) dx$

Solution:

$$F(x) = \frac{x^3}{3} + \frac{3x^2}{2} - 5x, F(3) - F(-2) = -\frac{35}{6}$$

Exercise:

Problem: $\int_{-2}^3 (t+2)(t-3)dt$

Exercise:

Problem: $\int_2^3 (t^2-9)(4-t^2)dt$

Solution:

$$F(x) = -\frac{t^5}{5} + \frac{13t^3}{3} - 36t, F(3) - F(2) = \frac{62}{15}$$

Exercise:

Problem: $\int_1^2 x^9 dx$

Exercise:

Problem: $\int_0^1 x^{99} dx$

Solution:

$$F(x) = \frac{x^{100}}{100}, F(1) - F(0) = \frac{1}{100}$$

Exercise:

Problem: $\int_4^8 (4t^{5/2} - 3t^{3/2})dt$

Exercise:

Problem: $\int_{1/4}^4 \left(x^2 - \frac{1}{x^2}\right)dx$

Solution:

$$F(x) = \frac{x^3}{3} + \frac{1}{x}, F(4) - F\left(\frac{1}{4}\right) = \frac{1125}{64}$$

Exercise:

Problem: $\int_1^2 \frac{2}{x^3} dx$

Exercise:

Problem: $\int_1^4 \frac{1}{2\sqrt{x}} dx$

Solution:

$$F(x) = \sqrt{x}, F(4) - F(1) = 1$$

Exercise:

Problem: $\int_1^4 \frac{2 - \sqrt{t}}{t^2} dt$

Exercise:

Problem: $\int_1^{16} \frac{dt}{t^{1/4}}$

Solution:

$$F(x) = \frac{4}{3}t^{3/4}, F(16) - F(1) = \frac{28}{3}$$

Exercise:

Problem: $\int_0^{2\pi} \cos \theta d\theta$

Exercise:

Problem: $\int_0^{\pi/2} \sin \theta d\theta$

Solution:

$$F(x) = -\cos x, F\left(\frac{\pi}{2}\right) - F(0) = 1$$

Exercise:

Problem: $\int_0^{\pi/4} \sec^2 \theta d\theta$

Exercise:

Problem: $\int_0^{\pi/4} \sec \theta \tan \theta$

Solution:

$$F(x) = \sec x, F\left(\frac{\pi}{4}\right) - F(0) = \sqrt{2} - 1$$

Exercise:

Problem: $\int_{\pi/3}^{\pi/4} \csc \theta \cot \theta d\theta$

Exercise:

Problem: $\int_{\pi/4}^{\pi/2} \csc^2 \theta d\theta$

Solution:

$$F(x) = -\cot(x), F\left(\frac{\pi}{2}\right) - F\left(\frac{\pi}{4}\right) = 1$$

Exercise:

Problem: $\int_1^2 \left(\frac{1}{t^2} - \frac{1}{t^3} \right) dt$

Exercise:

Problem: $\int_{-2}^{-1} \left(\frac{1}{t^2} - \frac{1}{t^3} \right) dt$

Solution:

$$F(x) = -\frac{1}{x} + \frac{1}{2x^2}, F(-1) - F(-2) = \frac{7}{8}$$

In the following exercises, use the evaluation theorem to express the integral as a function $F(x)$.

Exercise:

Problem: $\int_a^x t^2 dt$

Exercise:

Problem: $\int_1^x e^t dt$

Solution:

$$F(x) = e^x - e$$

Exercise:

Problem: $\int_0^x \cos t dt$

Exercise:

Problem: $\int_{-x}^x \sin t dt$

Solution:

$$F(x) = 0$$

In the following exercises, identify the roots of the integrand to remove absolute values, then evaluate using the Fundamental Theorem of Calculus, Part 2.

Exercise:

Problem: $\int_{-2}^3 |x| dx$

Exercise:

Problem: $\int_{-2}^4 |t^2 - 2t - 3| dt$

Solution:

$$\int_{-2}^{-1} (t^2 - 2t - 3) dt - \int_{-1}^3 (t^2 - 2t - 3) dt + \int_3^4 (t^2 - 2t - 3) dt = \frac{46}{3}$$

Exercise:

Problem: $\int_0^\pi |\cos t| dt$

Exercise:

Problem: $\int_{-\pi/2}^{\pi/2} |\sin t| dt$

Solution:

$$-\int_{-\pi/2}^0 \sin t dt + \int_0^{\pi/2} \sin t dt = 2$$

Exercise:**Problem:**

Suppose that the number of hours of daylight on a given day in Seattle is modeled by the function $-3.75 \cos\left(\frac{\pi t}{6}\right) + 12.25$, with t given in months and $t = 0$ corresponding to the winter solstice.

- What is the average number of daylight hours in a year?
- At which times t_1 and t_2 , where $0 \leq t_1 < t_2 < 12$, do the number of daylight hours equal the average number?
- Write an integral that expresses the total number of daylight hours in Seattle between t_1 and t_2 .
- Compute the mean hours of daylight in Seattle between t_1 and t_2 , where $0 \leq t_1 < t_2 < 12$, and then between t_2 and t_1 , and show that the average of the two is equal to the average day length.

Exercise:**Problem:**

Suppose the rate of gasoline consumption in the United States can be modeled by a sinusoidal function of the form $(11.21 - \cos\left(\frac{\pi t}{6}\right)) \times 10^9$ gal/mo.

- What is the average monthly consumption, and for which values of t is the rate at time t equal to the average rate?
- What is the number of gallons of gasoline consumed in the United States in a year?
- Write an integral that expresses the average monthly U.S. gas consumption during the part of the year between the beginning of April ($t = 3$) and the end of September ($t = 9$).

Solution:

- The average is 11.21×10^9 since $\cos\left(\frac{\pi t}{6}\right)$ has period 12 and integral 0 over any period. Consumption is equal to the average when $\cos\left(\frac{\pi t}{6}\right) = 0$, when $t = 3$, and when

$t = 9$. b. Total consumption is the average rate times duration:

$$11.21 \times 12 \times 10^9 = 1.35 \times 10^{11} \text{ c.}$$

$$10^9 \left(11.21 - \frac{1}{6} \int_3^9 \cos \left(\frac{\pi t}{6} \right) dt \right) = 10^9 \left(11.21 + \frac{2}{\pi} \right) = 11.84 \times 10^9$$

Exercise:

Problem:

Explain why, if f is continuous over $[a, b]$, there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t) dt.$$

Exercise:

Problem:

Explain why, if f is continuous over $[a, b]$ and is not equal to a constant, there is at least

one point $M \in [a, b]$ such that $f(M) = \frac{1}{b-a} \int_a^b f(t) dt$ and at least one point

$m \in [a, b]$ such that $f(m) < \frac{1}{b-a} \int_a^b f(t) dt$.

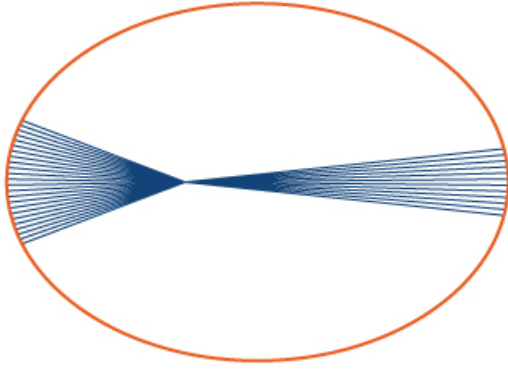
Solution:

If f is not constant, then its average is strictly smaller than the maximum and larger than the minimum, which are attained over $[a, b]$ by the extreme value theorem.

Exercise:

Problem:

Kepler's first law states that the planets move in elliptical orbits with the Sun at one focus. The closest point of a planetary orbit to the Sun is called the *perihelion* (for Earth, it currently occurs around January 3) and the farthest point is called the *aphelion* (for Earth, it currently occurs around July 4). Kepler's second law states that planets sweep out equal areas of their elliptical orbits in equal times. Thus, the two arcs indicated in the following figure are swept out in equal times. At what time of year is Earth moving fastest in its orbit? When is it moving slowest?



Exercise:

Problem:

A point on an ellipse with major axis length $2a$ and minor axis length $2b$ has the coordinates $(a \cos \theta, b \sin \theta)$, $0 \leq \theta \leq 2\pi$.

- Show that the distance from this point to the focus at $(-c, 0)$ is $d(\theta) = a + c \cos \theta$, where $c = \sqrt{a^2 - b^2}$.
- Use these coordinates to show that the average distance \bar{d} from a point on the ellipse to the focus at $(-c, 0)$, with respect to angle θ , is a .

Solution:

a. $d^2(\theta) = (a \cos \theta + c)^2 + b^2 \sin^2 \theta = a^2 + c^2 \cos^2 \theta + 2acc \cos \theta = (a + c \cos \theta)^2$; b.

$$\bar{d} = \frac{1}{2\pi} \int_0^{2\pi} (a + 2c \cos \theta) d\theta = a$$

Exercise:

Problem:

As implied earlier, according to Kepler's laws, Earth's orbit is an ellipse with the Sun at one focus. The perihelion for Earth's orbit around the Sun is 147,098,290 km and the aphelion is 152,098,232 km.

- By placing the major axis along the x -axis, find the average distance from Earth to the Sun.
- The classic definition of an astronomical unit (AU) is the distance from Earth to the Sun, and its value was computed as the average of the perihelion and aphelion distances. Is this definition justified?

Exercise:

Problem:

The force of gravitational attraction between the Sun and a planet is $F(\theta) = \frac{GmM}{r^2(\theta)}$, where m is the mass of the planet, M is the mass of the Sun, G is a universal constant, and $r(\theta)$ is the distance between the Sun and the planet when the planet is at an angle θ with the major axis of its orbit. Assuming that M , m , and the ellipse parameters a and b (half-lengths of the major and minor axes) are given, set up—but do not evaluate—an integral that expresses in terms of G , m , M , a , b the average gravitational force between the Sun and the planet.

Solution:

$$\text{Mean gravitational force} = \frac{GmM}{2} \int_0^{2\pi} \frac{1}{\left(a + 2\sqrt{a^2 - b^2}\cos\theta\right)^2} d\theta.$$

Exercise:**Problem:**

The displacement from rest of a mass attached to a spring satisfies the simple harmonic motion equation $x(t) = A\cos(\omega t - \phi)$, where ϕ is a phase constant, ω is the angular frequency, and A is the amplitude. Find the average velocity, the average speed (magnitude of velocity), the average displacement, and the average distance from rest (magnitude of displacement) of the mass.

Glossary

fundamental theorem of calculus

the theorem, central to the entire development of calculus, that establishes the relationship between differentiation and integration

fundamental theorem of calculus, part 1

uses a definite integral to define an antiderivative of a function

fundamental theorem of calculus, part 2

(also, **evaluation theorem**) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

mean value theorem for integrals

guarantees that a point c exists such that $f(c)$ is equal to the average value of the function

Integration Formulas and the Net Change Theorem

- Apply the basic integration formulas.
- Explain the significance of the net change theorem.
- Use the net change theorem to solve applied problems.
- Apply the integrals of odd and even functions.

In this section, we use some basic integration formulas studied previously to solve some key applied problems. It is important to note that these formulas are presented in terms of *indefinite* integrals. Although definite and indefinite integrals are closely related, there are some key differences to keep in mind. A definite integral is either a number (when the limits of integration are constants) or a single function (when one or both of the limits of integration are variables). An indefinite integral represents a family of functions, all of which differ by a constant. As you become more familiar with integration, you will get a feel for when to use definite integrals and when to use indefinite integrals. You will naturally select the correct approach for a given problem without thinking too much about it. However, until these concepts are cemented in your mind, think carefully about whether you need a definite integral or an indefinite integral and make sure you are using the proper notation based on your choice.

Basic Integration Formulas

Recall the integration formulas given in the [table in Antiderivatives](#) and the rule on properties of definite integrals. Let's look at a few examples of how to apply these rules.

Example:

Exercise:

Problem:

Integrating a Function Using the Power Rule

Use the power rule to integrate the function $\int_1^4 \sqrt{t}(1+t)dt$.

Solution:

The first step is to rewrite the function and simplify it so we can apply the power rule:

Equation:

$$\begin{aligned}\int_1^4 \sqrt{t}(1+t)dt &= \int_1^4 t^{1/2}(1+t)dt \\ &= \int_1^4 (t^{1/2} + t^{3/2})dt.\end{aligned}$$

Now apply the power rule:

Equation:

$$\begin{aligned}\int_1^4 (t^{1/2} + t^{3/2})dt &= \left(\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2}\right)\Big|_1^4 \\ &= \left[\frac{2}{3}(4)^{3/2} + \frac{2}{5}(4)^{5/2}\right] - \left[\frac{2}{3}(1)^{3/2} + \frac{2}{5}(1)^{5/2}\right] \\ &= \frac{256}{15}.\end{aligned}$$

Note:

Exercise:

Problem: Find the definite integral of $f(x) = x^2 - 3x$ over the interval $[1, 3]$.

Solution:

$$-\frac{10}{3}$$

Hint

Follow the process from [\[link\]](#) to solve the problem.

The Net Change Theorem

The **net change theorem** considers the integral of a *rate of change*. It says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity. The formula can be expressed in two ways. The second is more familiar; it is simply the definite integral.

Note:

Net Change Theorem

The new value of a changing quantity equals the initial value plus the integral of the rate of change:

Equation:

$$F(b) = F(a) + \int_a^b F'(x)dx$$

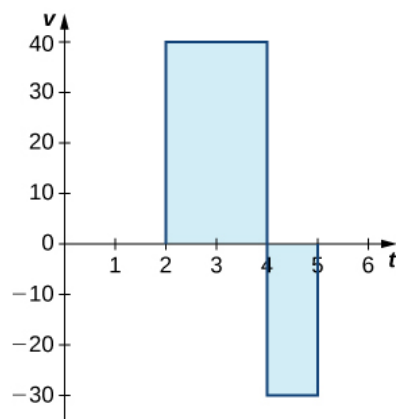
or

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Subtracting $F(a)$ from both sides of the first equation yields the second equation. Since they are equivalent formulas, which one we use depends on the application.

The significance of the net change theorem lies in the results. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral. To illustrate, let's apply the net change theorem to a velocity function in which the result is displacement.

We looked at a simple example of this in [The Definite Integral](#). Suppose a car is moving due north (the positive direction) at 40 mph between 2 p.m. and 4 p.m., then the car moves south at 30 mph between 4 p.m. and 5 p.m. We can graph this motion as shown in [\[link\]](#).



The graph shows speed versus time for the given motion of a car.

Just as we did before, we can use definite integrals to calculate the net displacement as well as the total distance traveled. The net displacement is given by

Equation:

$$\begin{aligned}\int_2^5 v(t)dt &= \int_2^4 40dt + \int_4^5 -30dt \\ &= 80 - 30 \\ &= 50.\end{aligned}$$

Thus, at 5 p.m. the car is 50 mi north of its starting position. The total distance traveled is given by

Equation:

$$\begin{aligned}\int_2^5 |v(t)|dt &= \int_2^4 40dt + \int_4^5 30dt \\ &= 80 + 30 \\ &= 110.\end{aligned}$$

Therefore, between 2 p.m. and 5 p.m., the car traveled a total of 110 mi.

To summarize, net displacement may include both positive and negative values. In other words, the velocity function accounts for both forward distance and backward distance. To find net displacement, integrate the velocity function over the interval. Total distance traveled, on the other hand, is always positive. To find the total distance traveled by an object, regardless of direction, we need to integrate the absolute value of the velocity function.

Example:

Exercise:

Problem:

Finding Net Displacement

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

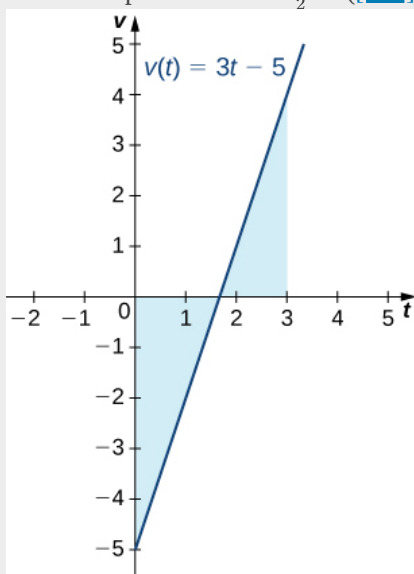
Solution:

Applying the net change theorem, we have

Equation:

$$\begin{aligned}\int_0^3 (3t - 5) dt &= \left. \frac{3t^2}{2} - 5t \right|_0^3 \\ &= \left[\frac{3(3)^2}{2} - 5(3) \right] - 0 \\ &= \frac{27}{2} - 15 \\ &= \frac{27}{2} - \frac{30}{2} \\ &= -\frac{3}{2}.\end{aligned}$$

The net displacement is $-\frac{3}{2}$ m ([link](#)).



The graph shows velocity versus time for a particle moving with a linear velocity function.

Example:

Exercise:

Problem:

Finding the Total Distance Traveled

Use [\[link\]](#) to find the total distance traveled by a particle according to the velocity function $v(t) = 3t - 5$ m/sec over a time interval $[0, 3]$.

Solution:

The total distance traveled includes both the positive and the negative values. Therefore, we must integrate the absolute value of the velocity function to find the total distance traveled.

To continue with the example, use two integrals to find the total distance. First, find the t -intercept of the function, since that is where the division of the interval occurs. Set the equation equal to zero and solve for t . Thus,

Equation:

$$\begin{aligned}3t - 5 &= 0 \\3t &= 5 \\t &= \frac{5}{3}.\end{aligned}$$

The two subintervals are $[0, \frac{5}{3}]$ and $[\frac{5}{3}, 3]$. To find the total distance traveled, integrate the absolute value of the function. Since the function is negative over the interval $[0, \frac{5}{3}]$, we have $|v(t)| = -v(t)$ over that interval. Over $[\frac{5}{3}, 3]$, the function is positive, so $|v(t)| = v(t)$. Thus, we have

Equation:

$$\begin{aligned}\int_0^3 |v(t)| dt &= \int_0^{5/3} -v(t) dt + \int_{5/3}^3 v(t) dt \\&= \int_0^{5/3} 5 - 3t dt + \int_{5/3}^3 3t - 5 dt \\&= \left(5t - \frac{3t^2}{2}\right) \Big|_0^{5/3} + \left(\frac{3t^2}{2} - 5t\right) \Big|_{5/3}^3 \\&= \left[5\left(\frac{5}{3}\right) - \frac{3(5/3)^2}{2}\right] - 0 + \left[\frac{27}{2} - 15\right] - \left[\frac{3(5/3)^2}{2} - \frac{25}{3}\right] \\&= \frac{25}{3} - \frac{25}{6} + \frac{27}{2} - 15 - \frac{25}{6} + \frac{25}{3} \\&= \frac{41}{6}.\end{aligned}$$

So, the total distance traveled is $\frac{41}{6}$ m.

Note:

Exercise:

Problem:

Find the net displacement and total distance traveled in meters given the velocity function $f(t) = \frac{1}{2}e^t - 2$ over the interval $[0, 2]$.

Solution:

Net displacement: $\frac{e^2-9}{2} \approx -0.8055$ m; total distance traveled: $4\ln 4 - 7.5 + \frac{e^2}{2} \approx 1.740$ m

Hint

Follow the procedures from [\[link\]](#) and [\[link\]](#). Note that $f(t) \leq 0$ for $t \leq \ln 4$, and $f(t) \geq 0$ for $t \geq \ln 4$.

Applying the Net Change Theorem

The net change theorem can be applied to the flow and consumption of fluids, as shown in [\[link\]](#).

Example:**Exercise:****Problem:****How Many Gallons of Gasoline Are Consumed?**

If the motor on a motorboat is started at $t = 0$ and the boat consumes gasoline at the rate of $5 - t^3$ gal/hr, how much gasoline is used in the first 2 hours?

Solution:

Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval $[0, 2]$. We have

Equation:

$$\begin{aligned}\int_0^2 (5 - t^3) dt &= \left(5t - \frac{t^4}{4} \right) \Big|_0^2 \\ &= \left[5(2) - \frac{(2)^4}{4} \right] - 0 \\ &= 10 - \frac{16}{4} \\ &= 6.\end{aligned}$$

Thus, the motorboat uses 6 gal of gas in 2 hours.

Example:**Exercise:****Problem:****Chapter Opener: Iceboats**



(credit: modification of work by Carter Brown, Flickr)

As we saw at the beginning of the chapter, top iceboat racers ([link](#)) can attain speeds of up to five times the wind speed. Andrew is an intermediate iceboater, though, so he attains speeds equal to only twice the wind speed. Suppose Andrew takes his iceboat out one morning when a light 5-mph breeze has been blowing all morning. As Andrew gets his iceboat set up, though, the wind begins to pick up. During his first half hour of iceboating, the wind speed increases according to the function $v(t) = 20t + 5$. For the second half hour of Andrew's outing, the wind remains steady at 15 mph. In other words, the wind speed is given by

Equation:

$$v(t) = \begin{cases} 20t + 5 & \text{for } 0 \leq t \leq \frac{1}{2} \\ 15 & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Recalling that Andrew's iceboat travels at twice the wind speed, and assuming he moves in a straight line away from his starting point, how far is Andrew from his starting point after 1 hour?

Solution:

To figure out how far Andrew has traveled, we need to integrate his velocity, which is twice the wind speed. Then

$$\text{Distance} = \int_0^1 2v(t) dt.$$

Substituting the expressions we were given for $v(t)$, we get

Equation:

$$\begin{aligned}
\int_0^1 2v(t)dt &= \int_0^{1/2} 2v(t)dt + \int_{1/2}^1 2v(t)dt \\
&= \int_0^{1/2} 2(20t + 5)dt + \int_{1/2}^1 2(15)dt \\
&= \int_0^{1/2} (40t + 10)dt + \int_{1/2}^1 30dt \\
&= [20t^2 + 10t]_0^{1/2} + [30t]_{1/2}^1 \\
&= \left(\frac{20}{4} + 5\right) - 0 + (30 - 15) \\
&= 25.
\end{aligned}$$

Andrew is 25 mi from his starting point after 1 hour.

Note:

Exercise:

Problem:

Suppose that, instead of remaining steady during the second half hour of Andrew's outing, the wind starts to die down according to the function $v(t) = -10t + 15$. In other words, the wind speed is given by

Equation:

$$v(t) = \begin{cases} 20t + 5 & \text{for } 0 \leq t \leq \frac{1}{2} \\ -10t + 15 & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Under these conditions, how far from his starting point is Andrew after 1 hour?

Solution:

17.5 mi

Hint

Don't forget that Andrew's iceboat moves twice as fast as the wind.

Integrating Even and Odd Functions

We saw in [Functions and Graphs](#) that an even function is a function in which $f(-x) = f(x)$ for all x in the domain—that is, the graph of the curve is unchanged when x is replaced with $-x$. The graphs of even functions are symmetric about the y -axis. An odd function is one in which $f(-x) = -f(x)$ for all x in the domain, and the graph of the function is symmetric about the origin.

Integrals of even functions, when the limits of integration are from $-a$ to a , involve two equal areas, because they are symmetric about the y -axis. Integrals of odd functions, when the limits of integration are similarly $[-a, a]$, evaluate to zero because the areas above and below the x -axis are equal.

Note:**Rule: Integrals of Even and Odd Functions**

For continuous even functions such that $f(-x) = f(x)$,

Equation:

$$\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx.$$

For continuous odd functions such that $f(-x) = -f(x)$,

Equation:

$$\int_{-a}^a f(x)dx = 0.$$

Example:**Exercise:****Problem:****Integrating an Even Function**

Integrate the even function $\int_{-2}^2 (3x^8 - 2)dx$ and verify that the integration formula for even functions holds.

Solution:

The symmetry appears in the graphs in [\[link\]](#). Graph (a) shows the region below the curve and above the x-axis. We have to zoom in to this graph by a huge amount to see the region. Graph (b) shows the region above the curve and below the x-axis. The signed area of this region is negative. Both views illustrate the symmetry about the y-axis of an even function. We have

Equation:

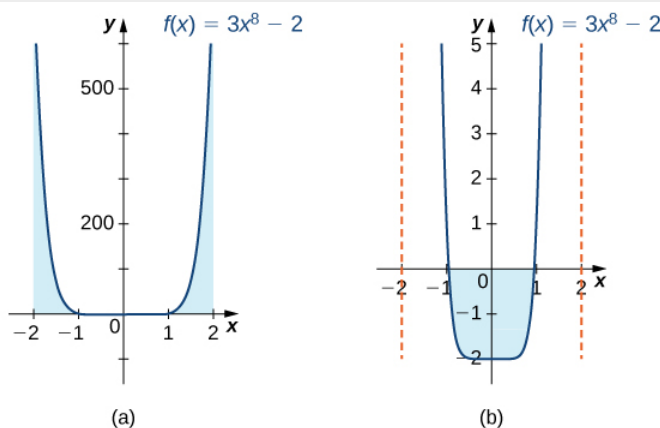
$$\begin{aligned} \int_{-2}^2 (3x^8 - 2)dx &= \left(\frac{x^9}{3} - 2x \right) \Big|_{-2}^2 \\ &= \left[\frac{(2)^9}{3} - 2(2) \right] - \left[\frac{(-2)^9}{3} - 2(-2) \right] \\ &= \left(\frac{512}{3} - 4 \right) - \left(-\frac{512}{3} + 4 \right) \\ &= \frac{1000}{3}. \end{aligned}$$

To verify the integration formula for even functions, we can calculate the integral from 0 to 2 and double it, then check to make sure we get the same answer.

Equation:

$$\begin{aligned} \int_0^2 (3x^8 - 2)dx &= \left(\frac{x^9}{3} - 2x \right) \Big|_0^2 \\ &= \frac{512}{3} - 4 \\ &= \frac{500}{3} \end{aligned}$$

Since $2 \cdot \frac{500}{3} = \frac{1000}{3}$, we have verified the formula for even functions in this particular example.



Graph (a) shows the positive area between the curve and the x -axis, whereas graph (b) shows the negative area between the curve and the x -axis. Both views show the symmetry about the y -axis.

Example:

Exercise:

Problem:

Integrating an Odd Function

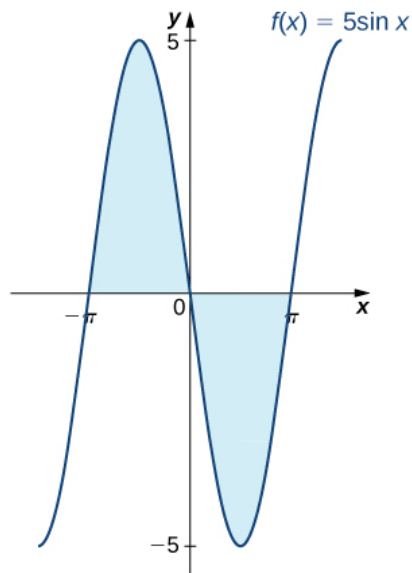
Evaluate the definite integral of the odd function $-5 \sin x$ over the interval $[-\pi, \pi]$.

Solution:

The graph is shown in [\[link\]](#). We can see the symmetry about the origin by the positive area above the x -axis over $[-\pi, 0]$, and the negative area below the x -axis over $[0, \pi]$. We have

Equation:

$$\begin{aligned}
 \int_{-\pi}^{\pi} -5 \sin x dx &= -5 (-\cos x) \Big|_{-\pi}^{\pi} \\
 &= 5 \cos x \Big|_{-\pi}^{\pi} \\
 &= [5 \cos \pi] - [5 \cos (-\pi)] \\
 &= -5 - (-5) \\
 &= 0.
 \end{aligned}$$



The graph shows areas between a curve and the x -axis for an odd function.

Note:

Exercise:

Problem: Integrate the function $\int_{-2}^2 x^4 dx$.

Solution:

$$\frac{64}{5}$$

Hint

Integrate an even function.

Key Concepts

- The net change theorem states that when a quantity changes, the final value equals the initial value plus the integral of the rate of change. Net change can be a positive number, a negative number, or zero.
- The area under an even function over a symmetric interval can be calculated by doubling the area over the positive x -axis. For an odd function, the integral over a symmetric interval equals zero, because half the area is negative.

Key Equations

- **Net Change Theorem**

$$F(b) = F(a) + \int_a^b F'(x)dx \text{ or } \int_a^b F'(x)dx = F(b) - F(a)$$

Use basic integration formulas to compute the following antiderivatives.

Exercise:

Problem: $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right) dx$

Solution:

$$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right) dx = \int x^{1/2} dx - \int x^{-1/2} dx = \frac{2}{3} x^{3/2} + C_1 - 2x^{1/2} + C_2 = \frac{2}{3} x^{3/2} - 2x^{1/2} + C$$

Exercise:

Problem: $\int \left(e^{2x} - \frac{1}{2} e^{x/2} \right) dx$

Exercise:

Problem: $\int \frac{dx}{2x}$

Solution:

$$\int \frac{dx}{2x} = \frac{1}{2} \ln |x| + C$$

Exercise:

Problem: $\int \frac{x-1}{x^2} dx$

Exercise:

Problem: $\int_0^\pi (\sin x - \cos x) dx$

Solution:

$$\int_0^\pi \sin x dx - \int_0^\pi \cos x dx = -\cos x \Big|_0^\pi - (\sin x) \Big|_0^\pi = (-(-1) + 1) - (0 - 0) = 2$$

Exercise:

Problem: $\int_0^{\pi/2} (x - \sin x) dx$

Exercise:

Problem:

Write an integral that expresses the increase in the perimeter $P(s)$ of a square when its side length s increases from 2 units to 4 units and evaluate the integral.

Solution:

$$P(s) = 4s, \text{ so } \frac{dP}{ds} = 4 \text{ and } \int_2^4 4ds = 8.$$

Exercise:**Problem:**

Write an integral that quantifies the change in the area $A(s) = s^2$ of a square when the side length doubles from S units to $2S$ units and evaluate the integral.

Exercise:**Problem:**

A regular N -gon (an N -sided polygon with sides that have equal length s , such as a pentagon or hexagon) has perimeter Ns . Write an integral that expresses the increase in perimeter of a regular N -gon when the length of each side increases from 1 unit to 2 units and evaluate the integral.

Solution:

$$\int_1^2 Nds = N$$

Exercise:**Problem:**

The area of a regular pentagon with side length $a > 0$ is pa^2 with $p = \frac{1}{4}\sqrt{5 + \sqrt{5 + 2\sqrt{5}}}$. The Pentagon in Washington, DC, has inner sides of length 360 ft and outer sides of length 920 ft. Write an integral to express the area of the roof of the Pentagon according to these dimensions and evaluate this area.

Exercise:**Problem:**

A dodecahedron is a Platonic solid with a surface that consists of 12 pentagons, each of equal area. By how much does the surface area of a dodecahedron increase as the side length of each pentagon doubles from 1 unit to 2 units?

Solution:

With p as in the previous exercise, each of the 12 pentagons increases in area from $2p$ to $4p$ units so the net increase in the area of the dodecahedron is $36p$ units.

Exercise:**Problem:**

An icosahedron is a Platonic solid with a surface that consists of 20 equilateral triangles. By how much does the surface area of an icosahedron increase as the side length of each triangle doubles from a unit to $2a$ units?

Exercise:**Problem:**

Write an integral that quantifies the change in the area of the surface of a cube when its side length doubles from s unit to $2s$ units and evaluate the integral.

Solution:

$$18s^2 = 6 \int_s^{2s} 2x dx$$

Exercise:

Problem:

Write an integral that quantifies the increase in the volume of a cube when the side length doubles from s unit to $2s$ units and evaluate the integral.

Exercise:

Problem:

Write an integral that quantifies the increase in the surface area of a sphere as its radius doubles from R unit to $2R$ units and evaluate the integral.

Solution:

$$12\pi R^2 = 8\pi \int_R^{2R} r dr$$

Exercise:

Problem:

Write an integral that quantifies the increase in the volume of a sphere as its radius doubles from R unit to $2R$ units and evaluate the integral.

Exercise:

Problem:

Suppose that a particle moves along a straight line with velocity $v(t) = 4 - 2t$, where $0 \leq t \leq 2$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 2$.

Solution:

$$d(t) = \int_0^t v(s) ds = 4t - t^2. \text{ The total distance is } d(2) = 4 \text{ m.}$$

Exercise:

Problem:

Suppose that a particle moves along a straight line with velocity defined by $v(t) = t^2 - 3t - 18$, where $0 \leq t \leq 6$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 6$.

Exercise:

Problem:

Suppose that a particle moves along a straight line with velocity defined by $v(t) = |2t - 6|$, where $0 \leq t \leq 6$ (in meters per second). Find the displacement at time t and the total distance traveled up to $t = 6$.

Solution:

$d(t) = \int_0^t v(s)ds$. For $t < 3$, $d(t) = \int_0^t (6 - 2t)dt = 6t - t^2$. For $t > 3$, $d(t) = d(3) + \int_3^t (2t - 6)dt = 9 + (t^2 - 6t)$. The total distance is $d(6) = 9$ m.

Exercise:

Problem:

Suppose that a particle moves along a straight line with acceleration defined by $a(t) = t - 3$, where $0 \leq t \leq 6$ (in meters per second). Find the velocity and displacement at time t and the total distance traveled up to $t = 6$ if $v(0) = 3$ and $d(0) = 0$.

Exercise:

Problem:

A ball is thrown upward from a height of 1.5 m at an initial speed of 40 m/sec. Acceleration resulting from gravity is -9.8 m/sec². Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball t seconds after it is thrown and before it returns to the ground.

Solution:

$$v(t) = 40 - 9.8t; h(t) = 1.5 + 40t - 4.9t^2 \text{ m/s}$$

Exercise:

Problem:

A ball is thrown upward from a height of 3 m at an initial speed of 60 m/sec. Acceleration resulting from gravity is -9.8 m/sec². Neglecting air resistance, solve for the velocity $v(t)$ and the height $h(t)$ of the ball t seconds after it is thrown and before it returns to the ground.

Exercise:

Problem:

The area $A(t)$ of a circular shape is growing at a constant rate. If the area increases from 4π units to 9π units between times $t = 2$ and $t = 3$, find the net change in the radius during that time.

Solution:

The net increase is 1 unit.

Exercise:

Problem:

A spherical balloon is being inflated at a constant rate. If the volume of the balloon changes from 36π in.³ to 288π in.³ between time $t = 30$ and $t = 60$ seconds, find the net change in the radius of the balloon during that time.

Exercise:

Problem:

Water flows into a conical tank with cross-sectional area πx^2 at height x and volume $\frac{\pi x^3}{3}$ up to height x . If water flows into the tank at a rate of 1 m³/min, find the height of water in the tank after 5 min. Find the change in height between 5 min and 10 min.

Solution:

At $t = 5$, the height of water is $x = \left(\frac{15}{\pi}\right)^{1/3}$ m.. The net change in height from $t = 5$ to $t = 10$ is $\left(\frac{30}{\pi}\right)^{1/3} - \left(\frac{15}{\pi}\right)^{1/3}$ m.

Exercise:

Problem:

A horizontal cylindrical tank has cross-sectional area $A(x) = 4(6x - x^2)m^2$ at height x meters above the bottom when $x \leq 3$.

- The volume V between heights a and b is $\int_a^b A(x)dx$. Find the volume at heights between 2 m and 3 m.
- Suppose that oil is being pumped into the tank at a rate of 50 L/min. Using the chain rule, $\frac{dx}{dt} = \frac{dx}{dV} \frac{dV}{dt}$, at how many meters per minute is the height of oil in the tank changing, expressed in terms of x , when the height is at x meters?
- How long does it take to fill the tank to 3 m starting from a fill level of 2 m?

Exercise:

Problem:

The following table lists the electrical power in gigawatts—the rate at which energy is consumed—used in a certain city for different hours of the day, in a typical 24-hour period, with hour 1 corresponding to midnight to 1 a.m.

Hour	Power	Hour	Power
1	28	13	48
2	25	14	49
3	24	15	49
4	23	16	50
5	24	17	50
6	27	18	50
7	29	19	46
8	32	20	43
9	34	21	42
10	39	22	40
11	42	23	37

Hour	Power	Hour	Power
12	46	24	34

Find the total amount of power in gigawatt-hours (gW-h) consumed by the city in a typical 24-hour period.

Solution:

The total daily power consumption is estimated as the sum of the hourly power rates, or 911 gW-h.

Exercise:

Problem:

The average residential electrical power use (in hundreds of watts) per hour is given in the following table.

Hour	Power	Hour	Power
1	8	13	12
2	6	14	13
3	5	15	14
4	4	16	15
5	5	17	17
6	6	18	19
7	7	19	18
8	8	20	17
9	9	21	16
10	10	22	16
11	10	23	13
12	11	24	11

- Compute the average total energy used in a day in kilowatt-hours (kWh).
- If a ton of coal generates 1842 kWh, how long does it take for an average residence to burn a ton of coal?
- Explain why the data might fit a plot of the form $p(t) = 11.5 - 7.5 \sin\left(\frac{\pi t}{12}\right)$.

Exercise:

Problem:

The data in the following table are used to estimate the average power output produced by Peter Sagan for each of the last 18 sec of Stage 1 of the 2012 Tour de France.

Second	Watts	Second	Watts
1	600	10	1200
2	500	11	1170
3	575	12	1125
4	1050	13	1100
5	925	14	1075
6	950	15	1000
7	1050	16	950
8	950	17	900
9	1100	18	780

Average Power OutputSource: sportsexercisengineering.com

Estimate the net energy used in kilojoules (kJ), noting that $1\text{ W} = 1\text{ J/s}$, and the average power output by Sagan during this time interval.

Solution:

17 kJ

Exercise:**Problem:**

The data in the following table are used to estimate the average power output produced by Peter Sagan for each 15-min interval of Stage 1 of the 2012 Tour de France.

Minutes	Watts	Minutes	Watts
15	200	165	170
30	180	180	220

Minutes	Watts	Minutes	Watts
45	190	195	140
60	230	210	225
75	240	225	170
90	210	240	210
105	210	255	200
120	220	270	220
135	210	285	250
150	150	300	400

Average Power OutputSource: sportsexercisengineering.com

Estimate the net energy used in kilojoules, noting that $1W = 1 \text{ J/s}$.

Exercise:

Problem:

The distribution of incomes as of 2012 in the United States in \$5000 increments is given in the following table. The k th row denotes the percentage of households with incomes between $5000xk$ and $5000xk + 4999$. The row $k = 40$ contains all households with income between \$200,000 and \$250,000 and $k = 41$ accounts for all households with income exceeding \$250,000.

0	3.5	21	1.5
1	4.1	22	1.4
2	5.9	23	1.3
3	5.7	24	1.3
4	5.9	25	1.1
5	5.4	26	1.0
6	5.5	27	0.75
7	5.1	28	0.8
8	4.8	29	1.0
9	4.1	30	0.6

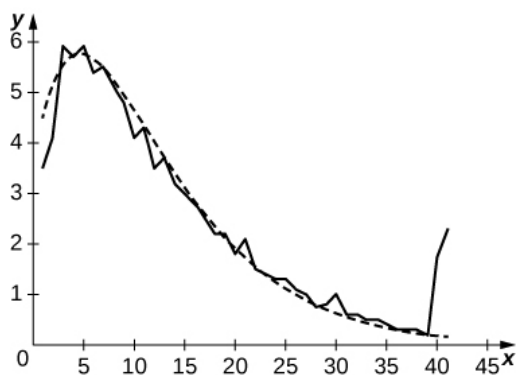
10	4.3	31	0.6
11	3.5	32	0.5
12	3.7	33	0.5
13	3.2	34	0.4
14	3.0	35	0.3
15	2.8	36	0.3
16	2.5	37	0.3
17	2.2	38	0.2
18	2.2	39	1.8
19	1.8	40	2.3
20	2.1	41	

Income DistributionsSource: <http://www.census.gov/prod/2013pubs/p60-245.pdf>

- Estimate the percentage of U.S. households in 2012 with incomes less than \$55,000.
- What percentage of households had incomes exceeding \$85,000?
- Plot the data and try to fit its shape to that of a graph of the form $a(x + c)e^{-b(x+c)}$ for suitable a, b, c .

Solution:

- 54.3%; b. 27.00%; c. The curve in the following plot is $2.35(t + 3)e^{-0.15(t+3)}$.



Exercise:

Problem:

Newton's law of gravity states that the gravitational force exerted by an object of mass M and one of mass m with centers that are separated by a distance r is $F = G \frac{mM}{r^2}$, with G an empirical constant

$G = 6.67 \times 10^{-11} \text{ m}^3 / (\text{kg} \cdot \text{s}^2)$. The work done by a variable force over an interval $[a, b]$ is defined as

$W = \int_a^b F(x) dx$. If Earth has mass 5.97219×10^{24} and radius 6371 km, compute the amount of work to elevate a polar weather satellite of mass 1400 kg to its orbiting altitude of 850 km above Earth.

Exercise:**Problem:**

For a given motor vehicle, the maximum achievable deceleration from braking is approximately 7 m/sec² on dry concrete. On wet asphalt, it is approximately 2.5 m/sec². Given that 1 mph corresponds to 0.447 m/sec, find the total distance that a car travels in meters on dry concrete after the brakes are applied until it comes to a complete stop if the initial velocity is 67 mph (30 m/sec) or if the initial braking velocity is 56 mph (25 m/sec). Find the corresponding distances if the surface is slippery wet asphalt.

Solution:

In dry conditions, with initial velocity $v_0 = 30$ m/s, $D = 64.3$ and, if $v_0 = 25$, $D = 44.64$. In wet conditions, if $v_0 = 30$, and $D = 180$ and if $v_0 = 25$, $D = 125$.

Exercise:**Problem:**

John is a 25-year old man who weighs 160 lb. He burns $500 - 50t$ calories/hr while riding his bike for t hours. If an oatmeal cookie has 55 cal and John eats $4t$ cookies during the t th hour, how many net calories has he lost after 3 hours riding his bike?

Exercise:**Problem:**

Sandra is a 25-year old woman who weighs 120 lb. She burns $300 - 50t$ cal/hr while walking on her treadmill. Her caloric intake from drinking Gatorade is $100t$ calories during the t th hour. What is her net decrease in calories after walking for 3 hours?

Solution:

225 cal

Exercise:**Problem:**

A motor vehicle has a maximum efficiency of 33 mpg at a cruising speed of 40 mph. The efficiency drops at a rate of 0.1 mpg/mph between 40 mph and 50 mph, and at a rate of 0.4 mpg/mph between 50 mph and 80 mph. What is the efficiency in miles per gallon if the car is cruising at 50 mph? What is the efficiency in miles per gallon if the car is cruising at 80 mph? If gasoline costs \$3.50/gal, what is the cost of fuel to drive 50 mi at 40 mph, at 50 mph, and at 80 mph?

Exercise:

Problem:

Although some engines are more efficient at given a horsepower than others, on average, fuel efficiency decreases with horsepower at a rate of $1/25$ mpg/horsepower. If a typical 50-horsepower engine has an average fuel efficiency of 32 mpg, what is the average fuel efficiency of an engine with the following horsepower: 150, 300, 450?

Solution:

$$E(150) = 28, E(300) = 22, E(450) = 16$$

Exercise:

Problem: [T] The following table lists the 2013 schedule of federal income tax versus taxable income.

Taxable Income Range	The Tax Is Of the Amount Over
\$0–\$8925	10%	\$0
\$8925–\$36,250	\$892.50 + 15%	\$8925
\$36,250–\$87,850	\$4,991.25 + 25%	\$36,250
\$87,850–\$183,250	\$17,891.25 + 28%	\$87,850
\$183,250–\$398,350	\$44,603.25 + 33%	\$183,250
\$398,350–\$400,000	\$115,586.25 + 35%	\$398,350
> \$400,000	\$116,163.75 + 39.6%	\$400,000

Federal Income Tax Versus Taxable IncomeSource: <http://www.irs.gov/pub/irs-prior/i1040tt--2013.pdf>.

Suppose that Steve just received a \$10,000 raise. How much of this raise is left after federal taxes if Steve's salary before receiving the raise was \$40,000? If it was \$90,000? If it was \$385,000?

Exercise:**Problem:**

[T] The following table provides hypothetical data regarding the level of service for a certain highway.

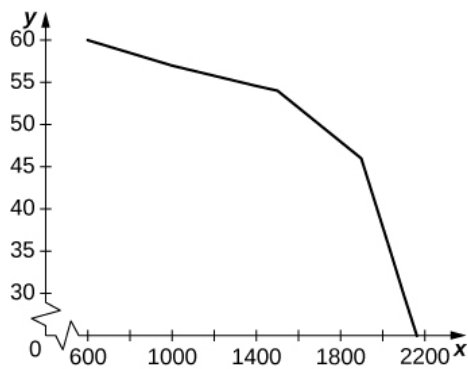
Highway Speed Range (mph)	Vehicles per Hour per Lane	Density Range (vehicles/mi)
> 60	< 600	< 10
60–57	600–1000	10–20

Highway Speed Range (mph)	Vehicles per Hour per Lane	Density Range (vehicles/mi)
57–54	1000–1500	20–30
54–46	1500–1900	30–45
46–30	1900–2100	45–70
<30	Unstable	70–200

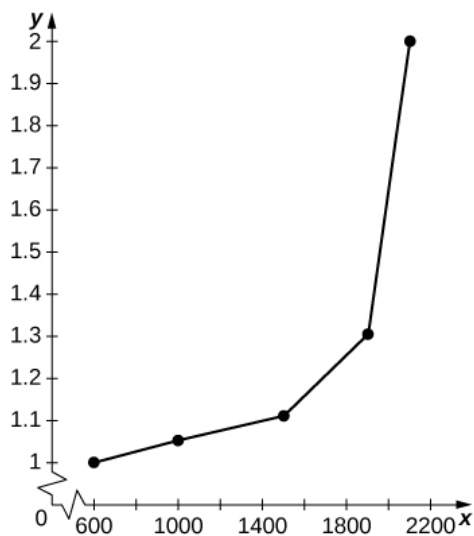
- Plot vehicles per hour per lane on the x -axis and highway speed on the y -axis.
- Compute the average decrease in speed (in miles per hour) per unit increase in congestion (vehicles per hour per lane) as the latter increases from 600 to 1000, from 1000 to 1500, and from 1500 to 2100. Does the decrease in miles per hour depend linearly on the increase in vehicles per hour per lane?
- Plot minutes per mile (60 times the reciprocal of miles per hour) as a function of vehicles per hour per lane. Is this function linear?

Solution:

a.



- b. Between 600 and 1000 the average decrease in vehicles per hour per lane is -0.0075 . Between 1000 and 1500 it is -0.006 per vehicles per hour per lane, and between 1500 and 2100 it is -0.04 vehicles per hour per lane. c.



The graph is nonlinear, with minutes per mile increasing dramatically as vehicles per hour per lane reach 2000.

For the next two exercises use the data in the following table, which displays bald eagle populations from 1963 to 2000 in the continental United States.

Year	Population of Breeding Pairs of Bald Eagles
1963	487
1974	791
1981	1188
1986	1875
1992	3749
1996	5094
2000	6471

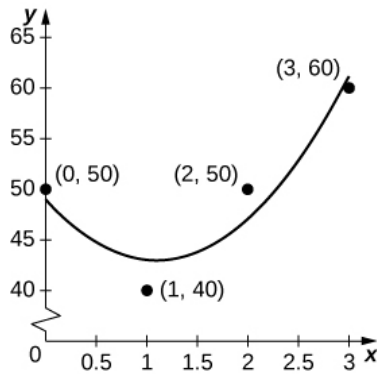
Population of Breeding Bald Eagle PairsSource: <http://www.fws.gov/Midwest/eagle/population/chtotfrs.html>.

Exercise:

Problem:

[T] The graph below plots the quadratic $p(t) = 6.48t^2 - 80.31t + 585.69$ against the data in preceding table, normalized so that $t = 0$ corresponds to 1963. Estimate the average number of bald eagles per year present for the 37 years by computing the average value of p over $[0, 37]$.

Time (hr)	Speed (mph)
0 (start)	50
1	40
2	50
3	60



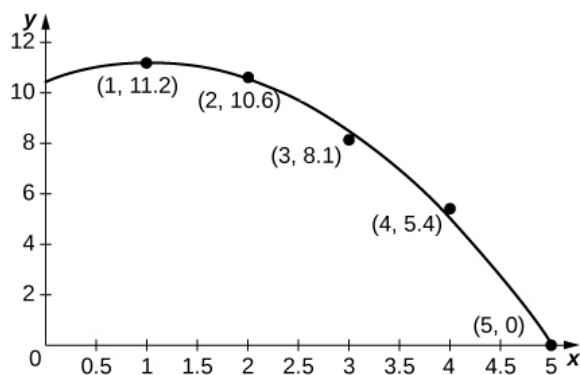
As a car accelerates, it does not accelerate at a constant rate; rather, the acceleration is variable. For the following exercises, use the following table, which contains the acceleration measured at every second as a driver merges onto a freeway.

Time (sec)	Acceleration (mph/sec)
1	11.2
2	10.6
3	8.1
4	5.4
5	0

Exercise:

Problem:

[T] The accompanying graph plots the best quadratic fit, $a(t) = -0.70t^2 + 1.44t + 10.44$, to the data from the preceding table. Compute the average value of $a(t)$ to estimate the average acceleration between $t = 0$ and $t = 5$.



Solution:

Average acceleration is $A = \frac{1}{5} \int_0^5 a(t) dt = -\frac{0.7(5^2)}{3} + \frac{1.44(5)}{2} + 10.44 \approx 8.2 \text{ mph/s}$

Exercise:

Problem:

[T] Using your acceleration equation from the previous exercise, find the corresponding velocity equation. Assuming the final velocity is 0 mph, find the velocity at time $t = 0$.

Exercise:

Problem:

[T] Using your velocity equation from the previous exercise, find the corresponding distance equation, assuming your initial distance is 0 mi. How far did you travel while you accelerated your car? (*Hint: You will need to convert time units.*)

Solution:

$$d(t) = \int_0^t |v(t)| dt = \int_0^t \left(\frac{7}{30}t^3 - 0.72t^2 - 10.44t + 41.033 \right) dt = \frac{7}{120}t^4 - 0.24t^3 - 5.22t^2 + 41.033t.$$

Then, $d(5) \approx 81.12 \text{ mph} \times \text{sec} \approx 119 \text{ feet}$.

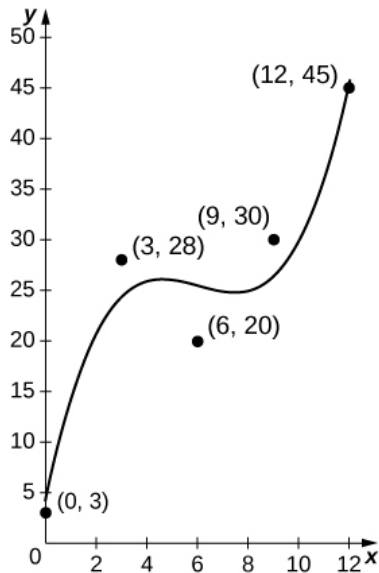
Exercise:

Problem:

[T] The number of hamburgers sold at a restaurant throughout the day is given in the following table, with the accompanying graph plotting the best cubic fit to the data, $b(t) = 0.12t^3 - 2.13t^2 + 12.13t + 3.91$, with $t = 0$ corresponding to 9 a.m. and $t = 12$ corresponding to 9 p.m. Compute the average value of $b(t)$ to estimate the average number of hamburgers sold per hour.

Hours Past Midnight	No. of Burgers Sold
9	3

Hours Past Midnight	No. of Burgers Sold
12	28
15	20
18	30
21	45

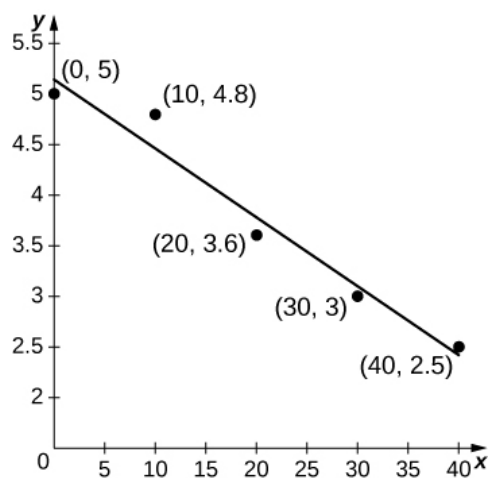


Exercise:

Problem:

[T] An athlete runs by a motion detector, which records her speed, as displayed in the following table. The best linear fit to this data, $\ell(t) = -0.068t + 5.14$, is shown in the accompanying graph. Use the average value of $\ell(t)$ between $t = 0$ and $t = 40$ to estimate the runner's average speed.

Minutes	Speed (m/sec)
0	5
10	4.8
20	3.6
30	3.0
40	2.5



Solution:

$$\frac{1}{40} \int_0^{40} (-0.068t + 5.14) dt = -\frac{0.068(40)}{2} + 5.14 = 3.78$$

Glossary

net change theorem

if we know the rate of change of a quantity, the net change theorem says the future quantity is equal to the initial quantity plus the integral of the rate of change of the quantity

Substitution

- Use substitution to evaluate indefinite integrals.
- Use substitution to evaluate definite integrals.

The Fundamental Theorem of Calculus gave us a method to evaluate integrals without using Riemann sums. The drawback of this method, though, is that we must be able to find an antiderivative, and this is not always easy. In this section we examine a technique, called **integration by substitution**, to help us find antiderivatives. Specifically, this method helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

At first, the approach to the substitution procedure may not appear very obvious. However, it is primarily a visual task—that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see? We are looking for an integrand of the form $f[g(x)]g'(x)dx$. For example, in the integral $\int (x^2 - 3)^3 2x dx$, we have $f(x) = x^3$, $g(x) = x^2 - 3$, and $g'(x) = 2x$.

Then,

Equation:

$$f[g(x)]g'(x) = (x^2 - 3)^3 (2x),$$

and we see that our integrand is in the correct form.

The method is called *substitution* because we substitute part of the integrand with the variable u and part of the integrand with du . It is also referred to as **change of variables** because we are changing variables to obtain an expression that is easier to work with for applying the integration rules.

Note:

Substitution with Indefinite Integrals

Let $u = g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g , and let $F(x)$ be an antiderivative of $f(x)$. Then,

Equation:

$$\begin{aligned}
 \int f[g(x)]g'(x)dx &= \int f(u)du \\
 &= F(u) + C \\
 &= F(g(x)) + C.
 \end{aligned}$$

Proof

Let f , g , u , and F be as specified in the theorem. Then

Equation:

$$\begin{aligned}
 \frac{d}{dx}F(g(x)) &= F'(g(x))g'(x) \\
 &= f[g(x)]g'(x).
 \end{aligned}$$

Integrating both sides with respect to x , we see that

Equation:

$$\int f[g(x)]g'(x)dx = F(g(x)) + C.$$

If we now substitute $u = g(x)$, and $du = g'(x)dx$, we get

Equation:

$$\begin{aligned}
 \int f[g(x)]g'(x)dx &= \int f(u)du \\
 &= F(u) + C \\
 &= F(g(x)) + C.
 \end{aligned}$$

□

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2x dx$. Rewrite the integral in terms of u :

Equation:

$$\int \underbrace{(x^2 - 3)}_u \underbrace{(2x dx)}_{du} = \int u^3 du.$$

Using the power rule for integrals, we have

Equation:

$$\int u^3 du = \frac{u^4}{4} + C.$$

Substitute the original expression for x back into the solution:

Equation:

$$\frac{u^4}{4} + C = \frac{(x^2 - 3)^4}{4} + C.$$

We can generalize the procedure in the following Problem-Solving Strategy.

Note:

Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u . Let's select $g(x)$ such that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$ into the integral.
3. We should now be able to evaluate the integral with respect to u . If the integral can't be evaluated we need to go back and select a different expression to use as u .
4. Evaluate the integral in terms of u .
5. Write the result in terms of x and the expression $g(x)$.

Example:

Exercise:

Problem:

Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative of $\int 6x(3x^2 + 4)^4 dx$.

Solution:

The first step is to choose an expression for u . We choose $u = 3x^2 + 4$. because then $du = 6x dx$. , and we already have du in the integrand. Write the integral in terms of u :

Equation:

$$\int 6x(3x^2 + 4)^4 dx = \int u^4 du.$$

Remember that du is the derivative of the expression chosen for u , regardless of what is inside the integrand. Now we can evaluate the integral with respect to u :

Equation:

$$\begin{aligned}\int u^4 du &= \frac{u^5}{5} + C \\ &= \frac{(3x^2+4)^5}{5} + C.\end{aligned}$$

Analysis

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand. Picking a value for C of 1, we let

$y = \frac{1}{5}(3x^2 + 4)^5 + 1$. We have

Equation:

$$y = \frac{1}{5}(3x^2 + 4)^5 + 1,$$

so

Equation:

$$\begin{aligned}y' &= \left(\frac{1}{5}\right)5(3x^2 + 4)^4 6x \\ &= 6x(3x^2 + 4)^4.\end{aligned}$$

This is exactly the expression we started with inside the integrand.

Note:

Exercise:

Problem: Use substitution to find the antiderivative of $\int 3x^2(x^3 - 3)^2 dx$.

Solution:

$$\int 3x^2(x^3 - 3)^2 dx = \frac{1}{3}(x^3 - 3)^3 + C$$

Hint

Let $u = x^3 - 3$.

Sometimes we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.

Example:

Exercise:

Problem:

Using Substitution with Alteration

Use substitution to find the antiderivative of $\int z\sqrt{z^2 - 5} dz$.

Solution:

Rewrite the integral as $\int z(z^2 - 5)^{1/2} dz$. Let $u = z^2 - 5$ and $du = 2z dz$.

Now we have a problem because $du = 2z dz$ and the original expression has only $z dz$. We have to alter our expression for du or the integral in u will be twice as large as it should be. If we multiply both sides of the du equation by $\frac{1}{2}$, we can solve this problem. Thus,

Equation:

$$\begin{aligned}
 u &= z^2 - 5 \\
 du &= 2z \, dz \\
 \frac{1}{2} du &= \frac{1}{2} (2z) dz = z \, dz.
 \end{aligned}$$

Write the integral in terms of u , but pull the $\frac{1}{2}$ outside the integration symbol:

Equation:

$$\int z(z^2 - 5)^{1/2} dz = \frac{1}{2} \int u^{1/2} du.$$

Integrate the expression in u :

Equation:

$$\begin{aligned}
 \frac{1}{2} \int u^{1/2} du &= \left(\frac{1}{2}\right) \frac{u^{3/2}}{\frac{3}{2}} + C \\
 &= \left(\frac{1}{2}\right) \left(\frac{2}{3}\right) u^{3/2} + C \\
 &= \frac{1}{3} u^{3/2} + C \\
 &= \frac{1}{3} (z^2 - 5)^{3/2} + C.
 \end{aligned}$$

Note:

Exercise:

Problem: Use substitution to find the antiderivative of $\int x^2(x^3 + 5)^9 dx$.

Solution:

$$\frac{(x^3+5)^{10}}{30} + C$$

Hint

Multiply the du equation by $\frac{1}{3}$.

Example:

Exercise:

Problem:

Using Substitution with Integrals of Trigonometric Functions

Use substitution to evaluate the integral $\int \frac{\sin t}{\cos^3 t} dt$.

Solution:

We know the derivative of $\cos t$ is $-\sin t$, so we set $u = \cos t$. Then $du = -\sin t dt$. Substituting into the integral, we have

Equation:

$$\int \frac{\sin t}{\cos^3 t} dt = - \int \frac{du}{u^3}.$$

Evaluating the integral, we get

Equation:

$$\begin{aligned} - \int \frac{du}{u^3} &= - \int u^{-3} du \\ &= - \left(-\frac{1}{2} \right) u^{-2} + C. \end{aligned}$$

Putting the answer back in terms of t , we get

Equation:

$$\begin{aligned} \int \frac{\sin t}{\cos^3 t} dt &= \frac{1}{2u^2} + C \\ &= \frac{1}{2\cos^2 t} + C. \end{aligned}$$

Note:

Exercise:

Problem: Use substitution to evaluate the integral $\int \frac{\cos t}{\sin^2 t} dt$.

Solution:

$$-\frac{1}{\sin t} + C$$

Hint

Use the process from [\[link\]](#) to solve the problem.

Sometimes we need to manipulate an integral in ways that are more complicated than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of the original variable. When we are done, u should be the only variable in the integrand. In some cases, this means solving for the original variable in terms of u . This technique should become clear in the next example.

Example:

Exercise:

Problem:

Finding an Antiderivative Using u -Substitution

Use substitution to find the antiderivative of $\int \frac{x}{\sqrt{x-1}} dx$.

Solution:

If we let $u = x - 1$, then $du = dx$. But this does not account for the x in the numerator of the integrand. We need to express x in terms of u . If $u = x - 1$, then $x = u + 1$. Now we can rewrite the integral in terms of u :

Equation:

$$\begin{aligned}
 \int \frac{x}{\sqrt{x-1}} dx &= \int \frac{u+1}{\sqrt{u}} du \\
 &= \int \sqrt{u} + \frac{1}{\sqrt{u}} du \\
 &= \int (u^{1/2} + u^{-1/2}) du.
 \end{aligned}$$

Then we integrate in the usual way, replace u with the original expression, and factor and simplify the result. Thus,

Equation:

$$\begin{aligned}
 \int (u^{1/2} + u^{-1/2}) du &= \frac{2}{3} u^{3/2} + 2u^{1/2} + C \\
 &= \frac{2}{3} (x-1)^{3/2} + 2(x-1)^{1/2} + C \\
 &= (x-1)^{1/2} \left[\frac{2}{3} (x-1) + 2 \right] + C \\
 &= (x-1)^{1/2} \left(\frac{2}{3} x - \frac{2}{3} + \frac{6}{3} \right) \\
 &= (x-1)^{1/2} \left(\frac{2}{3} x + \frac{4}{3} \right) \\
 &= \frac{2}{3} (x-1)^{1/2} (x+2) + C.
 \end{aligned}$$

Note:

Exercise:

Problem: Use substitution to evaluate the indefinite integral $\int \cos^3 t \sin t \, dt$.

Solution:

$$-\frac{\cos^4 t}{4} + C$$

Hint

Use the process from [\[link\]](#) to solve the problem.

Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires a change to the limits of integration. If we change variables in the integrand, the limits of integration change as well.

Note:**Substitution with Definite Integrals**

Let $u = g(x)$ and let g' be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$. Then,

Equation:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Although we will not formally prove this theorem, we justify it with some calculations here. From the substitution rule for indefinite integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

Equation:

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

Then

Equation:

$$\begin{aligned}\int_a^b f[g(x)]g'(x)dx &= F(g(x))\Big|_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a)) \\ &= F(u)\Big|_{u=g(a)}^{u=g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)du,\end{aligned}$$

and we have the desired result.

Example:

Exercise:

Problem:

Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate $\int_0^1 x^2(1 + 2x^3)^5 dx$.

Solution:

Let $u = 1 + 2x^3$, so $du = 6x^2 dx$. Since the original function includes one factor of x^2 and $du = 6x^2 dx$, multiply both sides of the du equation by $1/6$. Then,

Equation:

$$\begin{aligned} du &= 6x^2 dx \\ \frac{1}{6} du &= x^2 dx. \end{aligned}$$

To adjust the limits of integration, note that when $x = 0$, $u = 1 + 2(0) = 1$, and when $x = 1$, $u = 1 + 2(1) = 3$. Then

Equation:

$$\int_0^1 x^2(1 + 2x^3)^5 dx = \frac{1}{6} \int_1^3 u^5 du.$$

Evaluating this expression, we get

Equation:

$$\begin{aligned} \frac{1}{6} \int_1^3 u^5 du &= \left(\frac{1}{6}\right) \left(\frac{u^6}{6}\right) \Big|_1^3 \\ &= \frac{1}{36} \left[(3)^6 - (1)^6\right] \\ &= \frac{182}{9}. \end{aligned}$$

Note:

Exercise:

Problem: Use substitution to evaluate the definite integral $\int_{-1}^0 y(2y^2 - 3)^5 dy$.

Solution:

$$\frac{91}{3}$$

Hint

Use the steps from [\[link\]](#) to solve the problem.

Example:

Exercise:

Problem:

Using Substitution with an Exponential Function

Use substitution to evaluate $\int_0^1 xe^{4x^2+3} dx$.

Solution:

Let $u = 4x^2 + 3$. Then, $du = 8x dx$. To adjust the limits of integration, we note that when $x = 0$, $u = 3$, and when $x = 1$, $u = 7$. So our substitution gives

Equation:

$$\begin{aligned}\int_0^1 xe^{4x^2+3} dx &= \frac{1}{8} \int_3^7 e^u du \\ &= \frac{1}{8} e^u \Big|_3^7 \\ &= \frac{e^7 - e^3}{8} \\ &\approx 134.568.\end{aligned}$$

Note:

Exercise:

Problem: Use substitution to evaluate $\int_0^1 x^2 \cos\left(\frac{\pi}{2}x^3\right) dx$.

Solution:

$$\frac{2}{3\pi} \approx 0.2122$$

Hint

Use the process from [\[link\]](#) to solve the problem.

Substitution may be only one of the techniques needed to evaluate a definite integral. All of the properties and rules of integration apply independently, and trigonometric functions may need to be rewritten using a trigonometric identity before we can apply substitution. Also, we have the option of replacing the original expression for u after we find the antiderivative, which means that we do not have to change the limits of integration. These two approaches are shown in [\[link\]](#).

Example:**Exercise:****Problem:****Using Substitution to Evaluate a Trigonometric Integral**

Use substitution to evaluate $\int_0^{\pi/2} \cos^2 \theta \, d\theta$.

Solution:

Let us first use a trigonometric identity to rewrite the integral. The trig identity $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$ allows us to rewrite the integral as

Equation:

$$\int_0^{\pi/2} \cos^2 \theta \, d\theta = \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} \, d\theta.$$

Then,

Equation:

$$\begin{aligned}\int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta &= \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} d\theta + \int_0^{\pi/2} \cos 2\theta d\theta.\end{aligned}$$

We can evaluate the first integral as it is, but we need to make a substitution to evaluate the second integral. Let $u = 2\theta$. Then, $du = 2d\theta$, or $\frac{1}{2} du = d\theta$. Also, when $\theta = 0$, $u = 0$, and when $\theta = \pi/2$, $u = \pi$. Expressing the second integral in terms of u , we have

Equation:

$$\begin{aligned}\frac{1}{2} \int_0^{\pi/2} d\theta + \frac{1}{2} \int_0^{\pi/2} \cos 2\theta d\theta &= \frac{1}{2} \int_0^{\pi/2} d\theta + \frac{1}{2} \left(\frac{1}{2} \right) \int_0^{\pi} \cos u du \\ &= \frac{\theta}{2} \Big|_{\theta=0}^{\theta=\pi/2} + \frac{1}{4} \sin u \Big|_{u=0}^{u=\pi} \\ &= \left(\frac{\pi}{4} - 0 \right) + (0 - 0) = \frac{\pi}{4}.\end{aligned}$$

Key Concepts

- Substitution is a technique that simplifies the integration of functions that are the result of a chain-rule derivative. The term ‘substitution’ refers to changing variables or substituting the variable u and du for appropriate expressions in the integrand.
- When using substitution for a definite integral, we also have to change the limits of integration.

Key Equations

- **Substitution with Indefinite Integrals**

$$\int f[g(x)]g'(x)dx = \int f(u)du = F(u) + C = F(g(x)) + C$$

- **Substitution with Definite Integrals**

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Exercise:

Problem: Why is u -substitution referred to as *change of variable*?

Exercise:

Problem:

2. If $f = g \circ h$, when reversing the chain rule, $\frac{d}{dx}(g \circ h)(x) = g'(h(x))h'(x)$, should you take $u = g(x)$ or $u = h(x)$?

Solution:

$$u = h(x)$$

In the following exercises, verify each identity using differentiation. Then, using the indicated u -substitution, identify f such that the integral takes the form $\int f(u)du$.

Exercise:

Problem: $\int x\sqrt{x+1}dx = \frac{2}{15}(x+1)^{3/2}(3x-2) + C; u = x+1$

Exercise:

Problem:

$$\int \frac{x^2}{\sqrt{x-1}}dx (x > 1) = \frac{2}{15}\sqrt{x-1}(3x^2+4x+8) + C; u = x-1$$

Solution:

$$f(u) = \frac{(u+1)^2}{\sqrt{u}}$$

Exercise:

Problem: $\int x\sqrt{4x^2+9}dx = \frac{1}{12}(4x^2+9)^{3/2} + C; u = 4x^2+9$

Exercise:

Problem: $\int \frac{x}{\sqrt{4x^2 + 9}} dx = \frac{1}{4} \sqrt{4x^2 + 9} + C; u = 4x^2 + 9$

Solution:

$$du = 8x dx; f(u) = \frac{1}{8\sqrt{u}}$$

Exercise:

Problem: $\int \frac{x}{(4x^2 + 9)^2} dx = -\frac{1}{8(4x^2 + 9)}; u = 4x^2 + 9$

In the following exercises, find the antiderivative using the indicated substitution.

Exercise:

Problem: $\int (x + 1)^4 dx; u = x + 1$

Solution:

$$\frac{1}{5}(x + 1)^5 + C$$

Exercise:

Problem: $\int (x - 1)^5 dx; u = x - 1$

Exercise:

Problem: $\int (2x - 3)^{-7} dx; u = 2x - 3$

Solution:

$$-\frac{1}{12(3-2x)^6} + C$$

Exercise:

Problem: $\int (3x - 2)^{-11} dx; u = 3x - 2$

Exercise:

Problem: $\int \frac{x}{\sqrt{x^2 + 1}} dx; u = x^2 + 1$

Solution:

$$\sqrt{x^2 + 1} + C$$

Exercise:

Problem: $\int \frac{x}{\sqrt{1 - x^2}} dx; u = 1 - x^2$

Exercise:

Problem: $\int (x - 1)(x^2 - 2x)^3 dx; u = x^2 - 2x$

Solution:

$$\frac{1}{8} (x^2 - 2x)^4 + C$$

Exercise:

Problem: $\int (x^2 - 2x)(x^3 - 3x^2)^2 dx; u = x^3 - 3x^2$

Exercise:

Problem: $\int \cos^3 \theta d\theta; u = \sin \theta$ (*Hint:* $\cos^2 \theta = 1 - \sin^2 \theta$)

Solution:

$$\sin \theta - \frac{\sin^3 \theta}{3} + C$$

Exercise:

Problem: $\int \sin^3 \theta d\theta; u = \cos \theta$ (*Hint:* $\sin^2 \theta = 1 - \cos^2 \theta$)

In the following exercises, use a suitable change of variables to determine the indefinite integral.

Exercise:

Problem: $\int x(1-x)^{99} dx$

Solution:

$$\frac{(1-x)^{101}}{101} - \frac{(1-x)^{100}}{100} + C$$

Exercise:

Problem: $\int t(1-t^2)^{10} dt$

Exercise:

Problem: $\int (11x-7)^{-3} dx$

Solution:

$$-\frac{1}{22(7-11x^2)} + C$$

Exercise:

Problem: $\int (7x-11)^4 dx$

Exercise:

Problem: $\int \cos^3 \theta \sin \theta d\theta$

Solution:

$$-\frac{\cos^4\theta}{4} + C$$

Exercise:

Problem: $\int \sin^7\theta \cos\theta d\theta$

Exercise:

Problem: $\int \cos^2(\pi t) \sin(\pi t) dt$

Solution:

$$-\frac{\cos^3(\pi t)}{3\pi} + C$$

Exercise:

Problem: $\int \sin^2 x \cos^3 x dx$ (*Hint:* $\sin^2 x + \cos^2 x = 1$)

Exercise:

Problem: $\int t \sin(t^2) \cos(t^2) dt$

Solution:

$$-\frac{1}{4} \cos^2(t^2) + C$$

Exercise:

Problem: $\int t^2 \cos^2(t^3) \sin(t^3) dt$

Exercise:

Problem: $\int \frac{x^2}{(x^3 - 3)^2} dx$

Solution:

$$-\frac{1}{3(x^3-3)} + C$$

Exercise:

Problem: $\int \frac{x^3}{\sqrt{1-x^2}} dx$

Exercise:

Problem: $\int \frac{y^5}{(1-y^3)^{3/2}} dy$

Solution:

$$-\frac{2(y^3-2)}{3\sqrt{1-y^3}}$$

Exercise:

Problem: $\int \cos \theta (1 - \cos \theta)^{99} \sin \theta d\theta$

Exercise:

Problem: $\int (1 - \cos^3 \theta)^{10} \cos^2 \theta \sin \theta d\theta$

Solution:

$$\frac{1}{33} (1 - \cos^3 \theta)^{11} + C$$

Exercise:

Problem: $\int (\cos \theta - 1)(\cos^2 \theta - 2 \cos \theta)^3 \sin \theta d\theta$

Exercise:

Problem: $\int (\sin^2 \theta - 2 \sin \theta)(\sin^3 \theta - 3 \sin^2 \theta)^3 \cos \theta d\theta$

Solution:

$$\frac{1}{12} (\sin^3 \theta - 3\sin^2 \theta)^4 + C$$

In the following exercises, use a calculator to estimate the area under the curve using left Riemann sums with 50 terms, then use substitution to solve for the exact answer.

Exercise:

Problem: [T] $y = 3(1 - x)^2$ over $[0, 2]$

Exercise:

Problem: [T] $y = x(1 - x^2)^3$ over $[-1, 2]$

Solution:

$L_{50} = -8.5779$. The exact area is $-\frac{81}{8}$

Exercise:

Problem: [T] $y = \sin x(1 - \cos x)^2$ over $[0, \pi]$

Exercise:

Problem: [T] $y = \frac{x}{(x^2+1)^2}$ over $[-1, 1]$

Solution:

$L_{50} = -0.006399 \dots$ The exact area is 0.

In the following exercises, use a change of variables to evaluate the definite integral.

Exercise:

Problem: $\int_0^1 x\sqrt{1-x^2} dx$

Exercise:

Problem: $\int_0^1 \frac{x}{\sqrt{1+x^2}} dx$

Solution:

$$u = 1 + x^2, du = 2x dx, \frac{1}{2} \int_1^2 u^{-1/2} du = \sqrt{2} - 1$$

Exercise:

Problem: $\int_0^2 \frac{t}{\sqrt{5+t^2}} dt$

Exercise:

Problem: $\int_0^1 \frac{t^2}{\sqrt{1+t^3}} dt$

Solution:

$$u = 1 + t^3, du = 3t^2 dt, \frac{1}{3} \int_1^2 u^{-1/2} du = \frac{2}{3} (\sqrt{2} - 1)$$

Exercise:

Problem: $\int_0^{\pi/4} \sec^2 \theta \tan \theta d\theta$

Exercise:

Problem: $\int_0^{\pi/4} \frac{\sin \theta}{\cos^4 \theta} d\theta$

Solution:

$$u = \cos \theta, du = -\sin \theta d\theta, \int_{1/\sqrt{2}}^1 u^{-4} du = \frac{1}{3} (2\sqrt{2} - 1)$$

In the following exercises, evaluate the indefinite integral $\int f(x) dx$ with constant $C = 0$ using u -substitution. Then, graph the function and the antiderivative over the indicated interval. If possible, estimate a value of C that would need to be added to the antiderivative to make it equal to the definite integral $F(x) = \int_a^x f(t) dt$, with a the left endpoint of the given interval.

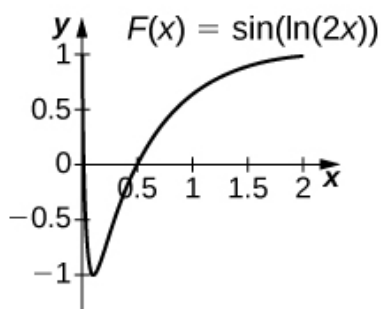
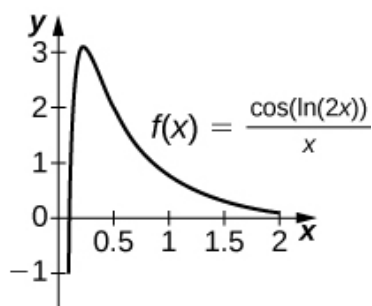
Exercise:

Problem: [T] $\int (2x + 1)e^{x^2+x-6} dx$ over $[-3, 2]$

Exercise:

Problem: [T] $\int \frac{\cos(\ln(2x))}{x} dx$ on $[0, 2]$

Solution:



The antiderivative is $y = \sin(\ln(2x))$. Since the antiderivative is not continuous at $x = 0$, one cannot find a value of C that would make $y = \sin(\ln(2x)) - C$ work as a definite integral.

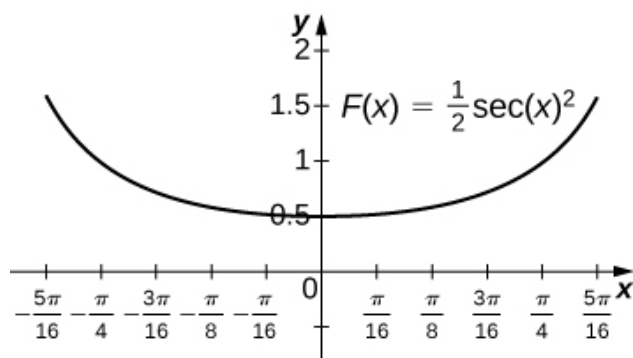
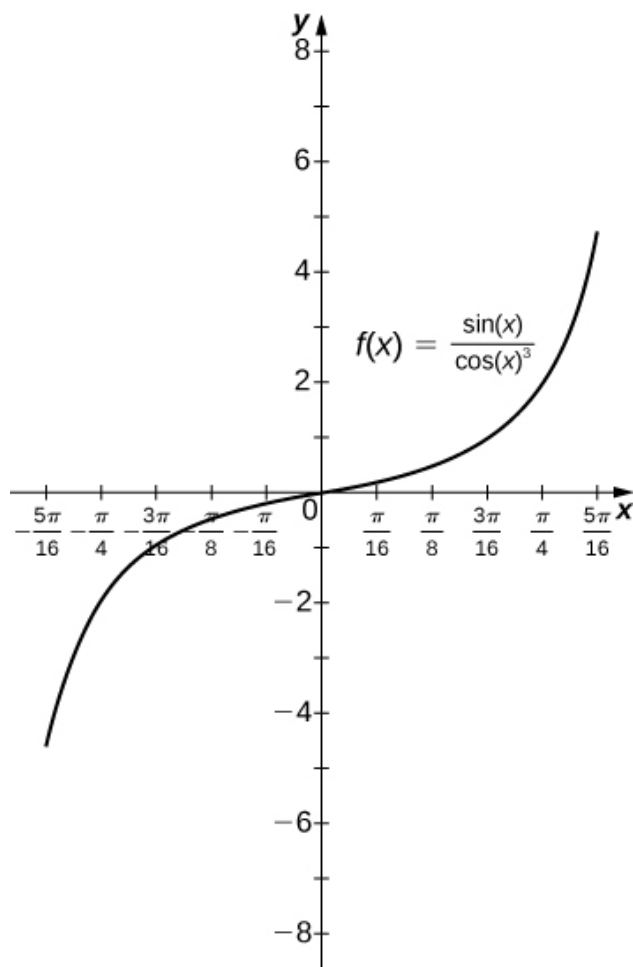
Exercise:

Problem: [T] $\int \frac{3x^2 + 2x + 1}{\sqrt{x^3 + x^2 + x + 4}} dx$ over $[-1, 2]$

Exercise:

Problem: [T] $\int \frac{\sin x}{\cos^3 x} dx$ over $[-\frac{\pi}{3}, \frac{\pi}{3}]$

Solution:



The antiderivative is $y = \frac{1}{2} \sec^2 x$. You should take $C = -2$ so that $F\left(-\frac{\pi}{3}\right) = 0$.

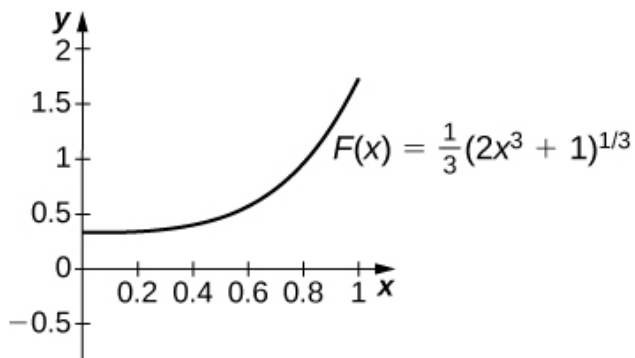
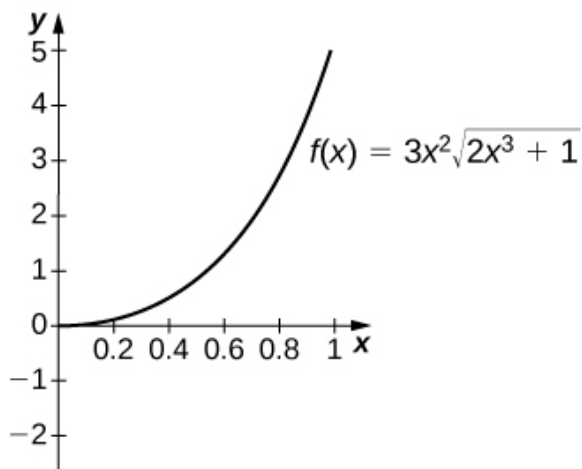
Exercise:

Problem: [T] $\int (x + 2)e^{-x^2 - 4x + 3} dx$ over $[-5, 1]$

Exercise:

Problem: [T] $\int 3x^2 \sqrt{2x^3 + 1} dx$ over $[0, 1]$

Solution:



The antiderivative is $y = \frac{1}{3}(2x^3 + 1)^{3/2}$. One should take $C = -\frac{1}{3}$.

Exercise:

Problem:

If $h(a) = h(b)$ in $\int_a^b g(h(x))h(x)dx$, what can you say about the value of the integral?

Exercise:

Problem:

Is the substitution $u = 1 - x^2$ in the definite integral $\int_0^2 \frac{x}{1 - x^2} dx$ okay? If not, why not?

Solution:

No, because the integrand is discontinuous at $x = 1$.

In the following exercises, use a change of variables to show that each definite integral is equal to zero.

Exercise:

Problem: $\int_0^\pi \cos^2(2\theta)\sin(2\theta)d\theta$

Exercise:

Problem: $\int_0^{\sqrt{\pi}} t \cos(t^2)\sin(t^2)dt$

Solution:

$u = \sin(t^2)$; the integral becomes $\frac{1}{2} \int_0^0 u du$.

Exercise:

Problem: $\int_0^1 (1 - 2t) dt$

Exercise:

Problem: $\int_0^1 \frac{1 - 2t}{\left(1 + \left(t - \frac{1}{2}\right)^2\right)} dt$

Solution:

$u = \left(1 + \left(t - \frac{1}{2}\right)^2\right)$; the integral becomes $-\int_{5/4}^{5/4} \frac{1}{u} du$.

Exercise:

Problem: $\int_0^\pi \sin\left(\left(t - \frac{\pi}{2}\right)^3\right) \cos\left(t - \frac{\pi}{2}\right) dt$

Exercise:

Problem: $\int_0^2 (1 - t) \cos(\pi t) dt$

Solution:

$u = 1 - t$; the integral becomes

$$\int_1^{-1} u \cos(\pi(1 - u)) du$$

$$= \int_1^{-1} u [\cos \pi \cos u - \sin \pi \sin u] du$$

$$= - \int_1^{-1} u \cos u du$$

$$= \int_{-1}^1 u \cos u du = 0$$

since the integrand is odd.

Exercise:

Problem: $\int_{\pi/4}^{3\pi/4} \sin^2 t \cos t dt$

Exercise:

Problem:

Show that the average value of $f(x)$ over an interval $[a, b]$ is the same as the average value of $f(cx)$ over the interval $[\frac{a}{c}, \frac{b}{c}]$ for $c > 0$.

Solution:

Setting $u = cx$ and $du = cdx$ gets you

$$\frac{1}{\frac{b}{c} - \frac{a}{c}} \int_{a/c}^{b/c} f(cx) dx = \frac{c}{b-a} \int_{u=a}^{u=b} f(u) \frac{du}{c} = \frac{1}{b-a} \int_a^b f(u) du.$$

Exercise:

Problem:

Find the area under the graph of $f(t) = \frac{t}{(1+t^2)^a}$ between $t = 0$ and $t = x$ where $a > 0$ and $a \neq 1$ is fixed, and evaluate the limit as $x \rightarrow \infty$.

Exercise:

Problem:

Find the area under the graph of $g(t) = \frac{t}{(1-t^2)^a}$ between $t = 0$ and $t = x$, where $0 < x < 1$ and $a > 0$ is fixed. Evaluate the limit as $x \rightarrow 1$.

Solution:

$$\int_0^x g(t) dt = \frac{1}{2} \int_{u=1-x^2}^1 \frac{du}{u^a} = \frac{1}{2(1-a)} u^{1-a} \Big|_{u=1-x^2}^1 = \frac{1}{2(1-a)} \left(1 - (1-x^2)^{1-a}\right).$$

As $x \rightarrow 1$ the limit is $\frac{1}{2(1-a)}$ if $a < 1$, and the limit diverges to $+\infty$ if $a > 1$.

Exercise:

Problem:

The area of a semicircle of radius 1 can be expressed as $\int_{-1}^1 \sqrt{1-x^2} dx$. Use the substitution $x = \cos t$ to express the area of a semicircle as the integral of a trigonometric function. You do not need to compute the integral.

Exercise:**Problem:**

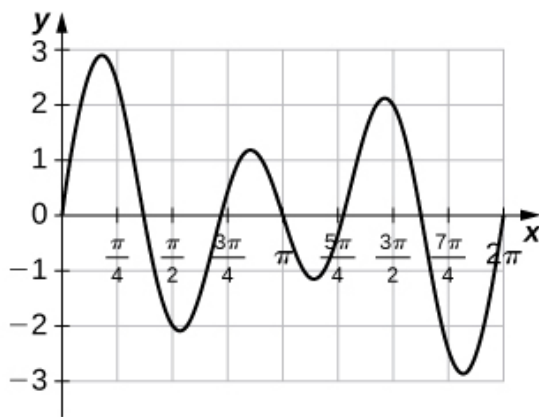
The area of the top half of an ellipse with a major axis that is the x -axis from $x = -1$ to a and with a minor axis that is the y -axis from $y = -b$ to b can be written as $\int_{-a}^a b \sqrt{1 - \frac{x^2}{a^2}} dx$. Use the substitution $x = a \cos t$ to express this area in terms of an integral of a trigonometric function. You do not need to compute the integral.

Solution:

$$\int_{t=\pi}^0 b \sqrt{1 - \cos^2 t} \times (-a \sin t) dt = \int_{t=0}^{\pi} ab \sin^2 t dt$$

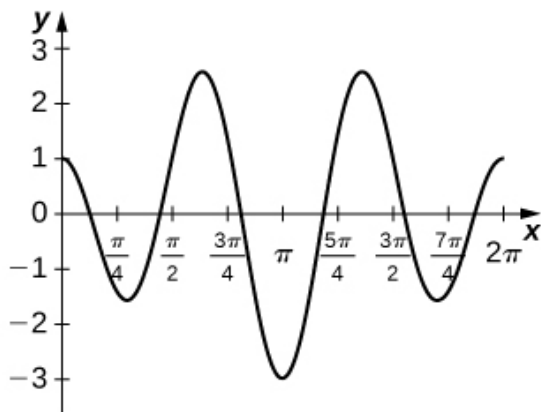
Exercise:**Problem:**

[T] The following graph is of a function of the form $f(t) = a \sin(nt) + b \sin(mt)$. Estimate the coefficients a and b , and the frequency parameters n and m . Use these estimates to approximate $\int_0^{\pi} f(t) dt$.



Exercise:**Problem:**

[T] The following graph is of a function of the form $f(x) = a \cos(nt) + b \cos(mt)$. Estimate the coefficients a and b and the frequency parameters n and m . Use these estimates to approximate $\int_0^\pi f(t) dt$.

**Solution:**

$$f(t) = 2 \cos(3t) - \cos(2t); \int_0^{\pi/2} (2 \cos(3t) - \cos(2t)) = -\frac{2}{3}$$

Glossary

change of variables

the substitution of a variable, such as u , for an expression in the integrand

integration by substitution

a technique for integration that allows integration of functions that are the result of a chain-rule derivative

Integrals Involving Exponential and Logarithmic Functions

- Integrate functions involving exponential functions.
- Integrate functions involving logarithmic functions.

Exponential and logarithmic functions are used to model population growth, cell growth, and financial growth, as well as depreciation, radioactive decay, and resource consumption, to name only a few applications. In this section, we explore integration involving exponential and logarithmic functions.

Integrals of Exponential Functions

The exponential function is perhaps the most efficient function in terms of the operations of calculus. The exponential function, $y = e^x$, is its own derivative and its own integral.

Note:

Rule: Integrals of Exponential Functions

Exponential functions can be integrated using the following formulas.

Equation:

$$\begin{aligned}\int e^x dx &= e^x + C \\ \int a^x dx &= \frac{a^x}{\ln a} + C\end{aligned}$$

Example:**Exercise:****Problem:**

Finding an Antiderivative of an Exponential Function

Find the antiderivative of the exponential function e^{-x} .

Solution:

Use substitution, setting $u = -x$, and then $du = -1dx$. Multiply the du equation by -1 , so you now have $-du = dx$. Then,

Equation:

$$\begin{aligned}\int e^{-x} dx &= -\int e^u du \\ &= -e^u + C \\ &= -e^{-x} + C.\end{aligned}$$

Note:

Exercise:

Problem:

Find the antiderivative of the function using substitution: $x^2 e^{-2x^3}$.

Solution:

$$\int x^2 e^{-2x^3} dx = -\frac{1}{6} e^{-2x^3} + C$$

Hint

Let u equal the exponent on e .

A common mistake when dealing with exponential expressions is treating the exponent on e the same way we treat exponents in polynomial expressions. We cannot use the power rule for the exponent on e . This can

be especially confusing when we have both exponentials and polynomials in the same expression, as in the previous checkpoint. In these cases, we should always double-check to make sure we're using the right rules for the functions we're integrating.

Example:

Exercise:

Problem:

Square Root of an Exponential Function

Find the antiderivative of the exponential function $e^x \sqrt{1 + e^x}$.

Solution:

First rewrite the problem using a rational exponent:

Equation:

$$\int e^x \sqrt{1 + e^x} dx = \int e^x (1 + e^x)^{1/2} dx.$$

Using substitution, choose $u = 1 + e^x$. $u = 1 + e^x$. Then, $du = e^x dx$. We have ([link](#))

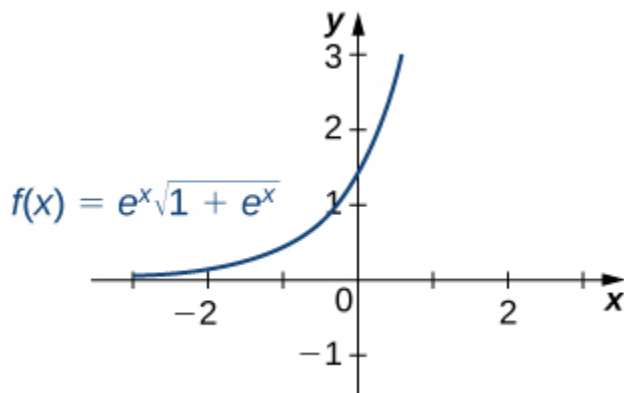
Equation:

$$\int e^x (1 + e^x)^{1/2} dx = \int u^{1/2} du.$$

Then

Equation:

$$\int u^{1/2} du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1 + e^x)^{3/2} + C.$$



The graph shows an exponential function times the square root of an exponential function.

Note:

Exercise:

Problem: Find the antiderivative of $e^x(3e^x - 2)^2$.

Solution:

$$\int e^x(3e^x - 2)^2 dx = \frac{1}{9}(3e^x - 2)^3$$

Hint

Let $u = 3e^x - 2$ $u = 3e^x - 2$.

Example:

Exercise:

Problem:**Using Substitution with an Exponential Function**

Use substitution to evaluate the indefinite integral $\int 3x^2 e^{2x^3} dx$.

Solution:

Here we choose to let u equal the expression in the exponent on e . Let $u = 2x^3$ and $du = 6x^2 dx$. Again, du is off by a constant multiplier; the original function contains a factor of $3x^2$, not $6x^2$. Multiply both sides of the equation by $\frac{1}{2}$ so that the integrand in u equals the integrand in x . Thus,

Equation:

$$\int 3x^2 e^{2x^3} dx = \frac{1}{2} \int e^u du.$$

Integrate the expression in u and then substitute the original expression in x back into the u integral:

Equation:

$$\frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{2x^3} + C.$$

Note:**Exercise:**

Problem: Evaluate the indefinite integral $\int 2x^3 e^{x^4} dx$.

Solution:

$$\int 2x^3 e^{x^4} dx = \frac{1}{2} e^{x^4}$$

Hint

Let $u = x^4$.

As mentioned at the beginning of this section, exponential functions are used in many real-life applications. The number e is often associated with compounded or accelerating growth, as we have seen in earlier sections about the derivative. Although the derivative represents a rate of change or a growth rate, the integral represents the total change or the total growth. Let's look at an example in which integration of an exponential function solves a common business application.

A price–demand function tells us the relationship between the quantity of a product demanded and the price of the product. In general, price decreases as quantity demanded increases. The marginal price–demand function is the derivative of the price–demand function and it tells us how fast the price changes at a given level of production. These functions are used in business to determine the price–elasticity of demand, and to help companies determine whether changing production levels would be profitable.

Example:**Exercise:****Problem:****Finding a Price–Demand Equation**

Find the price–demand equation for a particular brand of toothpaste at a supermarket chain when the demand is 50 tubes per week at \$2.35 per tube, given that the marginal price–demand function, $p'(x)$, for x number of tubes per week, is given as

Equation:

$$p'(x) = -0.015e^{-0.01x}.$$

If the supermarket chain sells 100 tubes per week, what price should it set?

Solution:

To find the price–demand equation, integrate the marginal price–demand function. First find the antiderivative, then look at the particulars. Thus,

Equation:

$$\begin{aligned} p(x) &= \int -0.015e^{-0.01x} dx \\ &= -0.015 \int e^{-0.01x} dx. \end{aligned}$$

Using substitution, let $u = -0.01x$ and $du = -0.01dx$. Then, divide both sides of the du equation by -0.01 . This gives

Equation:

$$\begin{aligned} \frac{-0.015}{-0.01} \int e^u du &= 1.5 \int e^u du \\ &= 1.5e^u + C \\ &= 1.5e^{-0.01x} + C. \end{aligned}$$

The next step is to solve for C . We know that when the price is \$2.35 per tube, the demand is 50 tubes per week. This means

Equation:

$$\begin{aligned} p(50) &= 1.5e^{-0.01(50)} + C \\ &= 2.35. \end{aligned}$$

Now, just solve for C :

Equation:

$$\begin{aligned}C &= 2.35 - 1.5e^{-0.5} \\&= 2.35 - 0.91 \\&= 1.44.\end{aligned}$$

Thus,

Equation:

$$p(x) = 1.5e^{-0.01x} + 1.44.$$

If the supermarket sells 100 tubes of toothpaste per week, the price would be

Equation:

$$p(100) = 1.5e^{-0.01(100)} + 1.44 = 1.5e^{-1} + 1.44 \approx 1.99.$$

The supermarket should charge \$1.99 per tube if it is selling 100 tubes per week.

Example:

Exercise:

Problem:

Evaluating a Definite Integral Involving an Exponential Function

Evaluate the definite integral $\int_1^2 e^{1-x} dx$.

Solution:

Again, substitution is the method to use. Let $u = 1 - x$, so $du = -1dx$ or $-du = dx$. Then $\int e^{1-x} dx = -\int e^u du$. Next, change the limits of integration. Using the equation $u = 1 - x$, we have

Equation:

$$u = 1 - (1) = 0$$

$$u = 1 - (2) = -1.$$

The integral then becomes

Equation:

$$\int_1^2 e^{1-x} dx = -\int_0^{-1} e^u du$$

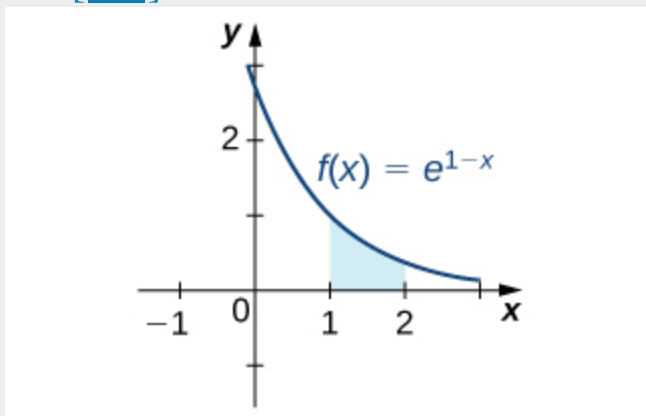
$$= \int_{-1}^0 e^u du$$

$$= e^u \Big|_{-1}^0$$

$$= e^0 - (e^{-1})$$

$$= -e^{-1} + 1.$$

See [\[link\]](#).



The indicated area can be calculated by evaluating a definite integral using substitution.

Note:

Exercise:

Problem: Evaluate $\int_0^2 e^{2x} dx$.

Solution:

$$\frac{1}{2} \int_0^4 e^u du = \frac{1}{2} (e^4 - 1)$$

Hint

Let $u = 2x$.

Example:

Exercise:

Problem:

Growth of Bacteria in a Culture

Suppose the rate of growth of bacteria in a Petri dish is given by $q(t) = 3^t$, where t is given in hours and $q(t)$ is given in thousands of bacteria per hour. If a culture starts with 10,000 bacteria, find a function $Q(t)$ that gives the number of bacteria in the Petri dish at any time t . How many bacteria are in the dish after 2 hours?

Solution:

We have

Equation:

$$Q(t) = \int 3^t dt = \frac{3^t}{\ln 3} + C.$$

Then, at $t = 0$ we have $Q(0) = 10 = \frac{1}{\ln 3} + C$, so $C \approx 9.090$ and we get

Equation:

$$Q(t) = \frac{3^t}{\ln 3} + 9.090.$$

At time $t = 2$, we have

Equation:

$$Q(2) = \frac{3^2}{\ln 3} + 9.090$$

Equation:

$$= 17.282.$$

After 2 hours, there are 17,282 bacteria in the dish.

Note:

Exercise:

Problem:

From [\[link\]](#), suppose the bacteria grow at a rate of $q(t) = 2^t$. Assume the culture still starts with 10,000 bacteria. Find $Q(t)$. How many bacteria are in the dish after 3 hours?

Solution:

$Q(t) = \frac{2^t}{\ln 2} + 8.557$. There are 20,099 bacteria in the dish after 3 hours.

Hint

Use the procedure from [\[link\]](#) to solve the problem.

Example:**Exercise:****Problem:****Fruit Fly Population Growth**

Suppose a population of fruit flies increases at a rate of $g(t) = 2e^{0.02t}$, in flies per day. If the initial population of fruit flies is 100 flies, how many flies are in the population after 10 days?

Solution:

Let $G(t)$ represent the number of flies in the population at time t . Applying the net change theorem, we have

Equation:

$$\begin{aligned}
 G(10) &= G(0) + \int_0^{10} 2e^{0.02t} dt \\
 &= 100 + \left[\frac{2}{0.02} e^{0.02t} \right]_0^{10} \\
 &= 100 + \left[100e^{0.02t} \right]_0^{10} \\
 &= 100 + 100e^{0.2} - 100 \\
 &\approx 122.
 \end{aligned}$$

There are 122 flies in the population after 10 days.

Note:

Exercise:

Problem:

Suppose the rate of growth of the fly population is given by $g(t) = e^{0.01t}$, and the initial fly population is 100 flies. How many flies are in the population after 15 days?

Solution:

There are 116 flies.

Hint

Use the process from [\[link\]](#) to solve the problem.

Example:

Exercise:

Problem:

Evaluating a Definite Integral Using Substitution

Evaluate the definite integral using substitution: $\int_1^2 \frac{e^{1/x}}{x^2} dx$.

Solution:

This problem requires some rewriting to simplify applying the properties. First, rewrite the exponent on e as a power of x , then bring the x^2 in the denominator up to the numerator using a negative exponent. We have

Equation:

$$\int_1^2 \frac{e^{1/x}}{x^2} dx = \int_1^2 e^{x^{-1}} x^{-2} dx.$$

Let $u = x^{-1}$, the exponent on e . Then

Equation:

$$\begin{aligned} du &= -x^{-2} dx \\ -du &= x^{-2} dx. \end{aligned}$$

Bringing the negative sign outside the integral sign, the problem now reads

Equation:

$$- \int e^u du.$$

Next, change the limits of integration:

Equation:

$$u = (1)^{-1} = 1$$

$$u = (2)^{-1} = \frac{1}{2}.$$

Notice that now the limits begin with the larger number, meaning we must multiply by -1 and interchange the limits. Thus,

Equation:

$$\begin{aligned}
 - \int_1^{1/2} e^u du &= \int_{1/2}^1 e^u du \\
 &= e^u \Big|_{1/2}^1 \\
 &= e - e^{1/2} \\
 &= e - \sqrt{e}.
 \end{aligned}$$

Note:

Exercise:

Problem:

Evaluate the definite integral using substitution: $\int_1^2 \frac{1}{x^3} e^{4x^{-2}} dx$.

Solution:

$$\int_1^2 \frac{1}{x^3} e^{4x^{-2}} dx = \frac{1}{8} [e^4 - e]$$

Hint

Let $u = 4x^{-2}$.

Integrals Involving Logarithmic Functions

Integrating functions of the form $f(x) = x^{-1}$ result in the absolute value of the natural log function, as shown in the following rule. Integral formulas for other logarithmic functions, such as $f(x) = \ln x$ and $f(x) = \log_a x$, are also included in the rule.

Note:

Rule: Integration Formulas Involving Logarithmic Functions

The following formulas can be used to evaluate integrals involving logarithmic functions.

Equation:

$$\begin{aligned}\int x^{-1} dx &= \ln |x| + C \\ \int \ln x \, dx &= x \ln x - x + C = x(\ln x - 1) + C \\ \int \log_a x \, dx &= \frac{x}{\ln a} (\ln x - 1) + C\end{aligned}$$

Example:

Exercise:

Problem:

Finding an Antiderivative Involving $\ln x$

Find the antiderivative of the function $\frac{3}{x-10}$.

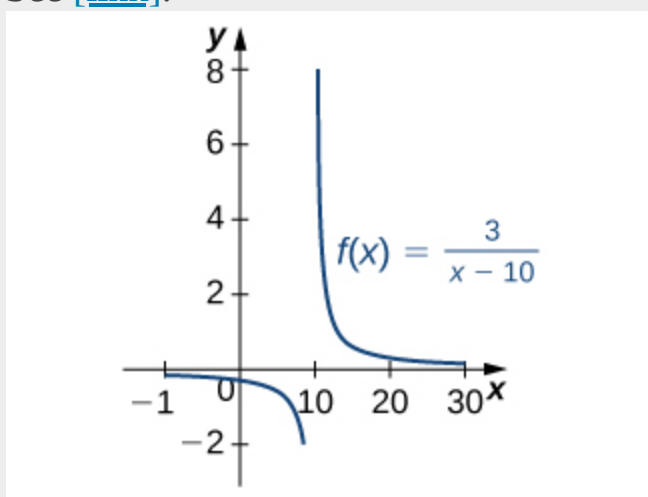
Solution:

First factor the 3 outside the integral symbol. Then use the u^{-1} rule. Thus,

Equation:

$$\begin{aligned}\int \frac{3}{x-10} dx &= 3 \int \frac{1}{x-10} dx \\ &= 3 \int \frac{du}{u} \\ &= 3 \ln |u| + C \\ &= 3 \ln |x-10| + C, x \neq 10.\end{aligned}$$

See [\[link\]](#).



The domain of this function is
 $x \neq 10$.

Note:

Exercise:

Problem: Find the antiderivative of $\frac{1}{x+2}$.

Solution:

$$\ln |x + 2| + C$$

Hint

Follow the pattern from [\[link\]](#) to solve the problem.

Example:**Exercise:****Problem:****Finding an Antiderivative of a Rational Function**

Find the antiderivative of $\frac{2x^3+3x}{x^4+3x^2}$.

Solution:

This can be rewritten as $\int (2x^3 + 3x)(x^4 + 3x^2)^{-1} dx$. Use substitution. Let $u = x^4 + 3x^2$, then $du = 4x^3 + 6x$. Alter du by factoring out the 2. Thus,

Equation:

$$\begin{aligned} du &= (4x^3 + 6x)dx \\ &= 2(2x^3 + 3x)dx \\ \frac{1}{2} du &= (2x^3 + 3x)dx. \end{aligned}$$

Rewrite the integrand in u :

Equation:

$$\int (2x^3 + 3x)(x^4 + 3x^2)^{-1} dx = \frac{1}{2} \int u^{-1} du.$$

Then we have

Equation:

$$\begin{aligned}\frac{1}{2} \int u^{-1} du &= \frac{1}{2} \ln |u| + C \\ &= \frac{1}{2} \ln |x^4 + 3x^2| + C.\end{aligned}$$

Example:

Exercise:

Problem:

Finding an Antiderivative of a Logarithmic Function

Find the antiderivative of the log function $\log_2 x$.

Solution:

Follow the format in the formula listed in the rule on integration formulas involving logarithmic functions. Based on this format, we have

Equation:

$$\int \log_2 x dx = \frac{x}{\ln 2} (\ln x - 1) + C.$$

Note:

Exercise:

Problem: Find the antiderivative of $\log_3 x$.

Solution:

$$\frac{x}{\ln 3} (\ln x - 1) + C$$

Hint

Follow [\[link\]](#) and refer to the rule on integration formulas involving logarithmic functions.

[\[link\]](#) is a definite integral of a trigonometric function. With trigonometric functions, we often have to apply a trigonometric property or an identity before we can move forward. Finding the right form of the integrand is usually the key to a smooth integration.

Example:**Exercise:****Problem:****Evaluating a Definite Integral**

Find the definite integral of $\int_0^{\pi/2} \frac{\sin x}{1 + \cos x} dx$.

Solution:

We need substitution to evaluate this problem. Let $u = 1 + \cos x$, , so $du = -\sin x \, dx$. Rewrite the integral in terms of u , changing the limits of integration as well. Thus,

Equation:

$$u = 1 + \cos(0) = 2$$

$$u = 1 + \cos\left(\frac{\pi}{2}\right) = 1.$$

Then

Equation:

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin x}{1 + \cos x} &= - \int_2^1 u^{-1} du \\ &= \int_1^2 u^{-1} du \\ &= \ln |u| \Big|_1^2 \\ &= [\ln 2 - \ln 1] \\ &= \ln 2.\end{aligned}$$

Key Concepts

- Exponential and logarithmic functions arise in many real-world applications, especially those involving growth and decay.
- Substitution is often used to evaluate integrals involving exponential functions or logarithms.

Key Equations

- **Integrals of Exponential Functions**

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

- **Integration Formulas Involving Logarithmic Functions**

$$\int x^{-1} dx = \ln |x| + C$$

$$\int \ln x \, dx = x \ln x - x + C = x (\ln x - 1) + C$$

$$\int \log_a x \, dx = \frac{x}{\ln a} (\ln x - 1) + C$$

In the following exercises, compute each indefinite integral.

Exercise:

Problem: $\int e^{2x} dx$

Exercise:

Problem: $\int e^{-3x} dx$

Solution:

$$-\frac{1}{3} e^{-3x} + C$$

Exercise:

Problem: $\int 2^x dx$

Exercise:

Problem: $\int 3^{-x} dx$

Solution:

$$-\frac{3^{-x}}{\ln 3} + C$$

Exercise:

Problem: $\int \frac{1}{2x} dx$

Exercise:

Problem: $\int \frac{2}{x} dx$

Solution:

$$\ln(x^2) + C$$

Exercise:

Problem: $\int \frac{1}{x^2} dx$

Exercise:

Problem: $\int \frac{1}{\sqrt{x}} dx$

Solution:

$$2\sqrt{x} + C$$

In the following exercises, find each indefinite integral by using appropriate substitutions.

Exercise:

Problem: $\int \frac{\ln x}{x} dx$

Exercise:

Problem: $\int \frac{dx}{x(\ln x)^2}$

Solution:

$$-\frac{1}{\ln x} + C$$

Exercise:

Problem: $\int \frac{dx}{x \ln x} \quad (x > 1)$

Exercise:

Problem: $\int \frac{dx}{x \ln x \ln(\ln x)}$

Solution:

$$\ln(\ln(\ln x)) + C$$

Exercise:

Problem: $\int \tan \theta \, d\theta$

Exercise:

Problem: $\int \frac{\cos x - x \sin x}{x \cos x} dx$

Solution:

$$\ln(x \cos x) + C$$

Exercise:

Problem: $\int \frac{\ln(\sin x)}{\tan x} dx$

Exercise:

Problem: $\int \ln(\cos x) \tan x dx$

Solution:

$$-\frac{1}{2} (\ln(\cos(x)))^2 + C$$

Exercise:

Problem: $\int x e^{-x^2} dx$

Exercise:

Problem: $\int x^2 e^{-x^3} dx$

Solution:

$$\frac{-e^{-x^3}}{3} + C$$

Exercise:

Problem: $\int e^{\sin x} \cos x dx$

Exercise:

Problem: $\int e^{\tan x} \sec^2 x dx$

Solution:

$$e^{\tan x} + C$$

Exercise:

Problem: $\int e^{\ln x} \frac{dx}{x}$

Exercise:

Problem: $\int \frac{e^{\ln(1-t)}}{1-t} dt$

Solution:

$$t + C$$

In the following exercises, verify by differentiation that

$\int \ln x \, dx = x (\ln x - 1) + C$, then use appropriate changes of variables to compute the integral.

Exercise:

Problem: $\int \ln x \, dx$ (*Hint:* $\int \ln x \, dx = \frac{1}{2} \int x \ln(x^2) \, dx$)

Exercise:

Problem: $\int x^2 \ln^2 x \, dx$

Solution:

$$\frac{1}{9} x^3 (\ln(x^3) - 1) + C$$

Exercise:

Problem: $\int \frac{\ln x}{x^2} dx$ (*Hint:* Set $u = \frac{1}{x}$.)

Exercise:

Problem: $\int \frac{\ln x}{\sqrt{x}} dx$ (*Hint:* Set $u = \sqrt{x}$.)

Solution:

$$2\sqrt{x}(\ln x - 2) + C$$

Exercise:

Problem:

Write an integral to express the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to e^x and evaluate the integral.

Exercise:

Problem:

Write an integral to express the area under the graph of $y = e^t$ between $t = 0$ and $t = \ln x$, and evaluate the integral.

Solution:

$$\int_0^{\ln x} e^t dt = e^t \Big|_0^{\ln x} = e^{\ln x} - e^0 = x - 1$$

In the following exercises, use appropriate substitutions to express the trigonometric integrals in terms of compositions with logarithms.

Exercise:

Problem: $\int \tan(2x) dx$

Exercise:

Problem: $\int \frac{\sin(3x) - \cos(3x)}{\sin(3x) + \cos(3x)} dx$

Solution:

$$-\frac{1}{3} \ln(\sin(3x) + \cos(3x))$$

Exercise:

Problem: $\int \frac{x \sin(x^2)}{\cos(x^2)} dx$

Exercise:

Problem: $\int x \csc(x^2) dx$

Solution:

$$-\frac{1}{2} \ln |\csc(x^2) + \cot(x^2)| + C$$

Exercise:

Problem: $\int \ln(\cos x) \tan x dx$

Exercise:

Problem: $\int \ln(\csc x) \cot x dx$

Solution:

$$-\frac{1}{2} (\ln(\csc x))^2 + C$$

Exercise:

Problem: $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

In the following exercises, evaluate the definite integral.

Exercise:

Problem: $\int_1^2 \frac{1 + 2x + x^2}{3x + 3x^2 + x^3} dx$

Solution:

$$\frac{1}{3} \ln \left(\frac{26}{7} \right)$$

Exercise:

Problem: $\int_0^{\pi/4} \tan x \, dx$

Exercise:

Problem: $\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x} dx$

Solution:

$$\ln \left(\sqrt{3} - 1 \right)$$

Exercise:

Problem: $\int_{\pi/6}^{\pi/2} \csc x \, dx$

Exercise:

Problem: $\int_{\pi/4}^{\pi/3} \cot x dx$

Solution:

$$\frac{1}{2} \ln \frac{3}{2}$$

In the following exercises, integrate using the indicated substitution.

Exercise:

Problem: $\int \frac{x}{x-100} dx; u = x - 100$

Exercise:

Problem: $\int \frac{y-1}{y+1} dy; u = y + 1$

Solution:

$$y - 2 \ln |y + 1| + C$$

Exercise:

Problem: $\int \frac{1-x^2}{3x-x^3} dx; u = 3x - x^3$

Exercise:

Problem: $\int \frac{\sin x + \cos x}{\sin x - \cos x} dx; u = \sin x - \cos x$

Solution:

$$\ln |\sin x - \cos x| + C$$

Exercise:

Problem: $\int e^{2x} \sqrt{1 - e^{2x}} dx; u = e^{2x}$

Exercise:

Problem: $\int \ln(x) \frac{\sqrt{1 - (\ln x)^2}}{x} dx; u = \ln x$

Solution:

$$-\frac{1}{3} (1 - (\ln x)^2)^{3/2} + C$$

In the following exercises, does the right-endpoint approximation overestimate or underestimate the exact area? Calculate the right endpoint estimate R_{50} and solve for the exact area.

Exercise:

Problem: [T] $y = e^x$ over $[0, 1]$

Exercise:

Problem: [T] $y = e^{-x}$ over $[0, 1]$

Solution:

Exact solution: $\frac{e-1}{e}$, $R_{50} = 0.6258$. Since f is decreasing, the right endpoint estimate underestimates the area.

Exercise:

Problem: [T] $y = \ln(x)$ over $[1, 2]$

Exercise:

Problem: [T] $y = \frac{x+1}{x^2+2x+6}$ over $[0, 1]$

Solution:

Exact solution: $\frac{2\ln(3)-\ln(6)}{2}$, $R_{50} = 0.2033$. Since f is increasing, the right endpoint estimate overestimates the area.

Exercise:

Problem: [T] $y = 2^x$ over $[-1, 0]$

Exercise:

Problem: [T] $y = -2^{-x}$ over $[0, 1]$

Solution:

Exact solution: $-\frac{1}{\ln(4)}$, $R_{50} = -0.7164$. Since f is increasing, the right endpoint estimate overestimates the area (the actual area is a larger negative number).

In the following exercises, $f(x) \geq 0$ for $a \leq x \leq b$. Find the area under the graph of $f(x)$ between the given values a and b by integrating.

Exercise:

Problem: $f(x) = \frac{\log_{10}(x)}{x}$; $a = 10, b = 100$

Exercise:

Problem: $f(x) = \frac{\log_2(x)}{x}$; $a = 32, b = 64$

Solution:

$$\frac{11}{2} \ln 2$$

Exercise:

Problem: $f(x) = 2^{-x}; a = 1, b = 2$

Exercise:

Problem: $f(x) = 2^{-x}; a = 3, b = 4$

Solution:

$$\frac{1}{\ln(65,536)}$$

Exercise:**Problem:**

Find the area under the graph of the function $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

Exercise:**Problem:**

Compute the integral of $f(x) = xe^{-x^2}$ and find the smallest value of N such that the area under the graph $f(x) = xe^{-x^2}$ between $x = N$ and $x = N + 10$ is, at most, 0.01.

Solution:

$$\int_N^{N+1} xe^{-x^2} dx = \frac{1}{2} \left(e^{-N^2} - e^{-(N+1)^2} \right). \text{ The quantity is less than } 0.01 \text{ when } N = 2.$$

Exercise:**Problem:**

Find the limit, as N tends to infinity, of the area under the graph of $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

Exercise:

Problem: Show that $\int_a^b \frac{dt}{t} = \int_{1/b}^{1/a} \frac{dt}{t}$ when $0 < a \leq b$.

Solution:

$$\int_a^b \frac{dx}{x} = \ln(b) - \ln(a) = \ln\left(\frac{1}{a}\right) - \ln\left(\frac{1}{b}\right) = \int_{1/b}^{1/a} \frac{dx}{x}$$

Exercise:**Problem:**

Suppose that $f(x) > 0$ for all x and that f and g are differentiable. Use the identity $f^g = e^{g \ln f}$ and the chain rule to find the derivative of f^g .

Exercise:**Problem:**

Use the previous exercise to find the antiderivative of

$h(x) = x^x (1 + \ln x)$ and evaluate $\int_2^3 x^x (1 + \ln x) dx$.

Solution:

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Exercise:**Problem:**

Show that if $c > 0$, then the integral of $1/x$ from ac to bc ($0 < a < b$) is the same as the integral of $1/x$ from a to b .

The following exercises are intended to derive the fundamental properties of the natural log starting from the *definition* $\ln(x) = \int_1^x \frac{dt}{t}$, using

properties of the definite integral and making no further assumptions.

Exercise:

Problem:

Use the identity $\ln(x) = \int_1^x \frac{dt}{t}$ to derive the identity $\ln\left(\frac{1}{x}\right) = -\ln x$.

Solution:

We may assume that $x > 1$, so $\frac{1}{x} < 1$. Then, $\int_1^{1/x} \frac{dt}{t}$. Now make the substitution $u = \frac{1}{t}$, so $du = -\frac{dt}{t^2}$ and $\frac{du}{u} = -\frac{dt}{t}$, and change endpoints: $\int_1^{1/x} \frac{dt}{t} = -\int_1^x \frac{du}{u} = -\ln x$.

Exercise:

Problem:

Use a change of variable in the integral $\int_1^{xy} \frac{1}{t} dt$ to show that $\ln xy = \ln x + \ln y$ for $x, y > 0$.

Exercise:

Problem:

Use the identity $\ln x = \int_1^x \frac{dt}{t}$ to show that $\ln(x)$ is an increasing function of x on $[0, \infty)$, and use the previous exercises to show that the range of $\ln(x)$ is $(-\infty, \infty)$. Without any further assumptions, conclude that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$.

Exercise:

Problem:

Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln(x)$, but keep in mind that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$. Call it E . Use the identity $\ln xy = \ln x + \ln y$ to deduce that $E(a + b) = E(a)E(b)$ for any real numbers a, b .

Exercise:**Problem:**

Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln x$, but keep in mind that $\ln x$ has an inverse function defined on $(-\infty, \infty)$. Call it E . Show that $E'(t) = E(t)$.

Solution:

$x = E(\ln(x))$. Then, $1 = \frac{E'(\ln x)}{x}$ or $x = E'(\ln x)$. Since any number t can be written $t = \ln x$ for some x , and for such t we have $x = E(t)$, it follows that for any t , $E'(t) = E(t)$.

Exercise:**Problem:**

The sine integral, defined as $S(x) = \int_0^x \frac{\sin t}{t} dt$ is an important quantity in engineering. Although it does not have a simple closed formula, it is possible to estimate its behavior for large x . Show that for $k \geq 1$, $|S(2\pi k) - S(2\pi(k+1))| \leq \frac{1}{k(2k+1)\pi}$.
(Hint: $\sin(t + \pi) = -\sin t$)

Exercise:

Problem:

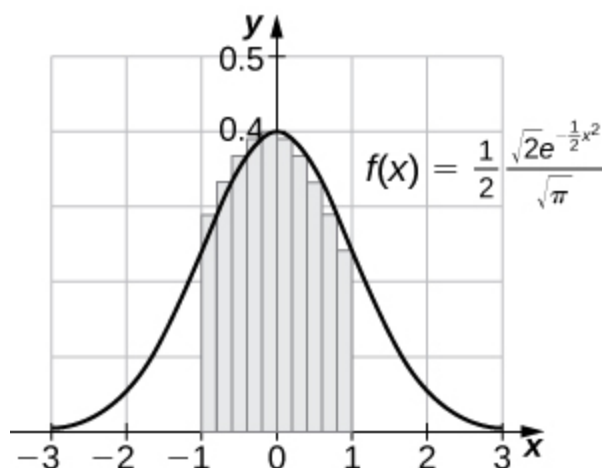
[T] The normal distribution in probability is given by

$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, where σ is the standard deviation and μ is the average. The *standard normal distribution* in probability, p_s , corresponds to $\mu = 0$ and $\sigma = 1$. Compute the left endpoint estimates

$$R_{10} \text{ and } R_{100} \text{ of } \int_{-1}^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Solution:

$$R_{10} = 0.6811, R_{100} = 0.6827$$

**Exercise:****Problem:**

[T] Compute the right endpoint estimates R_{50} and R_{100} of

$$\int_{-3}^5 \frac{1}{2\sqrt{2\pi}} e^{-(x-1)^2/8} dx.$$

Integrals Resulting in Inverse Trigonometric Functions

- Integrate functions resulting in inverse trigonometric functions

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall from [Functions and Graphs](#) that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also in [Derivatives](#), we developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

Integrals that Result in Inverse Sine Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

Note:

Rule: Integration Formulas Resulting in Inverse Trigonometric Functions

The following integration formulas yield inverse trigonometric functions:

1.

Equation:

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$$

2.

Equation:

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

3.

Equation:

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$$

Proof

Let $y = \sin^{-1} \frac{x}{a}$. Then $a \sin y = x$. Now let's use implicit differentiation.

We obtain

Equation:

$$\frac{d}{dx} (a \sin y) = \frac{d}{dx} (x)$$

$$a \cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a \cos y}.$$

For $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $\cos y \geq 0$. Thus, applying the Pythagorean identity

$\sin^2 y + \cos^2 y = 1$, we have $\cos y = \sqrt{1 - \sin^2 y}$. This gives

Equation:

$$\begin{aligned} \frac{1}{a \cos y} &= \frac{1}{a \sqrt{1 - \sin^2 y}} \\ &= \frac{1}{\sqrt{a^2 - a^2 \sin^2 y}} \\ &= \frac{1}{\sqrt{a^2 - x^2}}. \end{aligned}$$

Then for $-a \leq x \leq a$, we have

Equation:

$$\int \frac{1}{\sqrt{a^2 - u^2}} du = \sin^{-1} \left(\frac{u}{a} \right) + C.$$

□

Example:

Exercise:

Problem:

Evaluating a Definite Integral Using Inverse Trigonometric Functions

Evaluate the definite integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

Solution:

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have

Equation:

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x \Big|_0^1 \\ &= \sin^{-1} 1 - \sin^{-1} 0 \\ &= \frac{\pi}{2} - 0 \\ &= \frac{\pi}{2}. \end{aligned}$$

Note:

Exercise:

Problem: Find the antiderivative of $\int \frac{dx}{\sqrt{1-16x^2}}$.

Solution:

$$\frac{1}{4} \sin^{-1}(4x) + C$$

Hint

Substitute $u = 4x$

Example:

Exercise:

Problem:

Finding an Antiderivative Involving an Inverse Trigonometric Function

Evaluate the integral $\int \frac{dx}{\sqrt{4-9x^2}}$.

Solution:

Substitute $u = 3x$. Then $du = 3dx$ and we have

Equation:

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3} \int \frac{du}{\sqrt{4-u^2}}.$$

Applying the formula with $a = 2$, we obtain

Equation:

$$\begin{aligned}
 \int \frac{dx}{\sqrt{4-9x^2}} &= \frac{1}{3} \int \frac{du}{\sqrt{4-u^2}} \\
 &= \frac{1}{3} \sin^{-1} \left(\frac{u}{2} \right) + C \\
 &= \frac{1}{3} \sin^{-1} \left(\frac{3x}{2} \right) + C.
 \end{aligned}$$

Note:

Exercise:

Problem:

Find the indefinite integral using an inverse trigonometric function and substitution for $\int \frac{dx}{\sqrt{9-x^2}}$.

Solution:

$$\sin^{-1} \left(\frac{x}{3} \right) + C$$

Hint

Use the formula in the rule on integration formulas resulting in inverse trigonometric functions.

Example:

Exercise:

Problem:

Evaluating a Definite Integral

Evaluate the definite integral $\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}}$.

Solution:

The format of the problem matches the inverse sine formula. Thus,

Equation:

$$\begin{aligned}\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} &= \sin^{-1}u \Big|_0^{\sqrt{3}/2} \\ &= \left[\sin^{-1} \left(\frac{\sqrt{3}}{2} \right) \right] - [\sin^{-1}(0)] \\ &= \frac{\pi}{3}.\end{aligned}$$

Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out -1 and evaluate the integral using one of the formulas already provided. To close this section, we examine one more formula: the integral resulting in the inverse tangent function.

Example:**Exercise:****Problem:****Finding an Antiderivative Involving the Inverse Tangent Function**

Find an antiderivative of $\int \frac{1}{1+4x^2} dx$.

Solution:

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for $\tan^{-1}u + C$. So we use substitution, letting $u = 2x$, then $du = 2dx$ and $1/2du = dx$. Then, we have

Equation:

$$\frac{1}{2} \int \frac{1}{1 + u^2} du = \frac{1}{2} \tan^{-1}u + C = \frac{1}{2} \tan^{-1}(2x) + C.$$

Note:**Exercise:**

Problem: Use substitution to find the antiderivative of $\int \frac{dx}{25 + 4x^2}$.

Solution:

$$\frac{1}{10} \tan^{-1}\left(\frac{2x}{5}\right) + C$$

Hint

Use the solving strategy from [\[link\]](#) and the rule on integration formulas resulting in inverse trigonometric functions.

Example:**Exercise:****Problem:**

Applying the Integration Formulas

Find the antiderivative of $\int \frac{1}{9 + x^2} dx$.

Solution:

Apply the formula with $a = 3$. Then,

Equation:

$$\int \frac{dx}{9 + x^2} = \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) + C.$$

Note:

Exercise:

Problem: Find the antiderivative of $\int \frac{dx}{16 + x^2}$.

Solution:

$$\frac{1}{4} \tan^{-1} \left(\frac{x}{4} \right) + C$$

Hint

Follow the steps in [\[link\]](#).

Example:

Exercise:

Problem:

Evaluating a Definite Integral

Evaluate the definite integral $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2}$.

Solution:

Use the formula for the inverse tangent. We have

Equation:

$$\begin{aligned}\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} &= \tan^{-1} x \Big|_{\sqrt{3}/3}^{\sqrt{3}} \\ &= \left[\tan^{-1} (\sqrt{3}) \right] - \left[\tan^{-1} \left(\frac{\sqrt{3}}{3} \right) \right] \\ &= \frac{\pi}{6}.\end{aligned}$$

Note:

Exercise:

Problem: Evaluate the definite integral $\int_0^2 \frac{dx}{4+x^2}$.

Solution:

$$\frac{\pi}{8}$$

Hint

Follow the procedures from [\[link\]](#) to solve the problem.

Key Concepts

- Formulas for derivatives of inverse trigonometric functions developed in [Derivatives of Exponential and Logarithmic Functions](#) lead directly to integration formulas involving inverse trigonometric functions.
- Use the formulas listed in the rule on integration formulas resulting in inverse trigonometric functions to match up the correct format and make alterations as necessary to solve the problem.
- Substitution is often required to put the integrand in the correct form.

Key Equations

- **Integrals That Produce Inverse Trigonometric Functions**

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left(\frac{u}{a} \right) + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left(\frac{u}{a} \right) + C$$

In the following exercises, evaluate each integral in terms of an inverse trigonometric function.

Exercise:

Problem: $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1 - x^2}}$

Solution:

$$\sin^{-1} x \Big|_0^{\sqrt{3}/2} = \frac{\pi}{3}$$

Exercise:

Problem: $\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1 - x^2}}$

Exercise:

Problem: $\int_{\sqrt{3}}^1 \frac{dx}{\sqrt{1+x^2}}$

Solution:

$$\tan^{-1}x \Big|_{\sqrt{3}}^1 = -\frac{\pi}{12}$$

Exercise:

Problem: $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$

Exercise:

Problem: $\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}$

Solution:

$$\sec^{-1}x \Big|_1^{\sqrt{2}} = \frac{\pi}{4}$$

Exercise:

Problem: $\int_1^{2/\sqrt{3}} \frac{dx}{|x|\sqrt{x^2-1}}$

In the following exercises, find each indefinite integral, using appropriate substitutions.

Exercise:

Problem: $\int \frac{dx}{\sqrt{9-x^2}}$

Solution:

$$\sin^{-1}\left(\frac{x}{3}\right) + C$$

Exercise:

Problem: $\int \frac{dx}{\sqrt{1-16x^2}}$

Exercise:

Problem: $\int \frac{dx}{9+x^2}$

Solution:

$$\frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$$

Exercise:

Problem: $\int \frac{dx}{25+16x^2}$

Exercise:

Problem: $\int \frac{dx}{|x|\sqrt{x^2-9}}$

Solution:

$$\frac{1}{3} \sec^{-1}\left(\frac{x}{3}\right) + C$$

Exercise:

Problem: $\int \frac{dx}{|x|\sqrt{4x^2-16}}$

Exercise:**Problem:**

Explain the relationship $-\cos^{-1}t + C = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}t + C$.

Is it true, in general, that $\cos^{-1}t = -\sin^{-1}t$?

Solution:

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$. So, $\sin^{-1}t = \frac{\pi}{2} - \cos^{-1}t$. They differ by a constant.

Exercise:**Problem:**

Explain the relationship

$\sec^{-1}t + C = \int \frac{dt}{|t|\sqrt{t^2-1}} = -\csc^{-1}t + C$. Is it true, in general, that $\sec^{-1}t = -\csc^{-1}t$?

Exercise:**Problem:**

Explain what is wrong with the following integral: $\int_1^2 \frac{dt}{\sqrt{1-t^2}}$.

Solution:

$\sqrt{1-t^2}$ is not defined as a real number when $t > 1$.

Exercise:**Problem:**

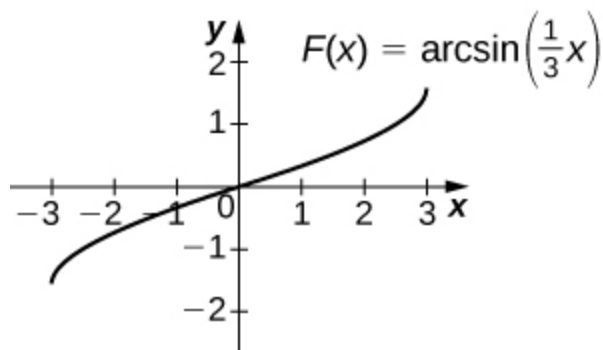
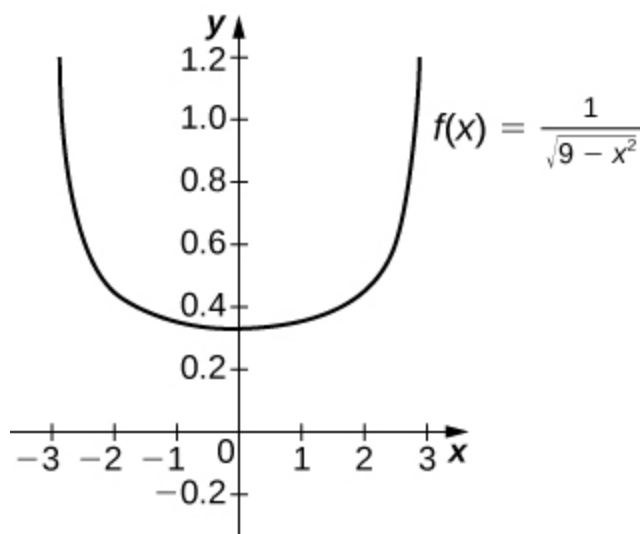
Explain what is wrong with the following integral: $\int_{-1}^1 \frac{dt}{|t|\sqrt{t^2-1}}$.

In the following exercises, solve for the antiderivative $\int f$ of f with $C = 0$, then use a calculator to graph f and the antiderivative over the given interval $[a, b]$. Identify a value of C such that adding C to the antiderivative recovers the definite integral $F(x) = \int_a^x f(t)dt$.

Exercise:

Problem: [T] $\int \frac{1}{\sqrt{9-x^2}} dx$ over $[-3, 3]$

Solution:



The antiderivative is $\sin^{-1}\left(\frac{x}{3}\right) + C$. Taking $C = \frac{\pi}{2}$ recovers the definite integral.

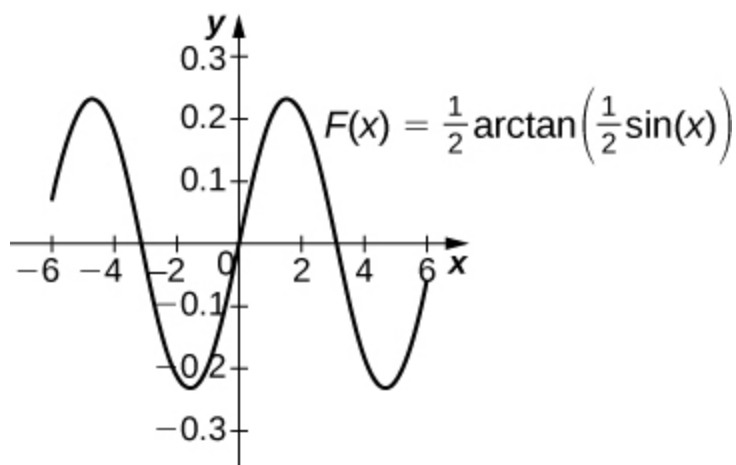
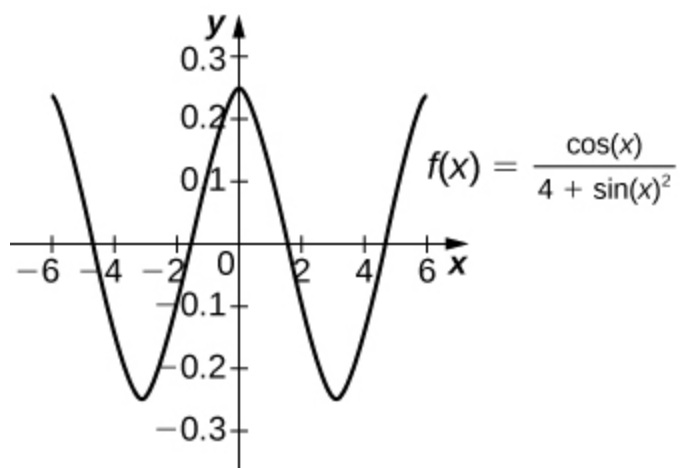
Exercise:

Problem: [T] $\int \frac{9}{9 + x^2} dx$ over $[-6, 6]$

Exercise:

Problem: [T] $\int \frac{\cos x}{4 + \sin^2 x} dx$ over $[-6, 6]$

Solution:



The antiderivative is $\frac{1}{2} \tan^{-1}\left(\frac{\sin x}{2}\right) + C$. Taking $C = \frac{1}{2} \tan^{-1}\left(\frac{\sin(6)}{2}\right)$ recovers the definite integral.

Exercise:

Problem: [T] $\int \frac{e^x}{1 + e^{2x}} dx$ over $[-6, 6]$

In the following exercises, compute the antiderivative using appropriate substitutions.

Exercise:

Problem: $\int \frac{\sin^{-1} t dt}{\sqrt{1-t^2}}$

Solution:

$$\frac{1}{2} (\sin^{-1} t)^2 + C$$

Exercise:

Problem: $\int \frac{dt}{\sin^{-1} t \sqrt{1-t^2}}$

Exercise:

Problem: $\int \frac{\tan^{-1} (2t)}{1+4t^2} dt$

Solution:

$$\frac{1}{4} (\tan^{-1} (2t))^2$$

Exercise:

Problem: $\int \frac{t \tan^{-1} (t^2)}{1+t^4} dt$

Exercise:

Problem: $\int \frac{\sec^{-1} \left(\frac{t}{2} \right)}{|t| \sqrt{t^2 - 4}} dt$

Solution:

$$\frac{1}{4} \left(\sec^{-1} \left(\frac{t}{2} \right) \right)^2 + C$$

Exercise:

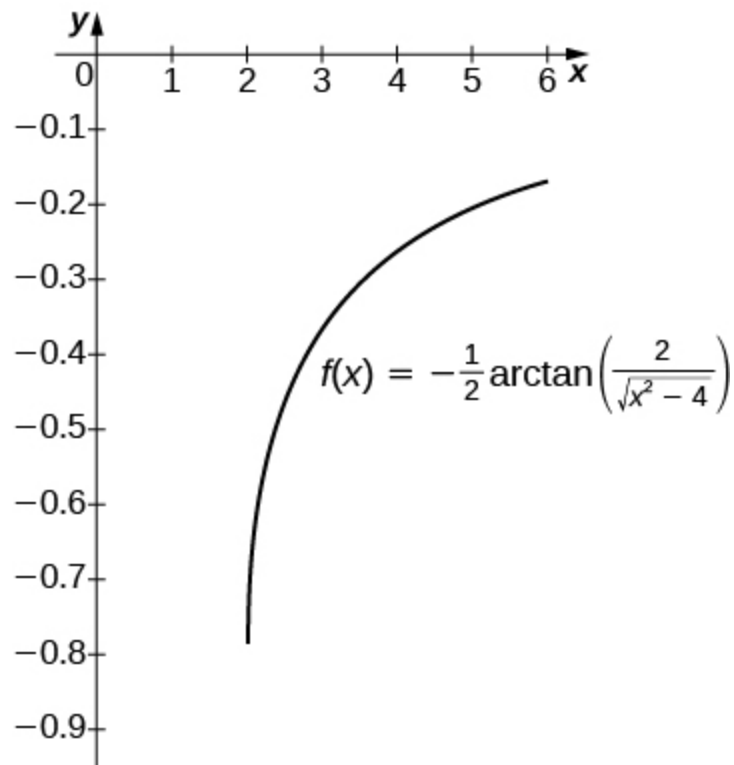
Problem: $\int \frac{t \sec^{-1}(t^2)}{t^2 \sqrt{t^4 - 1}} dt$

In the following exercises, use a calculator to graph the antiderivative $\int f$ with $C = 0$ over the given interval $[a, b]$. Approximate a value of C , if possible, such that adding C to the antiderivative gives the same value as the definite integral $F(x) = \int_a^x f(t) dt$.

Exercise:

Problem: [T] $\int \frac{1}{x \sqrt{x^2 - 4}} dx$ over $[2, 6]$

Solution:



The antiderivative is $\frac{1}{2} \sec^{-1}\left(\frac{x}{2}\right) + C$. Taking $C = 0$ recovers the definite integral over $[2, 6]$.

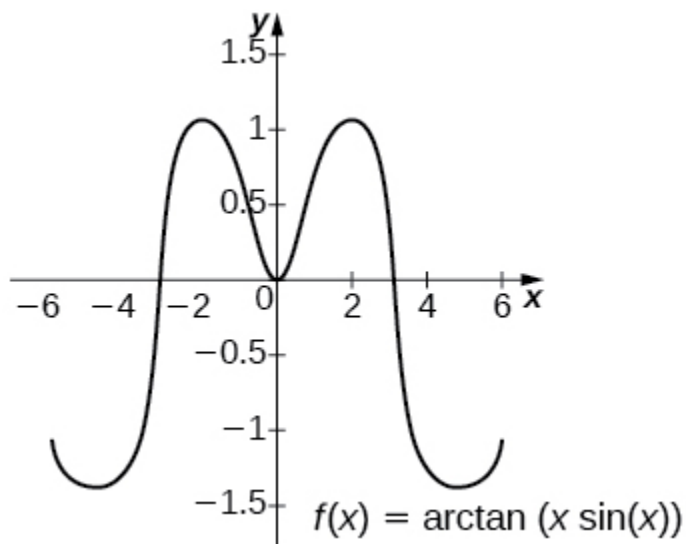
Exercise:

Problem: [T] $\int \frac{1}{(2x+2)\sqrt{x}} dx$ over $[0, 6]$

Exercise:

Problem: [T] $\int \frac{(\sin x + x \cos x)}{1 + x^2 \sin^2 x} dx$ over $[-6, 6]$

Solution:



The general antiderivative is $\tan^{-1}(x \sin x) + C$. Taking $C = -\tan^{-1}(6 \sin(6))$ recovers the definite integral.

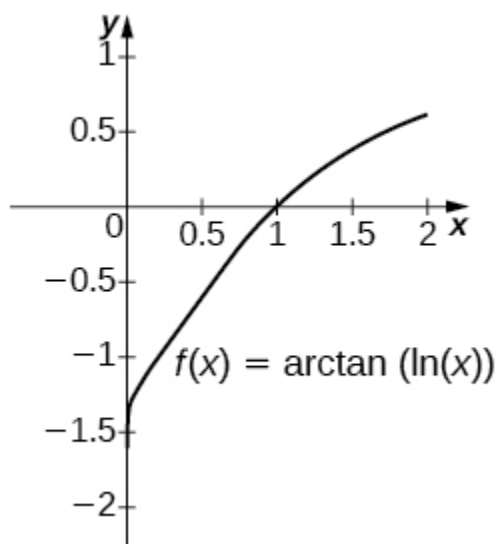
Exercise:

Problem: [T] $\int \frac{2e^{-2x}}{\sqrt{1 - e^{-4x}}} dx$ over $[0, 2]$

Exercise:

Problem: [T] $\int \frac{1}{x + x \ln^2 x} dx$ over $[0, 2]$

Solution:



The general antiderivative is $\tan^{-1}(\ln x) + C$. Taking $C = \frac{\pi}{2} = \tan^{-1}\infty$ recovers the definite integral.

Exercise:

Problem: [T] $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ over $[-1, 1]$

In the following exercises, compute each integral using appropriate substitutions.

Exercise:

Problem: $\int \frac{e^x}{\sqrt{1-e^{2t}}} dt$

Solution:

$$\sin^{-1}(e^t) + C$$

Exercise:

Problem: $\int \frac{e^t}{1 + e^{2t}} dt$

Exercise:

Problem: $\int \frac{dt}{t\sqrt{1 - \ln^2 t}}$

Solution:

$$\sin^{-1}(\ln t) + C$$

Exercise:

Problem: $\int \frac{dt}{t(1 + \ln^2 t)}$

Exercise:

Problem: $\int \frac{\cos^{-1}(2t)}{\sqrt{1 - 4t^2}} dt$

Solution:

$$-\frac{1}{2}(\cos^{-1}(2t))^2 + C$$

Exercise:

Problem: $\int \frac{e^t \cos^{-1}(e^t)}{\sqrt{1 - e^{2t}}} dt$

In the following exercises, compute each definite integral.

Exercise:

Problem: $\int_0^{1/2} \frac{\tan(\sin^{-1}t)}{\sqrt{1-t^2}} dt$

Solution:

$$\frac{1}{2} \ln\left(\frac{4}{3}\right)$$

Exercise:

Problem: $\int_{1/4}^{1/2} \frac{\tan(\cos^{-1}t)}{\sqrt{1-t^2}} dt$

Exercise:

Problem: $\int_0^{1/2} \frac{\sin(\tan^{-1}t)}{1+t^2} dt$

Solution:

$$1 - \frac{2}{\sqrt{5}}$$

Exercise:

Problem: $\int_0^{1/2} \frac{\cos(\tan^{-1}t)}{1+t^2} dt$

Exercise:

Problem:

For $A > 0$, compute $I(A) = \int_{-A}^A \frac{dt}{1+t^2}$ and evaluate $\lim_{A \rightarrow \infty} I(A)$, the area under the graph of $\frac{1}{1+t^2}$ on $[-\infty, \infty]$.

Solution:

$$2\tan^{-1}(A) \rightarrow \pi \text{ as } A \rightarrow \infty$$

Exercise:

Problem:

For $1 < B < \infty$, compute $I(B) = \int_1^B \frac{dt}{t\sqrt{t^2-1}}$ and evaluate $\lim_{B \rightarrow \infty} I(B)$, the area under the graph of $\frac{1}{t\sqrt{t^2-1}}$ over $[1, \infty)$.

Exercise:

Problem:

Use the substitution $u = \sqrt{2} \cot x$ and the identity $1 + \cot^2 x = \csc^2 x$ to evaluate $\int \frac{dx}{1 + \cos^2 x}$. (Hint: Multiply the top and bottom of the integrand by $\csc^2 x$.)

Solution:

Using the hint, one has $\int \frac{\csc^2 x}{\csc^2 x + \cot^2 x} dx = \int \frac{\csc^2 x}{1 + 2\cot^2 x} dx$.

Set $u = \sqrt{2}\cot x$. Then, $du = -\sqrt{2}\csc^2 x$ and the integral is $-\frac{1}{\sqrt{2}} \int \frac{du}{1 + u^2} = -\frac{1}{\sqrt{2}} \tan^{-1} u + C = \frac{1}{\sqrt{2}} \tan^{-1} (\sqrt{2}\cot x) + C$.

If one uses the identity $\tan^{-1} s + \tan^{-1} \left(\frac{1}{s}\right) = \frac{\pi}{2}$, then this can also be written $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x}{\sqrt{2}}\right) + C$.

Exercise:

Problem:

[T] Approximate the points at which the graphs of $f(x) = 2x^2 - 1$ and $g(x) = (1 + 4x^2)^{-3/2}$ intersect, and approximate the area between their graphs accurate to three decimal places.

Exercise:

Problem:

47. [T] Approximate the points at which the graphs of $f(x) = x^2 - 1$ and $f(x) = x^2 - 1$ intersect, and approximate the area between their graphs accurate to three decimal places.

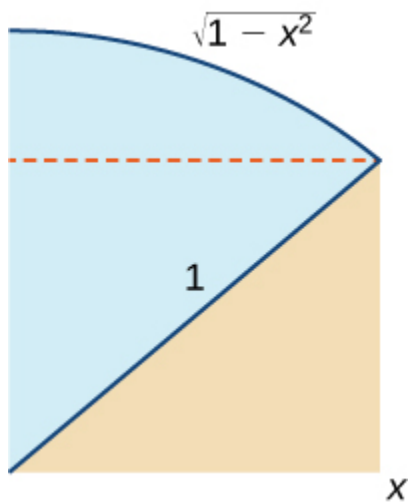
Solution:

$x \approx \pm 1.13525$. The left endpoint estimate with $N = 100$ is 2.796 and these decimals persist for $N = 500$.

Exercise:**Problem:**

Use the following graph to prove that

$$\int_0^x \sqrt{1-t^2} dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x.$$

**Chapter Review Exercises**

True or False. Justify your answer with a proof or a counterexample. Assume all functions f and g are continuous over their domains.

Exercise:

Problem:

If $f(x) > 0$, $f'(x) > 0$ for all x , then the right-hand rule underestimates the integral $\int_a^b f(x)$. Use a graph to justify your answer.

Solution:

False

Exercise:

Problem: $\int_a^b f(x)^2 dx = \int_a^b f(x) dx \int_a^b f(x) dx$

Exercise:

Problem:

If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x) \leq \int_a^b g(x)$.

Solution:

True

Exercise:

Problem: All continuous functions have an antiderivative.

Evaluate the Riemann sums L_4 and R_4 for the following functions over the specified interval. Compare your answer with the exact answer, when possible, or use a calculator to determine the answer.

Exercise:

Problem: $y = 3x^2 - 2x + 1$ over $[-1, 1]$

Solution:

$$L_4 = 5.25, R_4 = 3.25, \text{ exact answer: } 4$$

Exercise:

Problem: $y = \ln(x^2 + 1)$ over $[0, e]$

Exercise:

Problem: $y = x^2 \sin x$ over $[0, \pi]$

Solution:

$$L_4 = 5.364, R_4 = 5.364, \text{ exact answer: } 5.870$$

Exercise:

Problem: $y = \sqrt{x} + \frac{1}{x}$ over $[1, 4]$

Evaluate the following integrals.

Exercise:

Problem: $\int_{-1}^1 (x^3 - 2x^2 + 4x) dx$

Solution:

$$-\frac{4}{3}$$

Exercise:

Problem: $\int_0^4 \frac{3t}{\sqrt{1+6t^2}} dt$

Exercise:

Problem: $\int_{\pi/3}^{\pi/2} 2 \sec(2\theta) \tan(2\theta) d\theta$

Solution:

1

Exercise:

Problem: $\int_0^{\pi/4} e^{\cos^2 x} \sin x \cos x dx$

Find the antiderivative.

Exercise:

Problem: $\int \frac{dx}{(x+4)^3}$

Solution:

$$-\frac{1}{2(x+4)^2} + C$$

Exercise:

Problem: $\int x \ln(x^2) dx$

Exercise:

Problem: $\int \frac{4x^2}{\sqrt{1-x^6}} dx$

Solution:

$$\frac{4}{3} \sin^{-1}(x^3) + C$$

Exercise:

Problem: $\int \frac{e^{2x}}{1+e^{4x}} dx$

Find the derivative.

Exercise:

Problem: $\frac{d}{dt} \int_0^t \frac{\sin x}{\sqrt{1+x^2}} dx$

Solution:

$$\frac{\sin t}{\sqrt{1+t^2}}$$

Exercise:

Problem: $\frac{d}{dx} \int_1^{x^3} \sqrt{4-t^2} dt$

Exercise:

Problem: $\frac{d}{dx} \int_1^{\ln(x)} (4t + e^t) dt$

Solution:

$$4 \frac{\ln x}{x} + 1$$

Exercise:

Problem: $\frac{d}{dx} \int_0^{\cos x} e^{t^2} dt$

The following problems consider the historic average cost per gigabyte of RAM on a computer.

Year	5-Year Change (\$)
1980	0
1985	-5,468,750
1990	-755,495
1995	-73,005
2000	-29,768
2005	-918
2010	-177

Exercise:

Problem:

If the average cost per gigabyte of RAM in 2010 is \$12, find the average cost per gigabyte of RAM in 1980.

Solution:

\$6,328,113

Exercise:**Problem:**

The average cost per gigabyte of RAM can be approximated by the function $C(t) = 8,500,000(0.65)^t$, where t is measured in years since 1980, and C is cost in US\$. Find the average cost per gigabyte of RAM for 1980 to 2010.

Exercise:

Problem: Find the average cost of 1GB RAM for 2005 to 2010.

Solution:

\$73.36

Exercise:**Problem:**

The velocity of a bullet from a rifle can be approximated by $v(t) = 6400t^2 - 6505t + 2686$, where t is seconds after the shot and v is the velocity measured in feet per second. This equation only models the velocity for the first half-second after the shot:
 $0 \leq t \leq 0.5$. What is the total distance the bullet travels in 0.5 sec?

Exercise:**Problem:**

What is the average velocity of the bullet for the first half-second?

Solution:

$\frac{19117}{12}$ ft/sec, or 1593 ft/sec

Table of Integrals

Basic Integrals

$$1. \int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq -1$$

$$2. \int \frac{du}{u} = \ln |u| + C$$

$$3. \int e^u du = e^u + C$$

$$4. \int a^u du = \frac{a^u}{\ln a} + C$$

$$5. \int \sin u du = -\cos u + C$$

$$6. \int \cos u du = \sin u + C$$

$$7. \int \sec^2 u du = \tan u + C$$

$$8. \int \csc^2 u du = -\cot u + C$$

$$9. \int \sec u \tan u du = \sec u + C$$

$$10. \int \csc u \cot u du = -\csc u + C$$

$$11. \int \tan u du = \ln |\sec u| + C$$

$$12. \int \cot u \, du = \ln |\sin u| + C$$

$$13. \int \sec u \, du = \ln |\sec u + \tan u| + C$$

$$14. \int \csc u \, du = \ln |\csc u - \cot u| + C$$

$$15. \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$$

$$16. \int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

$$17. \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$$

Trigonometric Integrals

$$18. \int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$$

$$19. \int \cos^2 u \, du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$$

$$20. \int \tan^2 u \, du = \tan u - u + C$$

$$21. \int \cot^2 u \, du = -\cot u - u + C$$

$$22. \int \sin^3 u \, du = -\frac{1}{3}(2 + \sin^2 u) \cos u + C$$

$$23. \int \cos^3 u \, du = \frac{1}{3}(2 + \cos^2 u) \sin u + C$$

$$24. \int \tan^3 u \, du = \frac{1}{2} \tan^2 u + \ln |\cos u| + C$$

$$25. \int \cot^3 u \, du = -\frac{1}{2} \cot^2 u - \ln |\sin u| + C$$

$$26. \int \sec^3 u \, du = \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln |\sec u + \tan u| + C$$

$$27. \int \csc^3 u \, du = -\frac{1}{2} \csc u \cot u + \frac{1}{2} \ln |\csc u - \cot u| + C$$

$$28. \int \sin^n u \, du = -\frac{1}{n} \sin^{n-1} u \cos u + \frac{n-1}{n} \int \sin^{n-2} u \, du$$

$$29. \int \cos^n u \, du = \frac{1}{n} \cos^{n-1} u \sin u + \frac{n-1}{n} \int \cos^{n-2} u \, du$$

$$30. \int \tan^n u \, du = \frac{1}{n-1} \tan^{n-1} u - \int \tan^{n-2} u \, du$$

$$31. \int \cot^n u \, du = \frac{-1}{n-1} \cot^{n-1} u - \int \cot^{n-2} u \, du$$

$$32. \int \sec^n u \, du = \frac{1}{n-1} \tan u \sec^{n-2} u + \frac{n-2}{n-1} \int \sec^{n-2} u \, du$$

$$33. \int \csc^n u \, du = \frac{-1}{n-1} \cot u \csc^{n-2} u + \frac{n-2}{n-1} \int \csc^{n-2} u \, du$$

$$34. \int \sin au \sin bu \, du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$$

$$35. \int \cos au \cos bu \, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$$

$$36. \int \sin au \cos bu \, du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$$

$$37. \int u \sin u \, du = \sin u - u \cos u + C$$

$$38. \int u \cos u \, du = \cos u + u \sin u + C$$

$$39. \int u^n \sin u \, du = -u^n \cos u + n \int u^{n-1} \cos u \, du$$

$$40. \int u^n \cos u \, du = u^n \sin u - n \int u^{n-1} \sin u \, du$$

$$41. \begin{aligned} \int \sin^n u \cos^m u \, du &= -\frac{\sin^{n-1} u \cos^{m+1} u}{n+m} + \frac{n-1}{n+m} \int \sin^{n-2} u \cos^m u \, du \\ &= \frac{\sin^{n+1} u \cos^{m-1} u}{n+m} + \frac{m-1}{n+m} \int \sin^n u \cos^{m-2} u \, du \end{aligned}$$

Exponential and Logarithmic Integrals

$$42. \int u e^{au} \, du = \frac{1}{a^2} (au - 1) e^{au} + C$$

$$43. \int u^n e^{au} \, du = \frac{1}{a} u^n e^{au} - \frac{n}{a} \int u^{n-1} e^{au} \, du$$

$$44. \int e^{au} \sin bu \, du = \frac{e^{au}}{a^2 + b^2} (a \sin bu - b \cos bu) + C$$

$$45. \int e^{au} \cos bu \, du = \frac{e^{au}}{a^2 + b^2} (a \cos bu + b \sin bu) + C$$

$$46. \int \ln u \, du = u \ln u - u + C$$

$$47. \int u^n \ln u \, du = \frac{u^{n+1}}{(n+1)^2} [(n+1) \ln u - 1] + C$$

$$48. \int \frac{1}{u \ln u} du = \ln |\ln u| + C$$

Hyperbolic Integrals

$$49. \int \sinh u du = \cosh u + C$$

$$50. \int \cosh u du = \sinh u + C$$

$$51. \int \tanh u du = \ln \cosh u + C$$

$$52. \int \coth u du = \ln |\sinh u| + C$$

$$53. \int \operatorname{sech} u du = \tan^{-1} |\sinh u| + C$$

$$54. \int \operatorname{csch} u du = \ln \left| \tanh \frac{1}{2} u \right| + C$$

$$55. \int \operatorname{sech}^2 u du = \tanh u + C$$

$$56. \int \operatorname{csch}^2 u du = -\coth u + C$$

$$57. \int \operatorname{sech} u \tanh u du = -\operatorname{sech} u + C$$

$$58. \int \operatorname{csch} u \coth u du = -\operatorname{csch} u + C$$

Inverse Trigonometric Integrals

$$59. \int \sin^{-1} u \, du = u \sin^{-1} u + \sqrt{1-u^2} + C$$

$$60. \int \cos^{-1} u \, du = u \cos^{-1} u - \sqrt{1-u^2} + C$$

$$61. \int \tan^{-1} u \, du = u \tan^{-1} u - \frac{1}{2} \ln (1+u^2) + C$$

$$62. \int u \sin^{-1} u \, du = \frac{2u^2-1}{4} \sin^{-1} u + \frac{u\sqrt{1-u^2}}{4} + C$$

$$63. \int u \cos^{-1} u \, du = \frac{2u^2-1}{4} \cos^{-1} u - \frac{u\sqrt{1-u^2}}{4} + C$$

$$64. \int u \tan^{-1} u \, du = \frac{u^2+1}{2} \tan^{-1} u - \frac{u}{2} + C$$

$$65. \int u^n \sin^{-1} u \, du = \frac{1}{n+1} \left[u^{n+1} \sin^{-1} u - \int \frac{u^{n+1} \, du}{\sqrt{1-u^2}} \right], n \neq -1$$

$$66. \int u^n \cos^{-1} u \, du = \frac{1}{n+1} \left[u^{n+1} \cos^{-1} u + \int \frac{u^{n+1} \, du}{\sqrt{1-u^2}} \right], n \neq -1$$

$$67. \int u^n \tan^{-1} u \, du = \frac{1}{n+1} \left[u^{n+1} \tan^{-1} u - \int \frac{u^{n+1} \, du}{1+u^2} \right], n \neq -1$$

Integrals Involving $a^2 + u^2$, $a > 0$

$$68. \int \sqrt{a^2 + u^2} \, du = \frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln (u + \sqrt{a^2 + u^2}) + C$$

$$69. \int u^2 \sqrt{a^2 + u^2} \, du = \frac{u}{8} (a^2 + 2u^2) \sqrt{a^2 + u^2} - \frac{a^4}{8} \ln (u + \sqrt{a^2 + u^2}) + C$$

$$70. \int \frac{\sqrt{a^2 + u^2}}{u} du = \sqrt{a^2 + u^2} - a \ln \left| \frac{a + \sqrt{a^2 + u^2}}{u} \right| + C$$

$$71. \int \frac{\sqrt{a^2 + u^2}}{u^2} du = -\frac{\sqrt{a^2 + u^2}}{u} + \ln \left(u + \sqrt{a^2 + u^2} \right) + C$$

$$72. \int \frac{du}{\sqrt{a^2 + u^2}} = \ln \left(u + \sqrt{a^2 + u^2} \right) + C$$

$$73. \int \frac{u^2 du}{\sqrt{a^2 + u^2}} = \frac{u}{2} \left(\sqrt{a^2 + u^2} \right) - \frac{a^2}{2} \ln \left(u + \sqrt{a^2 + u^2} \right) + C$$

$$74. \int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a} \ln \left| \frac{\sqrt{a^2 + u^2} + a}{u} \right| + C$$

$$75. \int \frac{du}{u^2 \sqrt{a^2 + u^2}} = -\frac{\sqrt{a^2 + u^2}}{a^2 u} + C$$

$$76. \int \frac{du}{(a^2 + u^2)^{3/2}} = \frac{u}{a^2 \sqrt{a^2 + u^2}} + C$$

Integrals Involving $u^2 - a^2$, $a > 0$

$$77. \int \sqrt{u^2 - a^2} du = \frac{u}{2} \sqrt{u^2 - a^2} - \frac{a^2}{2} \ln \left| u + \sqrt{u^2 - a^2} \right| + C$$

$$78. \int u^2 \sqrt{u^2 - a^2} du = \frac{u}{8} (2u^2 - a^2) \sqrt{u^2 - a^2} - \frac{a^4}{8} \ln \left| u + \sqrt{u^2 - a^2} \right| + C$$

$$79. \int \frac{\sqrt{u^2 - a^2}}{u} du = \sqrt{u^2 - a^2} - a \cos^{-1} \frac{a}{|u|} + C$$

$$80. \int \frac{\sqrt{u^2 - a^2}}{u^2} du = -\frac{\sqrt{u^2 - a^2}}{u} + \ln \left| u + \sqrt{u^2 - a^2} \right| + C$$

$$81. \int \frac{du}{\sqrt{u^2 - a^2}} = \ln |u + \sqrt{u^2 - a^2}| + C$$

$$82. \int \frac{u^2 du}{\sqrt{u^2 - a^2}} = \frac{u}{2} \sqrt{u^2 - a^2} + \frac{a^2}{2} \ln |u + \sqrt{u^2 - a^2}| + C$$

$$83. \int \frac{du}{u^2 \sqrt{u^2 - a^2}} = \frac{\sqrt{u^2 - a^2}}{a^2 u} + C$$

$$84. \int \frac{du}{(u^2 - a^2)^{3/2}} = -\frac{u}{a^2 \sqrt{u^2 - a^2}} + C$$

Integrals Involving $a^2 - u^2$, $a > 0$

$$85. \int \sqrt{a^2 - u^2} du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C$$

$$86. \int u^2 \sqrt{a^2 - u^2} du = \frac{u}{8} (2u^2 - a^2) \sqrt{a^2 - u^2} + \frac{a^4}{8} \sin^{-1} \frac{u}{a} + C$$

$$87. \int \frac{\sqrt{a^2 - u^2}}{u} du = \sqrt{a^2 - u^2} - a \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| + C$$

$$88. \int \frac{\sqrt{a^2 - u^2}}{u^2} du = -\frac{1}{u} \sqrt{a^2 - u^2} - \sin^{-1} \frac{u}{a} + C$$

$$89. \int \frac{u^2 du}{\sqrt{a^2 - u^2}} = -\frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \frac{u}{a} + C$$

$$90. \int \frac{du}{u \sqrt{a^2 - u^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - u^2}}{u} \right| + C$$

$$91. \int \frac{du}{u^2 \sqrt{a^2 - u^2}} = -\frac{1}{a^2 u} \sqrt{a^2 - u^2} + C$$

$$92. \int (a^2 - u^2)^{3/2} du = -\frac{u}{8} (2u^2 - 5a^2) \sqrt{a^2 - u^2} + \frac{3a^4}{8} \sin^{-1} \frac{u}{a} + C$$

$$93. \int \frac{du}{(a^2 - u^2)^{3/2}} = -\frac{u}{a^2 \sqrt{a^2 - u^2}} + C$$

Integrals Involving $2au - u^2$, $a > 0$

$$94. \int \sqrt{2au - u^2} du = \frac{u-a}{2} \sqrt{2au - u^2} + \frac{a^2}{2} \cos^{-1} \left(\frac{a-u}{a} \right) + C$$

$$95. \int \frac{du}{\sqrt{2au - u^2}} = \cos^{-1} \left(\frac{a-u}{a} \right) + C$$

$$96. \int u \sqrt{2au - u^2} du = \frac{2u^2 - au - 3a^2}{6} \sqrt{2au - u^2} + \frac{a^3}{2} \cos^{-1} \left(\frac{a-u}{a} \right) + C$$

$$97. \int \frac{du}{u \sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C$$

Integrals Involving $a + bu$, $a \neq 0$

$$98. \int \frac{u du}{a + bu} = \frac{1}{b^2} (a + bu - a \ln |a + bu|) + C$$

$$99. \int \frac{u^2 du}{a + bu} = \frac{1}{2b^3} \left[(a + bu)^2 - 4a(a + bu) + 2a^2 \ln |a + bu| \right] + C$$

$$100. \int \frac{du}{u(a + bu)} = \frac{1}{a} \ln \left| \frac{u}{a + bu} \right| + C$$

$$101. \int \frac{du}{u^2(a + bu)} = -\frac{1}{au} + \frac{b}{a^2} \ln \left| \frac{a + bu}{u} \right| + C$$

$$102. \int \frac{u du}{(a + bu)^2} = \frac{a}{b^2(a + bu)} + \frac{1}{b^2} \ln |a + bu| + C$$

$$103. \int \frac{u \, du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2} \ln \left| \frac{a+bu}{u} \right| + C$$

$$104. \int \frac{u^2 \, du}{(a+bu)^2} = \frac{1}{b^3} \left(a + bu - \frac{a^2}{a+bu} - 2a \ln |a+bu| \right) + C$$

$$105. \int u \sqrt{a+bu} \, du = \frac{2}{15b^2} (3bu - 2a)(a+bu)^{3/2} + C$$

$$106. \int \frac{u \, du}{\sqrt{a+bu}} = \frac{2}{3b^2} (bu - 2a) \sqrt{a+bu} + C$$

$$107. \int \frac{u^2 \, du}{\sqrt{a+bu}} = \frac{2}{15b^3} (8a^2 + 3b^2 u^2 - 4abu) \sqrt{a+bu} + C$$

$$108. \int \frac{du}{u \sqrt{a+bu}} = \frac{1}{\sqrt{a}} \ln \left| \frac{\sqrt{a+bu} - \sqrt{a}}{\sqrt{a+bu} + \sqrt{a}} \right| + C, \quad \text{if } a > 0$$

$$= \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bu}{-a}} + C, \quad \text{if } a < 0$$

$$109. \int \frac{\sqrt{a+bu}}{u} \, du = 2\sqrt{a+bu} + a \int \frac{du}{u \sqrt{a+bu}}$$

$$110. \int \frac{\sqrt{a+bu}}{u^2} \, du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2} \int \frac{du}{u \sqrt{a+bu}}$$

$$111. \int u^n \sqrt{a+bu} \, du = \frac{2}{b(2n+3)} \left[u^n (a+bu)^{3/2} - na \int u^{n-1} \sqrt{a+bu} \, du \right]$$

$$112. \int \frac{u^n \, du}{\sqrt{a+bu}} = \frac{2u^n \sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)} \int \frac{u^{n-1} \, du}{\sqrt{a+bu}}$$

$$113. \int \frac{du}{u^n \sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)} \int \frac{du}{u^{n-1} \sqrt{a+bu}}$$

Table of Derivatives

General Formulas

$$1. \frac{d}{dx}(c) = 0$$

$$2. \frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

$$3. \frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

$$4. \frac{d}{dx}(x^n) = nx^{n-1}, \text{ for real numbers } n$$

$$5. \frac{d}{dx}(cf(x)) = cf'(x)$$

$$6. \frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

$$7. \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$8. \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Trigonometric Functions

$$9. \frac{d}{dx}(\sin x) = \cos x$$

$$10. \frac{d}{dx}(\tan x) = \sec^2 x$$

$$11. \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$12. \frac{d}{dx}(\cos x) = -\sin x$$

$$13. \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$14. \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Inverse Trigonometric Functions

$$15. \frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$16. \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$$

$$17. \frac{d}{dx} (\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$18. \frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$19. \frac{d}{dx} (\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$20. \frac{d}{dx} (\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Exponential and Logarithmic Functions

$$21. \frac{d}{dx} (e^x) = e^x$$

$$22. \frac{d}{dx} (\ln |x|) = \frac{1}{x}$$

$$23. \frac{d}{dx} (b^x) = b^x \ln b$$

$$24. \frac{d}{dx} (\log_b x) = \frac{1}{x \ln b}$$

Hyperbolic Functions

$$25. \frac{d}{dx} (\sinh x) = \cosh x$$

$$26. \frac{d}{dx} (\tanh x) = \operatorname{sech}^2 x$$

$$27. \frac{d}{dx} (\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$28. \frac{d}{dx} (\cosh x) = \sinh x$$

$$29. \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$30. \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Inverse Hyperbolic Functions

$$31. \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

$$32. \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} (|x| < 1)$$

$$33. \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}} \quad (0 < x < 1)$$

$$34. \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$$

$$35. \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2} \quad (|x| > 1)$$

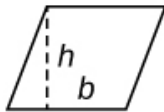
$$36. \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1+x^2}} \quad (x \neq 0)$$

Review of Pre-Calculus

Formulas from Geometry

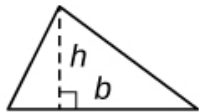
A = area, V = Volume, and S = lateral surface area

Parallelogram



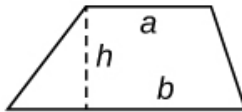
$$A = bh$$

Triangle



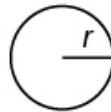
$$A = \frac{1}{2}bh$$

Trapezoid



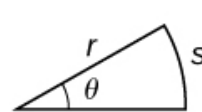
$$A = \frac{1}{2}(a + b)h$$

Circle



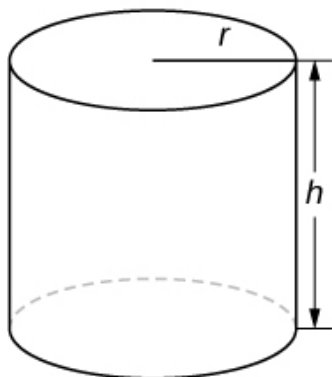
$$A = \pi r^2$$
$$C = 2\pi r$$

Sector



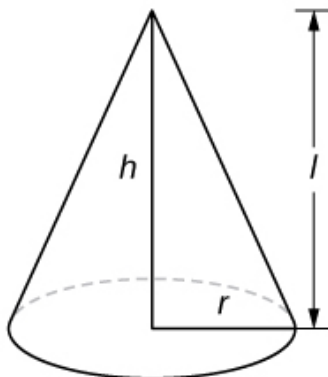
$$A = \frac{1}{2}r^2\theta$$
$$s = r\theta \text{ (}\theta \text{ in radians)}$$

Cylinder



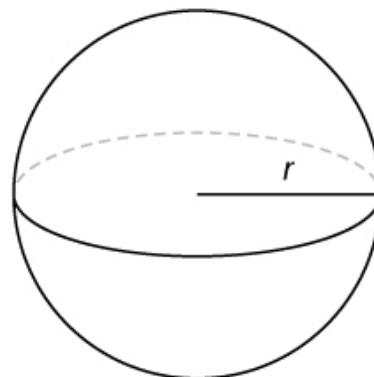
$$V = \pi r^2 h$$
$$S = 2\pi r h$$

Cone



$$V = \frac{1}{3}\pi r^2 h$$
$$S = \pi r l$$

Sphere



$$V = \frac{4}{3}\pi r^3$$
$$S = 4\pi r^2$$

Formulas from Algebra

Laws of Exponents

$$\begin{array}{lll}
x^m x^n & = & x^{m+n} \\
x^{-n} & = & \frac{1}{x^n} \\
x^{1/n} & = & \sqrt[n]{x} \\
x^{m/n} & = & \sqrt[n]{x^m} = (\sqrt[n]{x})^m
\end{array}
\qquad
\begin{array}{lll}
\frac{x^m}{x^n} & = & x^{m-n} \\
(xy)^n & = & x^n y^n \\
\sqrt[n]{xy} & = & \sqrt[n]{x} \sqrt[n]{y}
\end{array}
\qquad
\begin{array}{lll}
(x^m)^n & = & x^{mn} \\
\left(\frac{x}{y}\right)^n & = & \frac{x^n}{y^n} \\
\sqrt[n]{\frac{x}{y}} & = & \frac{\sqrt[n]{x}}{\sqrt[n]{y}}
\end{array}$$

Special Factorizations

$$\begin{array}{ll}
x^2 - y^2 & = (x + y)(x - y) \\
x^3 + y^3 & = (x + y)(x^2 - xy + y^2) \\
x^3 - y^3 & = (x - y)(x^2 + xy + y^2)
\end{array}$$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ca}}{2a}$.

Binomial Theorem

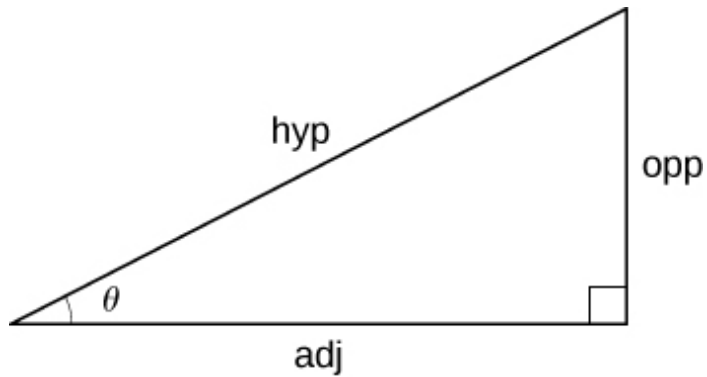
$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a b^{n-1} + b^n,$$

where $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 3 \cdot 2 \cdot 1} = \frac{n!}{k!(n-k)!}$

Formulas from Trigonometry

Right-Angle Trigonometry

$$\begin{aligned}\sin \theta &= \frac{\text{opp}}{\text{hyp}} & \csc \theta &= \frac{\text{hyp}}{\text{opp}} \\ \cos \theta &= \frac{\text{adj}}{\text{hyp}} & \sec \theta &= \frac{\text{hyp}}{\text{adj}} \\ \tan \theta &= \frac{\text{opp}}{\text{adj}} & \cot \theta &= \frac{\text{adj}}{\text{opp}}\end{aligned}$$



Trigonometric Functions of Important Angles

θ	Radians	$\sin \theta$	$\cos \theta$	$\tan \theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	—

Fundamental Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

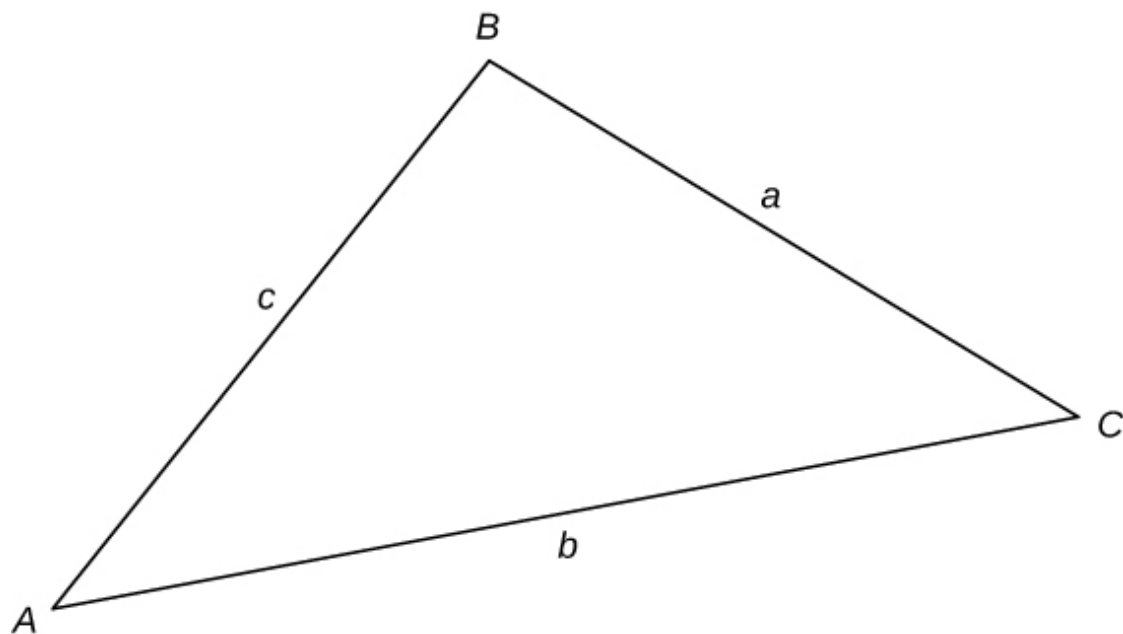
$$\sin(\theta + 2\pi) = \sin\theta$$

$$\cos(\theta + 2\pi) = \cos\theta$$

$$\tan(\theta + \pi) = \tan\theta$$

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Addition and Subtraction Formulas

$$\sin (x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin (x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos (x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos (x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double-Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Half-Angle Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$